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## HIGHER EXCITED STATES OF $\alpha+\alpha$ SYSTEM

**Abstract.** In this work we investigate the higher excited states of  $\alpha+\alpha$  system applying the complex scaling method. The low-lying  $0^+$ ,  $2^+$  and  $4^+$  states of  $\alpha+\alpha$  are measured well but the higher excited states  $6^+$ ,  $8^+$  and  $10^+$  of  $\alpha+\alpha$  are not available by experimentally and these higher excited states have been barely studied by theoretical approaches.

**Keywords:** Complex scaling method, alpha-alpha system.

### INTRODUCTION

The complex scaling method (CSM) [1-5] has been successfully utilized in the description of resonance states in light nuclei. The theory of the complex scaling was proposed mathematically [2] and it has been extensively applied to the atomic and nuclear physics [6-10].

In this work we investigate structure of  $\alpha+\alpha$  system. In particular, we focus on the its higher excited  $6^+$ ,  $8^+$  and  $10^+$  states because there is no experimental evidence for those higher states. But its low-lying  $0^+$ ,  $2^+$  and  $4^+$  states are experimentally well known. In addition, in this work we apply the CSM and harmonic oscillator wave function in order to calculate both low-lying and higher excited states of  $\alpha+\alpha$ .

### THEORETICAL FRAMEWORK

#### Complex Scaling Method

In the CSM the relative coordinate is rotated as like  $r \rightarrow re^{i\theta}$  in the complex coordinate plane. Therefore, the *Schrödinger* equation

$$\hat{H}|\psi\rangle = E|\psi\rangle \quad (1)$$

is rewritten as

$$\hat{H}(\theta)|\psi^\theta\rangle = E^\theta|\psi^\theta\rangle, \quad (2)$$

where  $\hat{H}(\theta)$  and  $\psi^\theta$  are the complex scaled Hamiltonian and wave function, respectively. The  $\theta$  is scaling angle being a real number,  $U(\theta)$  operate on a function  $\psi^\theta$ , that is

$$\psi^\theta = U(\theta)\psi(r) = e^{\frac{3}{2}i\theta} \psi(re^{i\theta}). \quad (3)$$

The eigenvalues and eigenstates are obtained by solving the complex scaled *Schrödinger* equation Eq.(2). The eigenvalues of resonance states are found as  $E^\theta = E_r - i\Gamma_r/2$ , where  $E_r$  is resonance energy and  $\Gamma_r$  is the width of resonance. More detailed explanation of the CSM is given in Refs.[1-2].

### Two body interaction

For the alpha-alpha system the Hamiltonian is expressed as

$$\hat{H} = \sum_{i=1}^2 \hat{T}_i - \hat{T}_{c.m.} + V_{\alpha\alpha}^{Nucl}(r) + V_{\alpha\alpha}^{Coul}(r). \quad (4)$$

Harmonic oscillator wave function for radial part is

$$\varphi_{nl}(r) = N_l^n \left( \frac{r}{b_F} \right)^l L_n^{l+\frac{1}{2}} \left( \left( \frac{r}{b_F} \right)^2 \right) \exp \left( -\frac{1}{2b_F^2} r^2 \right) Y_{lm}(r), \quad (5)$$

here  $L_n^{l+\frac{1}{2}}$  are Laguerre polynomials for the angular momentum  $l$  and  $N_l^n$  denotes the normalization

constants as given by  $N_l^n = \left\{ \frac{2\Gamma(n+1)}{b_F^3 \Gamma(l+n+\frac{3}{2})} \right\}^{1/2}$ . The size parameter of relative motion of two alpha-

cluster  $b_F$  is taken as 0.967 fm which corresponds to a single particle size parameter  $b_0 = 1.3975$  fm employed to fit the observed r.m.s. radius of  ${}^4\text{He}$  [11-12].

### Alpha-alpha potential

The  $\alpha+\alpha$  potential is constructed by the folding approach for the effective nucleon-nucleon interaction by the Schmid-Wildermuth [13-14] potential. An effective two-nucleon force is written as,

$$v_{ij} = V \{ W + BP_\sigma(ij) - HP_\tau(ij) - MP_\sigma(ij)P_\tau(ij) \} \cdot \exp(-\mu r^2) \quad (6)$$

where  $P_\sigma(ij)$  and  $P_\tau(ij)$  are the spin and isospin exchange operators. In this work we employ the Schmid-Wildermuth potential as a nucleon-nucleon force, which is given by following parameters:

$$\begin{aligned} V &= -72.98 \text{ MeV}; \mu = 0.46 \text{ fm}^{-2}; \\ W &= M = 0.4075; B = H = 0.0925. \end{aligned} \quad (7)$$

The folding potential of the alpha-alpha system is obtained from such a nucleon-nucleon force and also the Coulomb force.

Its explicit form is

$$V_{\alpha\alpha}^{Nucl} + V_{\alpha\alpha}^{Coul} = 2X_D \left[ \frac{2v_\alpha}{2v_\alpha + \frac{3\mu}{2}} \right]^{\frac{3}{2}} V \exp \left( -\frac{v_\alpha \mu}{v_\alpha + \frac{3\mu}{4}} r^2 \right) + \frac{4e^2}{r} \text{erf} \left( r \sqrt{\frac{4}{3} v_\alpha} \right), \quad (8)$$

where  $X_D = 2.445$  and  $\text{erf}(x)$  is the error function. We use a harmonic oscillator constant  $v_\alpha = \frac{M\omega}{2\hbar} = 0.2675 \text{ fm}^{-2}$  which is obtained by using  $r_{rms} = 1.63$  fm of the alpha-cluster.

In Eq. (8) the simplified notations can be applied:

$$V_0 = 2X_D V \left[ \frac{2v_\alpha}{2v_\alpha + \frac{3\mu}{2}} \right]^{\frac{3}{2}},$$

$$\beta = \frac{v_\alpha \mu}{v_\alpha + \frac{3\mu}{4}},$$

$$\alpha = \sqrt{\frac{4}{3}} v_\alpha.$$

**RESULTS**

**Complex Scaling Method**

Figure 1 displays the complex energy eigenvalues of  $4^+$  state which is obtained by diagonalization of Eq.(2) with  $N_{\max} = 50$  for  $\theta = 13^\circ$ . We can see all energies are on lines of  $\arg(E_0) = 2\theta$  which correspond to the branch cut of the complex energy plane. When we take larger values of  $\theta$ , we observe isolated energy points, being resonance states, whose positions are almost unchanged by varying  $\theta \geq \frac{1}{2} |\arg(E_\theta^R)|$ .

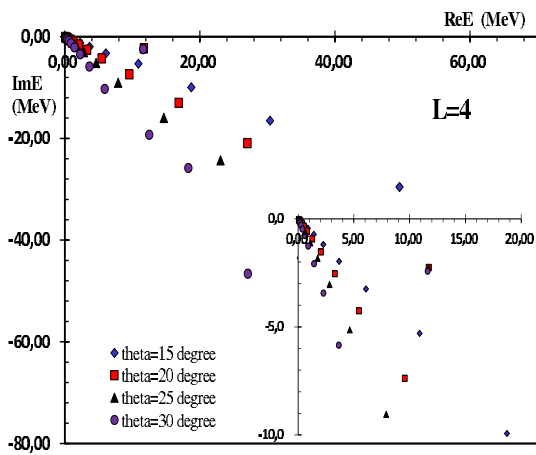


Figure 1. The resonance eigenvalues at  $J^\pi=4^+$  for the different  $\theta$

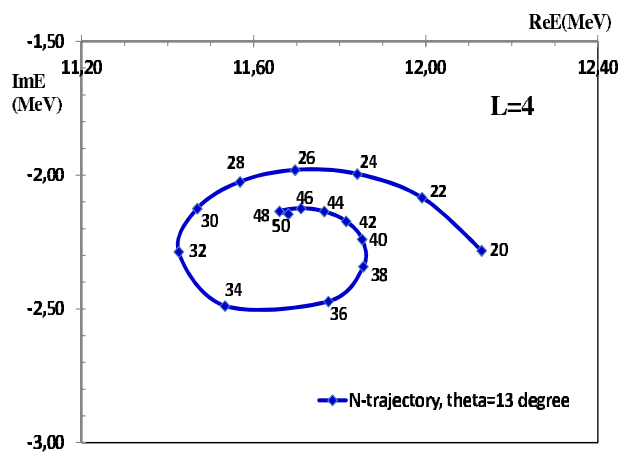


Figure 2.  $N$ -trajectory at  $J^\pi=4^+$ . The folding potential and harmonic oscillator wave function are used

In Fig. 2 presents the eigenvalues of  $4^+$  state which is calculated with  $N_{\max} = 20 \sim 50$ . This is called the  $N_{\max}$ -trajectory. As the number of basis states is increased, the  $N_{\max}$ -trajectory shows the spiral convergence. The radius of curvature of the spiral depends on the values of  $\theta$  and  $b_0$ . In calculation of the  $N_{\max}$ -trajectory, we fixed to  $\theta = 13^\circ$  and  $b_0 = 1.3975$  fm for  $4^+$ . The computed decay widths for the experimentally unknown  $6^+$ ,  $8^+$  and  $10^+$  higher states are rather large, however, they can be recognized as resonances on the complex energy plane.

Table I - Experimental and calculated resonance energies with corresponding decay widths of  $\alpha+\alpha$  system

States	Experimental data [12]		Present work	
	$E_r(MeV)$	$\Gamma_r(MeV)$	$E_r(MeV)$	$\Gamma_r(MeV)$
$4^+$	11.35	$\sim 3.5$	11.7	4.4
$6^+$	-	-	30.5	36.8
$8^+$	-	-	51.6	120
$10^+$	-	-	70.0	180

For the case of  $6^+$ ,  $8^+$  and  $10^+$  states, there are no measured data, however, we calculated a resonance energy with a broad decay width applying harmonic oscillator wave function. The calculated results are given in table 1.

### SUMMARY

Positions and widths of low-lying and higher excited states of  $^8\text{Be}$  are calculated by using the CSM and the two-body model. The result of recent calculation indicates that reasonably good agreement with measured data for  $4^+$  state. It is remarkable that the energies with decay widths of the higher excited  $6^+$ ,  $8^+$  and  $10^+$  states are calculated.

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### $\alpha+\alpha$ ЖҮЙЕСІНІҢ ЖОҒАРҒЫ ҚОЗҒАН КҮЙЛЕРІ

**Аннотация.** Бұл жұмыста кешенді масштабтау әдісін қолдана отырып  $\alpha+\alpha$  жүйесінің жоғары қозған күйлері зерттелді.  $\alpha+\alpha$ -ның төмен  $0^+$ ,  $2^+$  және  $4^+$  күйлері өлшенген, бірақ  $\alpha+\alpha$  жүйесінің  $6^+$ ,  $8^+$  және  $10^+$  жоғары қозған күйлерінің эксперименталды мәндері белгісіз болғандықтан, бұл жоғары қозған күйлер теориялық тәсілдермен есептелінді.

**Түйін сөздер:** кешенді масштабтау әдісі, альфа-альфа жүйесі.

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### ВЫСОКИЕ ВОЗБУЖДЕННЫЕ СОСТОЯНИЯ $\alpha+\alpha$ СИСТЕМЫ

**Аннотация.** В данной работе мы исследуем высокие возбужденные состояния  $\alpha+\alpha$  системы, применяя метод комплексного масштабирования. Низколежащие  $0^+$ ,  $2^+$  и  $4^+$  состояния  $\alpha+\alpha$  системы хорошо известны, но высокие возбужденные состояния  $6^+$ ,  $8^+$  и  $10^+$   $\alpha+\alpha$  системы не доступны экспериментально, поэтому эти высокие возбужденные состояния были изучены теоретическими подходами.

**Ключевые слова:** метод комплексного масштабирования, альфа-альфа-система.

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### **3D MODELING OF COMBUSTION THERMOCHEMICAL ACTIVATED FUEL**

**Abstract.** This article presents the results of numerical research of plasma thermochemical processes of preparing solid fuels for combustion in combustion chambers. During the numerical experiments have been applied the newest information technology and method of 3-D computer modeling of heat and mass transfer in the furnace space. Received the basic laws of convective heat and mass transfer in turbulent flows in the presence of chemical reactions using modern numerical methods, giving a complete description of the complex processes occurring in a real combustion chamber. Research of three-dimensional temperature and concentration fields has allowed establishing laws of the development of the combustion process in the entire volume of the object under study. Satisfactory agreement is obtained estimates with known results of natural experiments. Authors of article for the first time investigate influence of plasma thermochemical treatment of pulverized coal flows on the main characteristics of the physicochemical processes of solid fuel combustion. It has been established that the method of thermochemical activation of pulverized coal flows makes it possible to significantly optimize the process of burning low-grade high-ash Kazakhstan coal in the combustion chambers of the thermal power plants of Kazakhstan, significantly reduce emissions of harmful substances into the environment and create a way to obtain "clean" energy on energy facilities.

**Key words.** Heat and mass transfer, combustion, solid fuel, plasma activation, aerodynamic flow, concentration and temperature field, emissions of harmful substances.

#### **Introduction**

Studies of heat and mass transfer in high-temperature and chemically reacting environments are important at creation of new physical and chemical technologies, at design of aviation and rocket technics, by development new furnaces, gas turbines and internal combustion engines. In the conditions of depletion of natural energy resources and environmental pollution, the development of technological processes with the rational use of energy fuel and the solution of environmental problems are urgent and most important tasks for many thermophysical studies in this direction.

Heat and mass transfer processes in the presence of physical-chemical transformations and combustion are non-stationary, strongly non-isothermal with constant changes in the physical and chemical state of the environment, which greatly complicates them both experimental and theoretical study. Such flows are described by a complex system of non-autonomous nonlinear partial differential equations, in which the essential turbulence, multiphase nature of the medium, and source terms associated with the chemical kinetics of the processes occurring must be taken into account [1-5].

The main methods of study of such processes, particularly in the areas of real geometry are methods of numerical simulation and holding on to them through computational experiments adequately reflect the real physical processes, occurring in combustion chambers. Progress in the development of computational models, in the creation of effective computational algorithms and problem-oriented software packages allows us to solve many problems that have a huge practical application for various industries, especially for energy [6-11].

In this article is carried out of heat and mass transfer processes occurring in the areas of real geometry (combustion chamber), with the burning of energy fuel in them. The basic patterns and features of the formation of the aerodynamics of flow, velocity, temperature and concentration fields are established and the effect of plasma thermochemical treatment of pulverized coal on the main characteristics of the combustion process is shown.

This research builds on the achievements of modern thermal physics, on the use of new numerical methods for 3-D modeling, on the construction of efficient computational algorithms and new computational models, which make it possible to more accurately describe the actual physical and chemical processes occurring during at burning of power fuel in the combustion chambers of existing energy facilities [12-15].

### **Statement of the problem of burning thermochemically activated torch in the combustion chamber of an energy boiler**

Thermochemical preparation of pulverized coal for combustion is implemented in accordance with the following mechanism. Aero mixture is fed through the dust pipe to the burner. According to the traditional scheme, the air mixture is fed into the “cold” hot fire chamber (T about 350K), then it heats up and ignites. If the burner is equipped with a plasma torch, then the air mixture is heated by the plasma torch in the volume of the burner to the exit to the furnace. In this case, volatile coal is emitted and the coke residue is gasified. Gasification and volatile products begin to oxidize with oxygen of the primary air of the air mixture, which leads to additional heating of the reaction air of the air mixture. At the same time, carbon of the fuel is gasified to CO, and not to CO<sub>2</sub>, due to the existing coefficient of excess air in the air mixture (0.3-0.5 of the stoichiometric ratio) [13].

As a result, the output of plasma-fuel system (PFS) a high-temperature (about 1300K) reactive flow of carbon-containing particles and gaseous products of the plasma thermochemical preparation aero mixture. In this case, regardless of the quality of the original product, we get highly reactive two-component fuel. When mixed with secondary air in the combustion chamber, it is intensely ignited and burns steadily without fuel oil or gas, traditionally used to ignite and stabilize the combustion of low-grade pulverized coal [14-15].

To carry out computational experiments, the combustion chamber of the BKZ-420 boiler of the Almaty Thermal Power Plant-2 has been chosen [16-17]. The BKZ-420 steam boiler with a 420 t/h steam capacity is equipped with six rotary pulverized coal burners located in two levels with three burners on the front wall of the boiler (Fig. 1 a).

At computer simulation, the method of control volumes which is applied to numerically solve differential equations describing heat and mass transfer in the combustion chamber, and is described in detail in the following papers [18-21].

For this, the combustion chamber of the boiler under study is divided into small control volumes, the number of which depends on the geometry of the combustion chamber, its dimensions, and the location of the burner and plasma devices. For the BKZ-420 boiler, the calculated area is – 261008 control volumes (72x139x126) (Fig. 1 b).

When conducting computational experiments for the boiler BKZ-420 were investigated two modes of operation of the combustion chamber: 1) traditional combustion (burner works with standard pulverized coal burners); 2) the plasma torch is installed and operates on a pulverized coal flow in three burners: two extreme burners of the lower tier and a central burner of the upper tier [22-29].

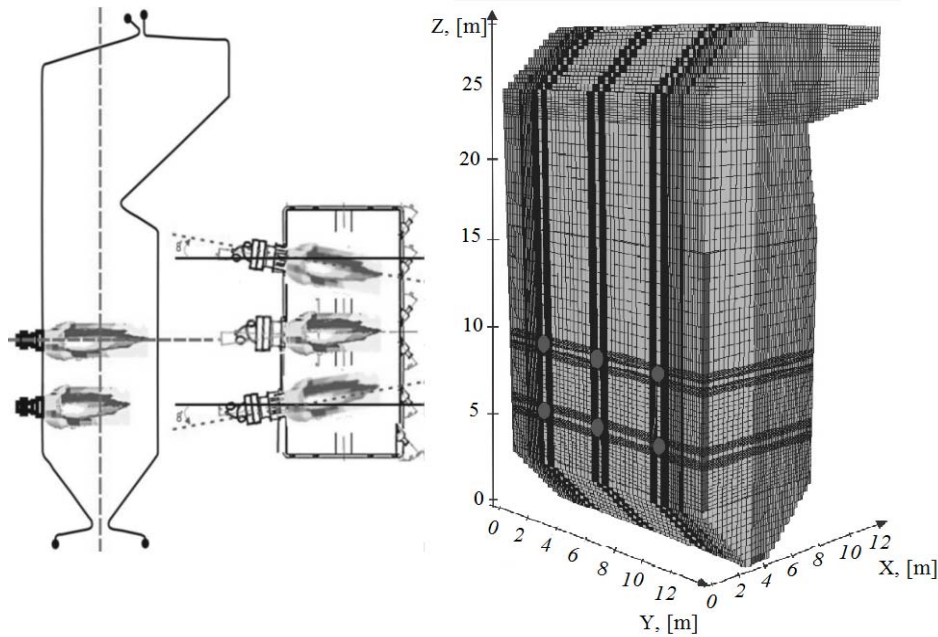


Fig. 1. General view of the combustion chamber of the boiler BKZ-420: configuration of torches (on left); breakdown into control volumes (on right)

### The results of computational experiments

Below are the results of 3D computer simulation of the study of the influence of thermochemical activation of pulverized coal flows on the combustion process of a pulverized coal flame in the combustion chamber of the BKZ-420 boiler.

The aerodynamics of the flow (distribution of the full velocity vector), temperature and concentration fields of nitrogen oxides NO over the entire combustion space of the combustion chamber were obtained for the traditional combustion of fuel and for the case when the plasma torch in the three burners affects the pulverized coal.

Figure 2 shows the aerodynamics of the flow and the distribution of the full velocity vector in the belt region of the burners of the combustion chamber. Analysis of the figure shows that the thermochemical activation of the pulverized coal flow has a significant impact on the flow field on the propagation of the reacting jet in the furnace volume, mixing processes in the jet, on the size and shape of the flame. This can be explained by the fact that, under the action of plasma activation of pulverized jets, turbulization of flows is intensified, and this in turn leads to acceleration of mass and heat exchange associated with increased mixture formation, additional heating of the air mixture and intensification of the burning process.

Figure 3 illustrates the temperature field in the sectional plane of the burners of the lower tier of the combustion chamber of the boiler BKZ-420. Compared with the combustion of a conventional pulverized coal stream, the average temperature in this area in the case of using thermochemically activated streams increases and is 1530°C without activation, and 1640°C for 3 activated streams.

It can be concluded that the plasma activation of the aerosol leads to its rapid heating, ignition and stable heating. At the same time, there is a shift of the combustion front to the location of plasma activation systems of coal streams. The region of high temperatures with increasing number of plasma-activated flows is shifted to the center of symmetry of the furnace, while at the side surfaces a higher temperature level.

Figure 4 shows a comparative analysis of the distribution of the average temperature in the cross section over the height of the combustion chamber of the BKZ-420 boiler. When plasma processing of fuel is observed, the location of the core of the torch is shifted and the length of the zone of maximum temperatures increases. The increase in flame temperature during the burning of 3 activated streams

occurs faster, and the temperature value differs to a greater extent in the burner belt. The minimum on the curves are related to the low temperature of the air mixture entering the combustion chamber through burners not equipped with plasma systems. In general, there is a lack of sharp temperature drops across the entire height of the combustion chamber, which indicates the stability of the combustion process.

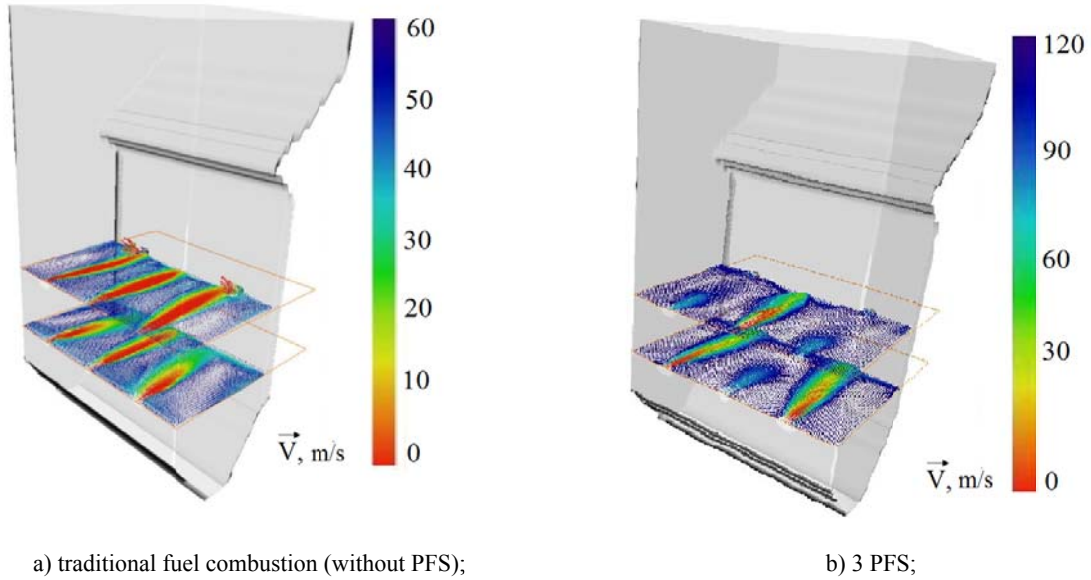


Fig. 2. Vector field of full velocity in the area of the burner belt of the combustion chamber of the boiler BKZ-420

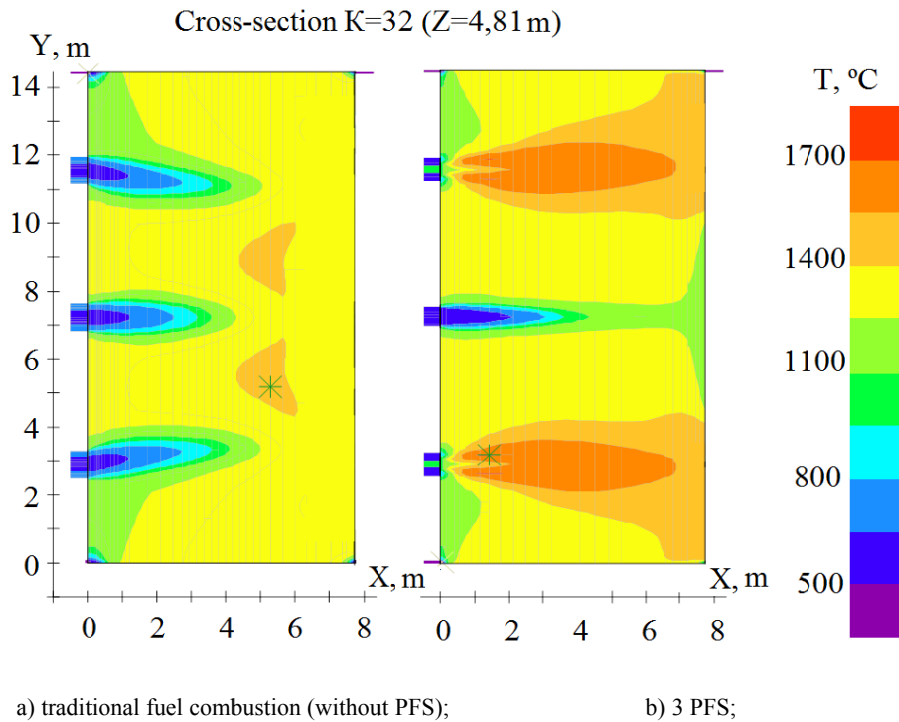
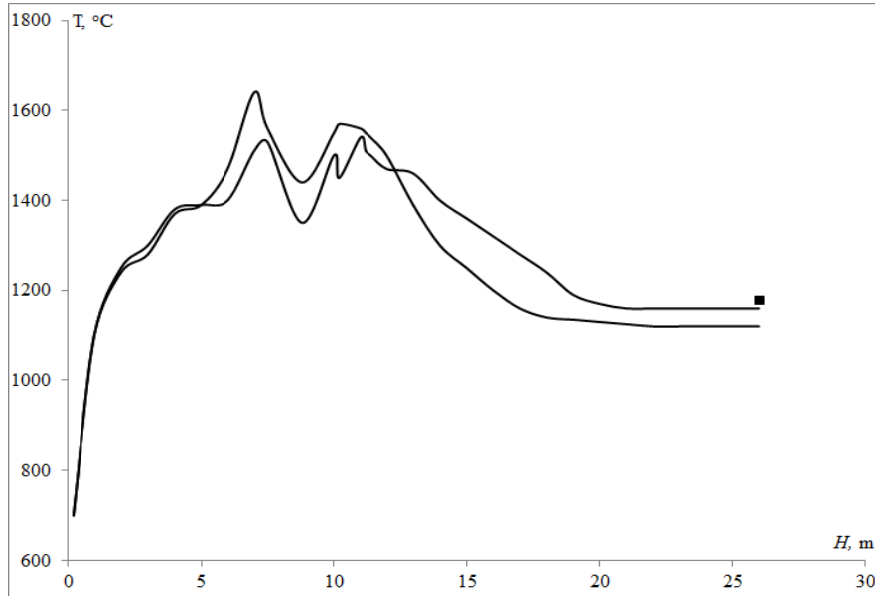
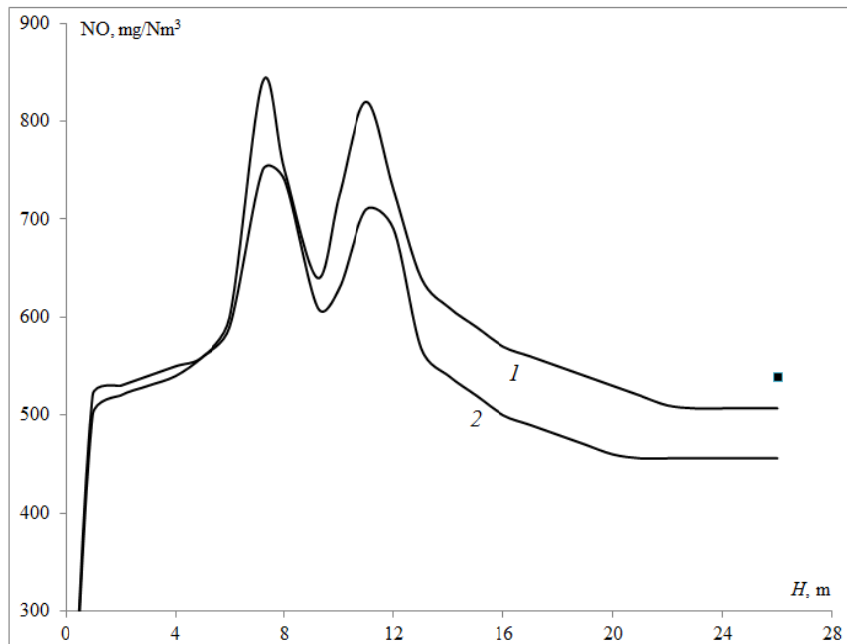


Fig. 3. Temperature field in the section plane of the burners of the lower tier of the combustion chamber of the boiler BKZ-420



1 – traditional fuel combustion (without PFS); 2 – 3 PFS;  
 ■ - experiment [30]

Fig. 4. Temperature distribution along the height of the combustion chamber of the boiler BKZ-420



1 – traditional fuel combustion (without PFS); 2 – 3 PFS;  
 ■ - experiment [30]

Fig. 5. The distribution of the concentration of nitrogen oxides NO at the height of the combustion chamber of the boiler BKZ-420

Figure 5 shows the fields of concentrations of nitrogen oxides NO along the height of the combustion chambers of the boiler BKZ-420. As can be seen from Figure 5, the main NO gas formation occurs in the region of propagation of flows from the burners. Thus character of the distribution of curves in this area is ambiguous, which indicates the complex nature of the process of formation of nitrogen oxides in this area

and the effect of plasma activation on the formation of these components. We see that the use of plasma torches leads to a decrease in the total concentration of NO from the furnace space and amounts to 507 mg/Nm<sup>3</sup> with traditional combustion, and with 3 thermochemically activated streams – 456 mg/Nm<sup>3</sup>. This, in turn, indicates that the reduction of nitrogen oxides NO at the exit from the furnace space when using PFS increases the ecological and economic indicators of energy facilities.

### Conclusion

According to the research, the following conclusions can be formulated:

- Presents the results of 3-D computer modeling of heat and mass transfer processes during combustion of pulverized coal, which was pre-processed in the PFS.
- Research of three-dimensional temperature and concentration fields has allowed establishing the basic patterns of development of heat and mass transfer processes in the entire volume of the combustion chamber of power boilers. Satisfactory agreement of the calculated data with the results of field experiments was obtained.
- It has been established that the method of thermochemical activation of pulverized coal flows makes it possible to significantly optimize the process of burning low-grade high-ash coal in the combustion chambers of Kazakhstan's thermal power plants and significantly reduce emissions of harmful substances into the environment.

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### АКТИВТЕНДІРІЛГЕН ТЕРМОХИМИЯЛЫҚ ОТЫННЫҢ ЖАНУЫН 3D МОДЕЛЬДЕУ

**Аннотация.** Мақалада жану камераларында қатты отынды жағуда плазма термохимиялық дайындау процестерін сандық зерттеу нәтижелері келтірілген. Сандық есептеу эксперименттерін жүргізу кезінде қазанның кеңістіктегі жылу және масса тасымалдау процестерінің ақпараттық технологиялары мен 3-D компьютерлік модельдері қолданылды. Нақты жану камерасында орын алатын турбулентті ағыстардағы конвективті жылу және массалық кешенді процестердің толық сипаттамасын беретін қазіргі заманғы сандық әдістермен химиялық реакциялардың қатысуымен алынды. Объектінің барлық көлеміндегі жану үдерісі үш өлшемді температура мен концентрация өрістерін зерттеуде даму моделін құруға мүмкіндік берді. Толық ауқымды эксперименттердің белгілі нәтижелерімен есептелген деректердің қанағаттанарлық сәйкестік алынды. Мақаланың авторлары қатты отын жануының физика-химиялық процестерінің негізгі сипаттамаларына шаңтозаңды көмір ағынын плазмалық термохимиялық өңдеудің әсерін зерттеді. Көмір ағынының термохимиялық активтендіру әдісі Қазақстанның жылу электр станцияларының жану камераларында жанғыш күлді көмірді көму үдерісін едәуір оңтайландыруға мүмкіндік береді, қоршаған ортаға зиянды заттар шығарындыларын едәуір азайтады және энергетикалық қондырғыларда «таза» энергетикаға қол жеткізуге мүмкіндік береді.

**Түйін сөздер.** Жылулық массатасымалдау, жану, көмір, плазмалық белсендіру, ағын аэродинамикасы, тұтқындар және температуралық өрісі, залалды заттардың шығарылу

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### 3D МОДЕЛИРОВАНИЕ ГОРЕНИЯ ТЕРМОХИМИЧЕСКИ АКТИВИРОВАННОГО ТОПЛИВА

**Аннотация.** В данной статье представлены результаты численных исследований процессов плазменной термохимической подготовки твердых топлив к сжиганию в камерах сгорания. При проведении вычислительных экспериментов были применены новейшие информационные технология и метод 3-D компьютерного моделирования процессов тепломассопереноса в топочном пространстве. Получены основные закономерности конвективного тепломассопереноса в турбулентных течениях при наличии химических реакций с использованием современных численных методов, дающих полное описание сложных процессов, имеющих место в реальной топочной камере. Исследование трехмерных температурных и концентрационных полей позволило установить закономерности развития процесса горения во всем объеме исследуемого объекта. Получено удовлетворительное согласие расчетных данных с известными результатами натуральных экспериментов. Авторами статьи впервые исследовано влияние плазменной термохимической обработки пылеугольных потоков на основные характеристики физико-химических процессов горения твердого топлива. Установлено, что метод термохимической активации пылеугольных потоков позволяет в значительной степени оптимизировать процесс сжигания низкосортных высокозольных казахстанских углей в топочных камерах ТЭС Казахстана, существенно снизить выбросы вредных веществ в окружающую среду и создать способ получения «чистой» энергии на энергетических объектах.

**Ключевые слова.** Тепломассоперенос, горение, твердое топливо, плазменная активация, аэродинамика течения, концентрационные и температурные поля, выбросы вредных веществ.

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## ABOUT SINGLE OPERATOR METHOD OF SOLUTION OF A SINGULARLY PERTURBED CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION $n$ – ORDER

**Abstract.** In this paper, by the method of the deviating argument, we obtain an asymptotic expansion of the solution of the Cauchy problem for an ordinary differential equation of  $n$  – th order with variable coefficients, with an estimate of the residual term through the right side of the equation. Many papers devoted to this topic are of an applied nature, and their estimates of the residual term are expressed in terms of  $O$  –large or  $o$  –small, so they have a theoretical value rather than applied, as they claim. The main advantage of the proposed method is the simplicity of its algorithm, and the residual term formula, explicitly expressed through the right side of the equation, and its evaluation.

**Keywords:** Singular value perturbation, spectral decomposition, deviating argument, residual term estimation, self - adjoint operator, Gilbert-Schmidt theorem, completely continuous operator, Friedrich's Lemma, Cauchy problem, asymptotic expansion, small parameter.

**1. Introduction.** Many problems of mechanics, physics, engineering and other fields of science lead to differential and integro-differential equations with a small parameter at the highest derivative. A systematic study of such equations (at present they are called singularly perturbed) began after the appearance of the fundamental works of A. N. Tikhonov [1-3], drew the attention of many researchers to equations with a small parameter at the highest derivative. In these works, a General formulation of the Cauchy problem for systems of nonlinear ordinary differential equations with a small parameter at the highest derivative is given, and theorems on the limit transition are proved, establishing a connection between the solution of the initial singularly perturbed Cauchy problem and the solution of the unperturbed problem obtained from the initial at zero value of the parameter.

One of the important problems of the theory of singularly perturbed equations is the construction of asymptotic expansions of solutions of equations by a small parameter.

Among the asymptotic methods developed for singularly perturbed problems, it should be noted a very effective method of the boundary functions proposed By M. I. Vishik and L. A. Lyusternik [4,5] for singularly perturbed linear and partial differential equations, as well as for the singularly perturbed nonlinear ordinary differential equations, and M. I. Imanaliev [8,9] for singularly perturbed nonlinear integro-differential equations. This method is now called the " Method of boundary layer function". Further development of this method is connected with the works of V. F. Butuzova [10,11], V. A. Tupchiev [12] and V. A. Trenogin [13].

S. A. Lomov [14,15] developed a method of regularization of singular perturbations, which allows to reduce the singularly perturbed problem to the regularly perturbed ones, with the help of which it is possible to develop the foundations of the General theory of singularly perturbed equations. The method is applicable to a wide range of problems for ordinary and partial differential equations.

In this paper, we propose a new method for solving singularly perturbed problems, which originates from the spectral theory of equations with a divergent argument. The essence of the method is as follows: the solution of the problem is decomposed into a Fourier series by eigenfunctions of the corresponding boundary value problem, then the coefficients of this series are transformed by integration in parts. As a result of these transformations, we obtain a new (recurrent) representation of the solution of the original problem. Further, by the method of mathematical induction it is possible to obtain an asymptotic expansion of the solution of the problem of interest to us. The remainder of the obtained decomposition is estimated by a priori estimates. With the help of direct calculations, the generality of the obtained recurrent formula is shown, and additional conditions that appeared in the course of research are removed. This work completes a series of studies devoted to the development of a spectral method for solving ill-posed problems [17-25].

**Problem statement.** Consider in the space  $H = L^2(0,1)$  the singularly perturbed Cauchy problem

$$L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \dots + a_n(x)y(x) = f(x), \quad (1)$$

$$y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0, \quad (2)$$

where  $a_i(x)$  - are real and sufficiently smooth functions on the interval  $[0,1]$ ,  $f(x) \in L^2(0,1)$ ,  $\varepsilon > 0$  - small parameter. The question is how the solution of this problem behaves as  $\varepsilon \rightarrow 0$ , depending on the behavior of the coefficients  $a_i(x)$ ,  $i = \overline{1, n}$  and the right part  $f(x)$ ? With the help of the spectral theory of the equation with deviating argument, to obtain the spectral decomposition of the solution of this problem in the space of the crane, and bring with it the asymptotic representation of the solution with the assessment of the remainder term through the right part of the equation.

## 2. Supporting proposals.

The Cauchy's problem (1) - (2) corresponds to the linear operator

$$L_\varepsilon y = \varepsilon y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \dots + a_n(x)y(x) = f(x),$$

with the range of definition

$$D(L_\varepsilon) = \{y(x) \in C^n[0,1]; y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0\},$$

and with the range of values  $R(L_\varepsilon) \subset C[0,1]$  contained in the linear variety of continuous functions.

We want to use theory of Hilbert – Schmidt on the spectral decomposition of a completely continuous and self-adjoint operator, therefore we give appropriate definitions and facts from the theory of linear operators.

**Lemma 2.1.** Let  $A$  be a densely defined operator in a Hilbert space  $H$ . Then

- (a)  $A^*$  exists and is closed;
- (b)  $A$  admits a closure if and only if  $D(A^*)$  is tight in  $H$ , and in this case  $\bar{A} = A^*$ ;
- (c) if  $A$  admits a closure, then  $(\bar{A})^* = A^*$ ;
- (d) if  $A$  - admits a closure and is invertible, then  $A^{-1}$  admits a closure and  $(\bar{A})^{-1} = \overline{A^{-1}}$ ;
- (e) the continuous operator always admits a closure to  $\overline{D(\bar{A})}$  by continuity.

**Lemma 2.2.**

(a) If a densely defined linear operator  $A$  in a Hilbert space  $H$  has a continuous inverse  $A^{-1}: H \rightarrow D(A)$ , then  $A^*$  - has a continuous inverse  $(A^*)^{-1}: H \rightarrow D(A^*)$  and  $(A^*)^{-1} = (A^{-1})^*$ ;

(b) If a linear operator  $A$  in a Hilbert space  $H$  is densely defined and closed and  $A^*$  has a bounded inverse, then  $A$  has a bounded inverse, and  $(A^{-1})^* = (A^*)^{-1}$ .

The proofs of this Lemmas 1, 2 are contained in many manuals on functional analysis [see, for example, 16].

**Definition 2.1.** An operator  $A^+$  is called formally adjoint to an operator  $A$  if for all  $u \in D(A)$  and  $v \in D(A^+)$  the equality

$$(Au, v) = (u, A^+v),$$

It is obvious that the operator  $A^*$  adjoint to  $A$  coincides with the formally adjoint operator which has a maximal domain.

**Lemma 2.3.** The operator formally adjoint to the operator  $L_\varepsilon$  has the following form:

$$L_\varepsilon^+ v = (-1)^n \varepsilon v^{(n)}(x) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [a_1(x)v(x)] +$$

$$+ (-1)^{n-2} \frac{d^{n-2}}{dx^{n-2}} [a_2(x)v(x)] + \dots - [a_{n-1}(x)v(x)]' + a_n(x)v(x),$$

$$D(L_\varepsilon^+) = \{v(x) \in C^n[0,1]; v(1) = 0, v'(1) = 0, \dots, v^{(n-1)}(1) = 0\}.$$

**Proof.** If  $u(x) \in D(L_\varepsilon)$  и  $v(x) \in D(L_\varepsilon^+)$ , so

$$\int_0^1 u^{(n)}(x)v(x)dx = \int_0^1 v(x)du^{(n-1)}(x) = v(x)u^{(n-1)}(x)|_0^1 -$$

$$- \int_0^1 v'(x)u^{(n-1)}(x)dx = - \int_0^1 v'(x)u^{(n-1)}(x)dx = \dots =$$

$$(-1)^n \int_0^1 u(x)v^{(n)}(x)dx;$$

$$\int_0^1 a_1(x)u^{(n-1)}(x)v(x)dx = \int_0^1 a_1(x)v(x)u^{(n-1)}(x)dx = a_1(x)v(x)u^{(n-2)}(x)|_0^1$$

$$- \int_0^1 [a_1(x)v(x)]' u^{(n-2)}(x)dx = - \int_0^1 [a_1(x)v(x)]' u^{(n-2)}(x)dx = \dots =$$

$$= (-1)^{n-1} \int_0^1 [a_1(x)v(x)]^{(n-1)} u(x)dx,$$

$$\int_0^1 a_2(x)u^{(n-2)}(x)v(x)dx = (-1)^{n-2} \int_0^1 u(x)[a_2(x)v(x)]^{(n-2)} dx, \dots$$

$$\int_0^1 a_{n-1}(x)u'(x)v(x)dx = \int_0^1 a_{n-1}(x)v(x)du(x) = a_{n-1}(x)v(x)u(x)|_0^1 -$$

$$- \int_0^1 [a_{n-1}(x)v(x)]' u(x)dx = - \int_0^1 u(x)[a_{n-1}(x)v(x)]' dx.$$

Therefore,

$$(L_\varepsilon u, v) = \int_0^1 \{ \varepsilon (-1)^n v^{(n)}(x) + (-1)^{n-1} [a_1(x)v(x)]^{(n-1)} +$$

$$(-1)^{n-2} [a_2(x)v(x)]^{(n-2)} + \dots + \dots - [a_{n-1}(x)v(x)]' +$$

$$+ a_n(x)v(x) \} u(x) dx = (u, L^+ v), \Rightarrow$$

$$L^+ v(x) = (-1)^n \varepsilon v^{(n)}(x) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [a_1(x)v(x)] +$$

$$(-1)^{n-2} \cdot \frac{d^{n-2}}{dx^{n-2}} [a_2(x)v(x)] + \dots - [a_{n-1}(x)v(x)]' + a_n(x)v(x),$$

that's what was required to prove.

By virtue of the Friedrich's Lemma, the linear variety of infinitely differentiable and finite functions  $C_0^\infty(0,1)$  is tight in space  $H$ , therefore both operators  $L_\varepsilon$  and  $L_\varepsilon^+$  are tightly defined in this space. Then, by Lemma 1, the operator  $L_\varepsilon^*$  exists, and is closed,  $L_\varepsilon$  permits the circuit, and  $\overline{L_\varepsilon} = L_\varepsilon^{**}$ ,  $(\overline{L_\varepsilon})^* = L_\varepsilon^*$ . By virtue of one of the theorems of the theory of Hilbert spaces there is a formula:

$$H = \overline{R(L_\varepsilon^*)} \otimes N(L_\varepsilon^{**}) = \overline{R(L_\varepsilon^*)} \otimes N(\overline{L_\varepsilon}),$$

therefore, for the existence of the inverse operator  $(\overline{L_\varepsilon})^{-1}$  is necessary and sufficient fulfillment of the equality  $\overline{R(L_\varepsilon^*)} = H$ . Then, by virtue of paragraph (d) of Lemma 1, we have the formula  $(\overline{L_\varepsilon})^{-1} = \overline{(L_\varepsilon^{-1})}$ , i.e. the inverse operator to the closure of the operator  $L_\varepsilon$  can be found using the closure of the operator  $L_\varepsilon^{-1}$ , which exists due to the existence of the operator  $(\overline{L_\varepsilon})^{-1}$ . If  $D(\overline{L_\varepsilon})^{-1} = H$ , then by the Banach

theorem about closed graph operator  $(\bar{L}_\varepsilon)^{-1}$  is restricted in space  $H$ . But the problem is precisely that. Without further information, we cannot confirm this.

We show that  $\overline{R(L_\varepsilon^*)} = H$ , note for this that  $L_\varepsilon^+ \subset L_\varepsilon^*$ , therefore  $R(L_\varepsilon^+) \subset R(L_\varepsilon^*)$ . If  $\overline{R(L_\varepsilon^*)} = H$ , then the required statement follows.

**Lemma 2.4.** If the function is  $K(x, t)$ , for a fixed value of  $t$ , the first variable  $x$  is the solution of the Cauchy problem of the following homogeneous equation:

$$\begin{aligned} & \left[ \varepsilon \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x) \right] K(x, t) = 0, \\ & K(x, t)|_{t=x} = 0, \frac{\partial K}{\partial x} \Big|_{t=x} = 0, \dots, \frac{\partial^{(n-2)} K}{\partial x^{n-2}} \Big|_{t=x} = 0, \varepsilon \frac{\partial^{(n-1)} K}{\partial x^{n-1}} \Big|_{t=x} = 1, \end{aligned} \quad (3)$$

that function

$$y(x) = y(x, \varepsilon, f) = \int_0^x K(x, t) f(t) dt = \int_0^1 \theta(x-t) K(x, t) f(t) dt \quad (4)$$

for any continuous function  $f(t)$ , is the solution of the Cauchy problem(1)-(2).

**Proof.** If  $f(t)$  is continuous on the  $[0,1]$  segment, then the function (4) is continuously differentiable and the formula holds:

$$y'(x) = K(x, x) \cdot f(x) + \int_0^x \frac{\partial K}{\partial x} f(t) dt,$$

therefore with condition (3), we have

$$y'(x) = \int_0^x \frac{\partial K}{\partial x} f(t) dt,$$

we have

$$y^{(m)}(x) = \int_0^x \frac{\partial^m K}{\partial x^m} f(t) dt, \quad 1 \leq m \leq n-1;$$

$$y^{(n)}(x) = \frac{\partial^{n-1} K}{\partial x^{n-1}} \Big|_{t=x} \cdot f(x) + \int_0^x \frac{\partial^n K}{\partial x^n} f(t) dt.$$

Consequently,

$$\begin{aligned} \varepsilon y^{(n)}(x) + \sum_{k=1}^n a_k(x) y^{(n-k)}(x) &= \varepsilon \frac{\partial^{n-1} K}{\partial x^{n-1}} \Big|_{t=x} \cdot f(x) + \\ &+ \int_0^x \left[ \varepsilon \frac{\partial^n K}{\partial x^n} + \sum_{k=1}^n a_k(x) \frac{\partial^{n-k} K}{\partial x^{n-k}} \right] f(t) dt = f(x). \end{aligned}$$

**Definition 2.2.** The function  $K(x, t) \cdot \theta(x-t)$  is called the Cauchy kernel of the integral operator (4), where  $\theta(x)$  is the Heaviside function.

We will return to the study of Cauchy kernel properties a little later, and now we will deal with the solvability of the Cauchy problem. It would be tempting to deduce this statement from the formula  $\overline{R(L_\varepsilon^*)} = H$ , but from the form of the formally adjoint operator  $L_\varepsilon^+$ , it is obvious that this path requires a certain smoothness of the coefficients of the equation  $L_\varepsilon y = f$ , so we will deal with the equation itself  $L_\varepsilon y = f$ , especially since we have already shown the dense solvability of this equation. The uniqueness of the solution found follows from the following a priori estimates.

### Lemma 2.5

If  $a_1(x)$  is a continuous function on the interval  $[0,1]$ , satisfying the condition

(a)  $a_1(x) \geq \alpha > 0, \forall x \in [0,1]$

and on the domain of definition of the operator  $L_\varepsilon$  the inequality holds:

(b)  $\left( \sum_{k=2}^n a_k(x) y^{(n-k)}(x), y^{(n-1)}(x) \right) \geq 0, \forall y \in D(L_\varepsilon)$ ;

then the following a priori estimates take place:

$$\|y\| \leq \|\dot{y}\| \leq \|y'\| \dots \leq \|y^{(n-2)}\| \leq \|y^{(n-1)}\| \leq \frac{\|L_\varepsilon y\|}{\alpha}. \tag{6}$$

**Proof.** Multiplying both sides of equation (1), scalar by the function  $y^{(n-1)}(x)$ , we have

$$(L_\varepsilon y, y^{(n-1)}) = (\varepsilon y^n, y^{(n-1)}) + (a_1 y^{(n-1)}, y^{(n-1)}) + \left( \sum_{k=2}^n a_k(x) y^{(n-k)}, y^{(n-1)} \right) = (f, y^{(n-1)})$$

then

$$\begin{aligned} (\varepsilon y^n, y^{(n-1)}) &= \varepsilon \int_0^1 y^{(n)} y^{(n-1)}(x) dx = \varepsilon \int_0^1 y^{(n-1)}(x) d y^{(n-1)}(x) = \\ &= \frac{\varepsilon [y^{(n-1)}(x)]^2}{2} \Big|_0^1 = \frac{\varepsilon [y^{(n-1)}(1)]^2}{2} \geq 0, \end{aligned}$$

so

$$\begin{aligned} \alpha \|y^{(n-1)}\|^2 &\leq \int_0^1 a_1(x) [y^{(n-1)}(x)]^2 dx \leq (L_\varepsilon y, y^{(n-1)}) \leq (f, y^{(n-1)}) \leq \\ &\leq \|f\| \cdot \|y^{(n-1)}\|; \alpha \|y^{(n-1)}\| \leq \|f\|; \|y^{(n-1)}(x)\| \leq \frac{\|f\|}{\alpha} = \frac{\|L_\varepsilon y\|}{\alpha}, \end{aligned}$$

As,  $y(0) = 0$ , then  $y(x) = \int_0^x \dot{y}(t) dt$ ,

$$\begin{aligned} |y(x)| &\leq \left( \int_0^x 1^2 dt \right)^{\frac{1}{2}} \left( \int_0^1 \dot{y}^2(t) dt \right)^{\frac{1}{2}}; |y(x)|^2 \leq \int_0^x dt \int_0^x \dot{y}^2(t) dt \leq x \int_0^x \dot{y}^2(t) dt, \\ |y(x)|^2 &\leq x \int_0^x \dot{y}^2(t) dt \leq \int_0^1 \dot{y}^2(t) dt, \Rightarrow \|y\|^2 \leq \|\dot{y}\|^2, \Rightarrow \|y\| \leq \|\dot{y}\|. \end{aligned}$$

In a similar way, we have

$$\|\dot{y}\| \leq \|y'\| \dots \leq \|y^{(n-2)}\| \leq \|y^{(n-1)}\| \leq \frac{\|L_\varepsilon y\|}{\alpha}. \tag{5}$$

Thus, there is an inverse operator  $L_\varepsilon^{-1}$ , therefore, by virtue of item (d) of Lemma 1,  $L_\varepsilon^{-1}$  admits a closure and  $\overline{(L_\varepsilon^{-1})} = (\overline{L_\varepsilon})^{-1}$ . Lemma 4 that  $R(L_\varepsilon)$  coincides with the linear diversity continuous on  $[0,1]$  functions that is dense in the space  $H$ , therefore  $\overline{R(L_\varepsilon)} = H$ , but  $L_\varepsilon \subset \overline{L_\varepsilon}, \Rightarrow R(L_\varepsilon) \subset R(\overline{L_\varepsilon})$ , so a fortiori  $\overline{R(\overline{L_\varepsilon})} = H$ . By virtue of a priori estimates (5)  $\overline{R(\overline{L_\varepsilon})} = R(\overline{L_\varepsilon})$ . Indeed, if  $y \in D(\overline{L_\varepsilon})$ , then there is a sequence  $\{y_n\} \in D(L_\varepsilon)$ , such that  $y_n \rightarrow y, L_\varepsilon y_n = f_n \rightarrow f$ , then it follows from (5) that the sequence  $\{y_n^{(k)}\} (k = 1, 2, \dots, n-1), n = 1, 2, \dots$  in the space  $H$ , which means that  $y(x) \in W_2^{n-1}[0,1]$  и  $y_n(x) \rightarrow y(x)$  in the space  $W_2^{n-1}[0,1]$ . Passing to the limit in inequalities (5), we get:

$$\|y\| \leq \|\dot{y}\| \leq \dots \leq \|y^{(n-1)}\| \leq \frac{\|\overline{L_\varepsilon} y\|}{\alpha}. \tag{6}$$

If  $z(x) \in \overline{R(\overline{L_\varepsilon})}$ , then there exists a sequence  $\{z_n(x)\} \subset R(\overline{L_\varepsilon})$ , such that  $z_n(x) \rightarrow z(x)$  in  $H$ . Then the sequence  $z_n(x) = \overline{L_\varepsilon} y_n$  fundamental in  $H$ , and in virtue of a priori estimates, the sequence  $\{y_k\}$  is fundamental in the space of  $W_2^{n-1}[0,1]$ , which means that  $y_k \rightarrow y, y'_k \rightarrow y', \dots, y_k^{(n-1)} \rightarrow y^{(n-1)}, \overline{L_\varepsilon} y_n \rightarrow z(x)$ , i.e. there exists a function  $y(x) \in D(\overline{L_\varepsilon})$  such that  $\overline{L_\varepsilon} y_n = z(x)$ , that is,  $z(x) \in R(\overline{L_\varepsilon})$ , as required.

Thus,  $(\overline{L_\varepsilon})^{-1}$  exists and is defined on the whole space  $H$ , since

$$D(\overline{L_\varepsilon})^{-1} = R(\overline{L_\varepsilon}) = \overline{R(\overline{L_\varepsilon})} = H,$$

then, by the Banach theorem, the operator  $(\bar{L}_\varepsilon)^{-1}$  is bounded; this is also obvious from the obtained inequalities (6). By virtue of clause c) of Lemma 1,  $(\bar{L}_\varepsilon)^* = L_\varepsilon^*$ , therefore by virtue of clause a) of Lemma 2, the operator  $L_\varepsilon^*$  has a continuous inverse

$$(L_\varepsilon^*)^{-1}: H \rightarrow D(L_\varepsilon^*) \text{ and } (L_\varepsilon^*)^{-1} = [(L_\varepsilon^{-1})^{-1}]^* = \left( (\bar{L}_\varepsilon^{-1}) \right)^* = (L_\varepsilon^{-1})^*.$$

From inequalities (6) it follows that

$$\sqrt{\|y\|^2 + \|\dot{y}\|^2} \leq \left( \frac{\|\bar{L}_\varepsilon y\|^2}{\alpha^2} + \frac{\|\bar{L}_\varepsilon \dot{y}\|^2}{\alpha^2} \right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{\alpha} \|\bar{L}_\varepsilon y\|,$$

those, the operator  $(\bar{L}_\varepsilon)^{-1}$  translates a bounded set into a compact one, therefore it is completely continuous, by Schauder's theory, the operator  $(L_\varepsilon^{-1})^*$  is also completely continuous.

**Definition 2.3.** The closures of the operator  $L_\varepsilon$  are called the Cauchy operator and denote by  $C_\varepsilon$ , i.e.  $C_\varepsilon = \bar{L}_\varepsilon$ .

**Lemma 2.6.** Under the conditions of Lemma 5, the Cauchy operator  $C_\varepsilon$  is bounded invertible, and the inverse operator  $C_\varepsilon^{-1}$  is completely continuous in the space  $H$ , moreover, the equality  $(L_\varepsilon^*)^{-1} = (L_\varepsilon^{-1})^*$ .

**Lemma 2.7.** If

$$\begin{cases} L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{m=0}^{n-1} a_{n-m}(x) y^{(m)}(x), \\ D(L_\varepsilon) = \{y(x) \in C^n[0,1], y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0\}, \\ L_\varepsilon^+ z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x) z(x)]^{(k)}, \\ D(L_\varepsilon^+) = \{z(x) \in C^n[0,1]; z(1) = 0, z'(1) = 0, \dots, z^{(n-1)}(1) = 0\}, \end{cases}$$

and the operator  $S$  is defined by the equality  $Su(x) = u(1-x)$ , then the equality  $SL_\varepsilon = L_\varepsilon^+ S$  holds if and only if

$$a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), \quad m = 0, 1, 2, \dots, n-1. \quad (7)$$

**Proof.** According to the Leibniz's formula

$$[a_{n-k} v]^{(k)} = \sum_{m=0}^k c_k^m v^{(m)} a_{n-k}^{(k-m)},$$

therefore

$$\begin{aligned} \sum_{k=0}^{n-1} (-1)^k [a_{n-k} v(x)]^{(k)} &= \sum_{k=0}^{n-1} (-1)^k \sum_{m=0}^k c_k^m a_{n-k}^{(k-m)}(x) v^{(m)}(x) = \\ &= \sum_{m=0}^{n-1} \left[ \sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x) \right] \cdot v^{(m)}(x). \end{aligned}$$

Therefore,

$$L_\varepsilon^+ z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{m=0}^{n-1} \left[ \sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x) \right] \cdot z^{(m)}(x).$$

Then

$$\begin{aligned}
 SL_\varepsilon y &= \varepsilon y^{(n)}(1-x) + \sum_{m=0}^{n-1} a_{n-m}(1-x)y^{(m)}(1-x), \\
 L_\varepsilon^+ Sy &= (-1)^n \varepsilon (Sy)^{(n)} + \sum_{m=0}^{n-1} \left[ \sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x) \right] \cdot (Sy)^{(m)} = \\
 &= \varepsilon y^{(n)}(1-x) + \sum_{m=0}^{n-1} \left[ \sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x) \right] (-1)^m y^{(m)}(1-x).
 \end{aligned}$$

Equating the corresponding coefficients of these expressions, we obtain the formula (7).

**Lemma 2.8.** If

$$(a) Su(x) = u(1-x);$$

$$(b) L_\varepsilon y(x) = \varepsilon y^{(n)} + \sum_{m=0}^{n-1} a_{n-m}(x)y^{(m)}(x); y \in D(L_\varepsilon);$$

$$(c) a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0, 1, 2, \dots, n-1,$$

then the operator  $SL_\varepsilon$  is symmetric in the space  $H$ .

**Proof.** If  $y(x) \in D(L_\varepsilon)$ , i.e.  $y(x) \in C^n[0,1]$ , and  $y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0$ , then  $Sy(x) = y(1-x) \in D(L_\varepsilon^+)$ . In fact, it is obvious that  $Sy(x) \in C^n[0,1]$ , and  $[Sy(x)]^{(m)} = (-1)^m y^{(m)}(1-x)$ , therefore,  $[Sy(x)]^{(m)}|_{x=1} = 0, m = 0, 1, 2, \dots, n-1$ . By virtue of the previous lemma, the equality  $SL_\varepsilon = L_\varepsilon^+ S$ , holds; therefore, for all  $u(x)$  and  $v(x) \in D(L_\varepsilon)$  we have  $(SL_\varepsilon u, v) = (L_\varepsilon u, Sv) = (u, L_\varepsilon^+ Sv) = (u, SL_\varepsilon v)$ .

**Lemma 2.9.** If

$$(a) Su(x) = u(1-x);$$

$$(b) L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{m=0}^{n-1} a_{n-m}(x)y^{(m)}(x);$$

$$y(x) \in D(L_\varepsilon) = \{y(x) \in C^n[0,1]; y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0\},$$

then equality takes place

$$\overline{SL_\varepsilon} = S\overline{L_\varepsilon},$$

where the bar ( $\overline{\phantom{x}}$ ), as usual, means the operator's closure operation.

**Proof.**

Suppose that the operator  $SL_\varepsilon$  is not closable, then there exists a sequence  $u_n \in D(L_\varepsilon)$  such that  $u_n \rightarrow 0, SL_\varepsilon u_n \rightarrow f \in H$  and  $f \neq 0$ . Then  $u_n \in D(L_\varepsilon), L_\varepsilon u_n = SSL_\varepsilon u_n \rightarrow Sf \neq 0$ , therefore  $L_\varepsilon$  is also not closable. Similarly, the non-closure of the operator  $L_\varepsilon$  implies the non-closure of the operator  $SL_\varepsilon$ . Therefore, the operator  $SL_\varepsilon$  is closable if and only if we close the operator  $L_\varepsilon$ . If  $u \in D(\overline{L_\varepsilon})$ , then there is a sequence  $u_n \in D(SL_\varepsilon)$  such that  $u_n \rightarrow u, L_\varepsilon u_n \rightarrow \overline{L_\varepsilon} u$ , therefore,  $u_n \in D(SL_\varepsilon)$  and  $SL_\varepsilon u_n \rightarrow S\overline{L_\varepsilon} u$ . Therefore,  $u \in D(\overline{SL_\varepsilon})$  and  $S\overline{L_\varepsilon} u = \overline{SL_\varepsilon} u$ . Thus, if  $u \in D(\overline{L_\varepsilon})$ , then  $u \in D(\overline{SL_\varepsilon})$  and the equality  $\overline{SL_\varepsilon} u = S\overline{L_\varepsilon} u$  holds.

Conversely, let  $u \in D(\overline{SL_\varepsilon})$ . Then there is a sequence  $u_n \in D(SL_\varepsilon)$ , such that  $u_n \rightarrow u$  and  $SL_\varepsilon u_n \rightarrow \overline{SL_\varepsilon} u$ , since  $D(SL_\varepsilon) = D(L_\varepsilon)$ , then  $u_n \in D(L_\varepsilon)$  and  $L_\varepsilon u_n = S\underline{SL_\varepsilon} u_n \rightarrow S(\overline{SL_\varepsilon} u)$ , and this means that  $u \in D(\overline{L_\varepsilon})$  and  $\overline{L_\varepsilon} u = S\overline{SL_\varepsilon} u$ , i.e.  $S\overline{L_\varepsilon} = \overline{SL_\varepsilon}$ .

**Consequence 2.1.** The operator  $SL_\varepsilon$  is self-adjoint in the essential in space, i.e.  $(\overline{SL_\varepsilon})^* = \overline{SL_\varepsilon}$ .

**Proof.** The operator  $SL_\varepsilon$  is symmetric, therefore, the operator  $\overline{SL_\varepsilon}$  is also symmetric. Since  $\overline{SL_\varepsilon} = S\overline{L_\varepsilon}$  and  $R(\overline{L_\varepsilon}) = H$ , then  $R(\overline{SL_\varepsilon}) = H, \Rightarrow R(S\overline{L_\varepsilon}) = H$ . From the symmetry of the operator  $SL_\varepsilon$  it

follows that  $SL_\varepsilon \subset (SL_\varepsilon)^*$ , passing to the closure, and taking into account the closedness of the adjoint operator, we get the inclusion  $\overline{SL_\varepsilon} \subset (SL_\varepsilon)^*$ , and since  $R(\overline{SL_\varepsilon}) = H$ , then  $R(SL_\varepsilon)^* = H$  and  $R(\overline{SL_\varepsilon}) = R(SL_\varepsilon)^*$ . Therefore, taking into account the invertibility of these operators, we have

$$D(\overline{SL_\varepsilon}) = D(SL_\varepsilon)^* \text{ и поэтому } \overline{SL_\varepsilon} = (SL_\varepsilon)^*.$$

So  $(\overline{SL_\varepsilon})^* = (SL_\varepsilon)^{**} = \overline{SL_\varepsilon}$ , that's what was required to prove.

**Lemma 2.10.** If the  $k$ -th coefficient of equation (1) is  $n - k$  ( $k = 0, 1, 2, \dots, n$ ) times continuously differentiable on the interval  $[0, 1]$ , and satisfies the following conditions:

(a)  $a_1(x) \geq \alpha > 0$ ;

(b)  $\left(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$ ;

(c)  $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0, 1, 2, \dots, n-1$ ,

where  $c_k^m$  are binomial coefficients, then the  $SC_\varepsilon$  operator is self-adjoint in the space  $H$ , and has a completely continuous inverse, where  $C_\varepsilon$  is the Cauchy operator.

### 3. Main Results.

**Theorem 3.1.** If the  $k$ -th coefficient of equation (1)  $n - k$  ( $k = 0, 1, 2, \dots, n$ ) is continuously differentiable on the interval  $[0, 1]$  and satisfies the following conditions:

(a)  $a_1(x) \geq \alpha > 0$ ;

(b)  $\left(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$ ;

(c)  $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0, 1, 2, \dots, n-1$ ,

where  $c_k^m$  are binomial coefficients, then the Cauchy's problem (1) - (2) is strongly solvable and this strong solution has the following representation:

$$y(x, \varepsilon, f) = \sum_{n=1}^{\infty} \frac{(Sf, \varphi_n)}{\lambda_n} \varphi_n(x), \quad (8)$$

where  $\lambda_n$  ( $n = 1, 2, \dots$ ) - are eigenvalues, and  $\varphi_n(x)$  ( $n = 1, 2, \dots$ ) are eigenfunctions of the operator  $SL_\varepsilon$ , the  $S$  operator is defined by the formula:

$$Su(x) = u(1-x).$$

**Proof.** By Lemma 10, the operator  $(SC_\varepsilon)^{-1}$  is completely continuous and self-adjoint, therefore, according to the Hilbert - Schmidt theorem, the decomposition takes place

$$(SC_\varepsilon)^{-1}f = \sum_{n=1}^{\infty} \frac{(f, \varphi_n)}{\lambda_n} \varphi_n(x) + \varphi_0(x),$$

where  $\varphi_0(x) \in \ker(SC_\varepsilon)^{-1}$  and  $\{\varphi_n(x)\}$  ( $n = 1, 2, \dots$ ) - are orthonormal eigenvectors of the operator  $(SC_\varepsilon)^{-1}$ , and  $\lambda_n^{-1}$  ( $n = 1, 2, \dots$ ) the corresponding eigenvalues of the same operator, then  $(SC_\varepsilon)^{-1}\varphi_0 = 0, \Rightarrow \varphi_0 = SC_\varepsilon(SC_\varepsilon)^{-1}\varphi_0 = 0$ , therefore

$$(SC_\varepsilon)^{-1}f(x) = \sum_{n=1}^{\infty} \frac{(f, \varphi_n)}{\lambda_n} \varphi_n(x).$$

If for some function  $f(x) \in H$  there is the equality  $(f, \varphi_n) = 0$  ( $n = 1, 2, \dots$ ), then  $(SC_\varepsilon)^{-1}f = 0, \Rightarrow f = 0$ , the system of eigenfunctions  $\{\varphi_n\}$  is complete in  $H$ . Since, by virtue of the self-adjointness of the operator  $(SC_\varepsilon)^{-1}$ , this system is orthogonal, after normalization it forms an orthonormal basis of the space  $H$ .

We now return to the Cauchy's problem (1) - (2).

$$L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{k=1}^n a_k(x) y^{(n-k)}(x) = f(x), x \in (0, 1],$$

$$y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0,$$



or in operator form:

$$L_\varepsilon y = f.$$

Acting by the operator  $S$  on both sides of the equation, we get

$$SL_\varepsilon y = Sf.$$

Therefore, for all  $y(x) \in D(L_\varepsilon)$ , the equality

$$SC_\varepsilon y = S\bar{L}_\varepsilon y = SL_\varepsilon y = Sf,$$

So

$$\begin{aligned} y(x) = y(x, \varepsilon, f) &= (SC_\varepsilon)^{-1}Sf = \sum_{n=1}^{\infty} ((SC_\varepsilon)^{-1}Sf, \varphi_n)\varphi_n(x) = \\ &= \sum_{n=1}^{\infty} (Sf, (SC_\varepsilon)^{-1}\varphi_n)\varphi_n(x) = \sum_{n=1}^{\infty} \frac{(Sf, \varphi_n)}{\lambda_n} \varphi_n(x). \end{aligned}$$

The equation for the functions is:

$$(SC_\varepsilon)^{-1}\varphi_n = \frac{\varphi_n(x)}{\lambda_n}, \lambda_n \neq 0, n = 1, 2, \dots$$

or

$$C_\varepsilon^{-1}S\varphi_n = \frac{\varphi_n(x)}{\lambda_n}, (\bar{L}_\varepsilon)^{-1}S\varphi_n = \frac{\varphi_n(x)}{\lambda_n}.$$

So  $\psi_n(x) = S\varphi_n(x)$ , we have

$$(\bar{L}_\varepsilon)^{-1}\psi_n(x) = \frac{S\psi_n(x)}{\lambda_n}$$

or

$$S\psi_n(x) = \lambda_n(\bar{L}_\varepsilon)^{-1}\psi_n(x) = \lambda_n(\bar{L}_\varepsilon^{-1})\psi_n(x) = \lambda_n \int_0^x K(x, t) \psi_n(t) dt.$$

If  $\psi_n(t) \in L^2(0,1)$ , then from this equation we see that  $[S\psi_n \in W_2^n[0,1]$ , then  $\varphi_n(x) \in W_2^n[0,1]$ . If  $\varphi_n(x) \in W_2^n[0,1]$ , then  $S\psi_n(x) \in W_2^{2n}[0,1]$ , then  $\varphi_n(x) \in W_2^{2n}[0,1]$ , continuing this process, we obtain that  $\varphi_n(x) \in C^\infty$ , i.e. infinitely differentiable. Therefore, any function belongs to the domain  $D(L_\varepsilon)$ , therefore

$$\lambda_n \varphi_n = SC_\varepsilon \varphi_n = S\bar{L}_\varepsilon \varphi_n = SL_\varepsilon \varphi_n, =>$$

$$L_\varepsilon \varphi_n(x) = \lambda_n S\varphi_n(x), n = 1, 2, \dots$$

Therefore,

$$\begin{cases} \varepsilon \varphi_m^{(n)}(x) + a_1(x)\varphi_m^{(n-1)}(x) + \dots + a_n(x)\varphi_m(x) = \lambda_m \varphi_m(1-x) = \lambda_m S\varphi_m(x), \\ \varphi_m(0) = 0, \varphi_m'(0) = 0, \dots, \varphi_m^{(n-1)}(0) = 0, \end{cases}$$

For further clarity, we will study the properties of the operator  $B$ , where

$$\begin{cases} Bu(x) = a_1(x)u^{(n-1)}(x) + a_2(x)u^{(n-2)}(x) + \dots + a_n(x)u(x), \\ D(B) = \{u(x) \in C^{n-1}(0,1) \cap C^{n-2}[0,1]; u(0) = 0, u'(0) = 0, \dots, u^{(n-2)}(0) = 0, \end{cases}$$

denote these equations as (9)-(10).

**Theorem 3.2.**

(a) If  $k$ -th coefficient of operator (9)-(10)  $n - k$  times is continuously differentiable on the interval  $[0,1]$ , then one of the formally adjoint operators of operator  $B$  has the form:

$$\begin{cases} B^+v = \sum_{k=1}^n (-1)^{n-k} [a_k(x)v(x)]^{(n-k)}, \\ v(1) = 0, v'(1) = 0, \dots, v^{(n-2)}(1) = 0; \end{cases}$$

(b) if there is equality

$$a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), \quad m = 0, 1, 2, \dots, n-1,$$

where  $c_k^m$  - are binomial coefficients then

$$SBu = B^+Su, \quad \forall u(x) \in D(B).$$

**Proof.** If  $u \in D(B)$  and  $v(x) \in D(B^+)$ , so

$$\begin{aligned} \text{(a)} \quad (Bu, v) &= \left( \sum_{k=1}^n a_k(x)u^{(n-k)}, v \right) = \sum_{k=1}^n (a_k(x)u^{(n-k)}, v); \\ (a_k u^{(n-k)}, v) &= (a_k v, u^{(n-k)}) = \int_0^1 a_k v u^{(n-k)} dx = \int_0^1 a_k v du^{(n-k-1)} = \\ &= a_k v u^{(n-k-1)} \Big|_0^1 - \int_0^1 (a_k v)' u^{(n-k-1)} dx = a_k v u^{(n-k-1)} \Big|_0^1 - \\ &- \int_0^1 (a_k v)' du^{(n-k-2)} dx = a_k v u^{(n-k-1)} \Big|_0^1 - (a_k v)' u^{(n-k-2)} \Big|_0^1 + \\ &+ \int_0^1 (a_k v)'' u^{(n-k-2)} dx = \dots = \sum_{m=0}^{n-k-1} (a_k v)^m (-1)^m u^{(n-k-1-m)} \Big|_0^1 + \\ &+ (-1)^{n-k} \int_0^1 (a_k v)^{(n-k)} u(x) dx, \end{aligned}$$

Therefore,

$$\begin{aligned} (Bu, v) &= \sum_{k=1}^n \sum_{m=0}^{n-k-1} (a_k v)^m (-1)^m u^{(n-k-1-m)} \Big|_0^1 + \sum_{k=1}^n (-1)^{n-k} (u, (a_k v)^{(n-k)}) \\ &= [u, v] + \left( u, \sum_{k=1}^n (-1)^{n-k} (a_k v)^{(n-k)} \right), \end{aligned}$$

where

$$[u, v] = \sum_{k=1}^n \sum_{m=0}^{n-k-1} (a_k v)^m (-1)^m u^{(n-k-1-m)}(x) \Big|_0^1.$$

If

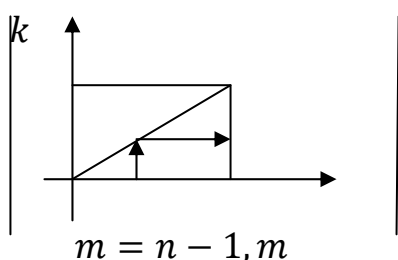
$$\begin{aligned} u(0) = 0, u'(0) = 0, \dots, u^{(n-2)}(0) = 0; \\ v(1) = 0, v'(1) = 0, \dots, v^{(n-2)}(1) = 0, \end{aligned}$$

therefore  $[u, v] = 0$ , so  $(Bu, v) = (u, B^+v)$ , where

$$B^+v = \sum_{k=1}^n (-1)^{n-k} (a_k v)^{(n-k)}.$$

(b) If  $u(x) \in D(B)$ , we have it  $Su(x) \in D(B^+)$  and expression  $B^+Su$  it has sense

$$\begin{aligned}
 B^+Su &= \sum_{k=1}^n (-1)^{n-k} (a_k Su)^{(n-k)} = \left| \begin{matrix} n-k = m, \\ k = n-m \end{matrix} \right| = \\
 &= \sum_{m=n-1}^0 (-1)^m (a_{n-m} Su)^{(m)} = \sum_{m=0}^{n-1} (-1)^m (a_{n-m} Su)^{(m)} = \\
 &= \sum_{m=0}^{n-1} (-1)^m \sum_{k=0}^m a_{n-m}^{(m-k)} c_m^k (Su)^{(k)} = \sum_{m=0}^{n-1} (-1)^{m+k} \sum_{k=0}^m a_{n-m}^{(m-k)} c_m^k Su^{(k)} = \\
 &= \left( \sum_{k=0}^{n-1} \sum_{m=k}^{n-1} (-1)^{m+k} a_{n-m}^{(m-k)} c_m^k \right) Su^{(k)};
 \end{aligned}$$



$$\begin{aligned}
 SBu &= S \sum_{k=0}^{n-1} a_{n-k} u^{(k)} = \sum_{k=0}^{n-1} a_{n-k} (1-x) Su^{(k)} = \\
 &= \sum_{k=0}^{n-1} \left( \sum_{m=1}^{n-1} (-1)^{m+k} c_m^k a_{n-m}^{(m-k)} \right) Su^{(k)} = B^+Su.
 \end{aligned}$$

By Lemma 10, the operator  $\overline{SB}$  is in the space  $H$ . Therefore,  $(\overline{SB})^* = \overline{SB} = S\overline{B}$ ,  $(S\overline{B})^* = S\overline{B}$ ,  $(\overline{B})^*S^* = S\overline{B}$ ,  $(\overline{B})^*S = S\overline{B}$ ,  $B^*S = S\overline{B}$ ,  $B^* = S\overline{B}S$ ,  $(B^*)^{-1} = S^{-1}(\overline{B})^{-1}S^{-1} = S(\overline{B})^{-1}S$ ,  $S(B^*)^{-1} = (\overline{B})^{-1}S$ ,  $S(B^{-1})^* = (\overline{B^{-1}})S$ ,  $(B^{-1})^*S = S(\overline{B^{-1}})$ , that's what was required to prove.

**Theorem 3.3.** If the  $k$  - th coefficient of equation (1) is  $n - k$  ( $k = 0, 1, 2, \dots, n$ ) it is continuously different once on the interval  $[0, 1]$  and satisfies the following conditions:

- (a)  $a_1(x) \geq \alpha > 0$ ;
- (b)  $\left( \sum_{k=2}^n a_k(x) y^{(n-k)}(x), y^{(n-1)}(x) \right) \geq 0, \forall y \in D(L_\varepsilon)$ ;
- (c)  $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0, 1, 2, \dots, n-1$ ,

where  $c_k^m$  -are binomial coefficients, then the formula holds:

$$\psi(x) = \sum_{m=1}^{\infty} \frac{\varepsilon \varphi_m(1)}{\lambda_m} \varphi_m(x), \tag{11}$$

where

$$\begin{aligned}
 \varepsilon \varphi_m^{(n)}(x) + a_1(x) \varphi_m^{(n-1)}(x) + \dots + a_n(x) \varphi_m(x) &= \lambda_m S \varphi_m(x); \\
 \varphi_m(0) = 0, \varphi_m'(0) = 0, \dots, \varphi_m^{(n-1)}(0) &= 0,
 \end{aligned}$$

and  $\psi(x)$  - is a solution of a homogeneous equation

$$\varepsilon\psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \dots + a_n(x)\psi(x) = 0 \quad (12)$$

satisfying the initial conditions:

$$\psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1.$$

**Proof.** Noting that  $\psi(x) \in D(B)$  we rewrite equations (12) in the form  $\varepsilon\psi^{(n)}(x) + B\psi(x) = 0$ . Noting also that  $\varphi_m(x) \in D(B)$  and  $\varphi_m'(x) \in D(B)$  we rewrite the equations of the functions as

$$\varepsilon\varphi_m^{(n)}(x) + B\varphi_m(x) = \lambda_m S\varphi_m(x), S\varphi_m(x) = \varphi_m(1-x).$$

Using these two formulas, we calculate the Fourier coefficients of this function  $SB\psi$ .

$$\begin{aligned} (SB\psi, \varphi_m) &= (SB\psi, \lambda_m B^{-1}S\varphi_m - \varepsilon B^{-1}\varphi_m^{(n)}) = \lambda_m (SB\psi, B^{-1}S\varphi_m) - \\ &- \varepsilon (SB\psi, B^{-1}\varphi_m^{(n)}) = \lambda_m ((B^{-1})^* SB\psi, S\varphi_m) - \varepsilon ((B^{-1})^* SB\psi, \varphi_m^{(n)}) = \\ &= \lambda_m (S\overline{B^{-1}}B\psi, S\varphi_m) - \varepsilon (S\overline{B^{-1}}B\psi, \varphi_m^{(n)}) = |B\psi \in D(B^{-1}) \subset D(\overline{B^{-1}})| = \\ &= \lambda_m (SB^{-1}B\psi, S\varphi_m) - \varepsilon (SB^{-1}B\psi, \varphi_m^{(n)}) = \lambda_m (\psi, \varphi_m) - \varepsilon (S\psi, \varphi_m^{(n)}); \end{aligned}$$

Using integration by parts, we transform the scalar product

$$(S\psi, \varphi_m^{(n)}).$$

$$\begin{aligned} (S\psi, \varphi_m^{(n)}) &= \int_0^1 S\psi d\varphi_m^{(n-1)}(x) = S\psi \cdot \varphi_m^{(n-1)}(x) \Big|_0^1 - \int_0^1 (S\psi)' \varphi_m^{(n-1)}(x) dx = \\ &= \psi(1-x) \varphi_m^{(n-1)}(x) \Big|_0^1 - \int_0^1 (S\psi)' \varphi_m^{(n-1)}(x) dx = - \int_0^1 (S\psi)' \varphi_m^{(n-1)}(x) dx; \\ (S\psi, \varphi_m^{(n)}) &= \sum_{k=0}^{n-1} (-1)^k (S\psi)^{(k)} \varphi_m^{(n-1-k)}(x) \Big|_0^1 + (-1)^n \int_0^1 (S\psi)^{(n)} \varphi_m(x) dx = \\ &= \sum_{k=0}^{n-1} (-1)^k (-1)^k \psi^{(k)}(1-x) \varphi_m^{(n-1-k)}(x) \Big|_0^1 + (-1)^{n+n} \cdot \\ &\cdot \int_0^1 \psi^{(n)}(1-x) \varphi_m(x) dx = \sum_{k=0}^{n-1} \psi^{(k)}(1-x) \varphi_m^{(n-1-k)}(x) \Big|_0^1 + \\ &+ \int_0^1 \psi^{(n)}(1-x) \varphi_m(x) dx = \psi^{(n-1)}(0) \varphi_m(1) + (S\psi^{(n)}, \varphi_m) = \\ &= \varphi_m(1) + (S\psi^{(n)}, \varphi_m); \end{aligned}$$

Operated by the operator S on both sides of the equation

$$\varepsilon\psi^{(n)}(x) + B\psi(x) = 0,$$

we have

$$\varepsilon S\psi^{(n)} + SB\psi(x) = 0, S\psi^{(n)} = -\frac{SB\psi}{\varepsilon}.$$

Therefore,

$$(S\psi^{(n)}, \varphi_m) = \left( -\frac{SB\psi}{\varepsilon}, \varphi_m \right),$$

so

$$(S\psi, \varphi_m^{(n)}) = \varphi_m(1) - \left(\frac{SB\psi}{\varepsilon}, \varphi_m\right).$$

So

$$\left(\frac{SB\psi}{\varepsilon}, \varphi_m\right) = \lambda_m(\psi, \varphi_m) - \varepsilon\varphi_m(1) + (SB\psi, \varphi_m), \Rightarrow \lambda_m(\psi, \varphi_m) = \varepsilon\varphi_m(1),$$

$$(\psi, \varphi_m) = \frac{\varepsilon\varphi_m(1)}{\lambda_m}, \psi(x) = \sum_{m=1}^{\infty} (\psi, \varphi_m) \cdot \varphi_m(x) = \sum_{m=1}^{\infty} \frac{\varepsilon\varphi_m(1)}{\lambda_m} \varphi_m(x),$$

that's what was required to prove.

Apparently, the formula (11) has independent value, for example, for the solution of inverse problems.

Now, using the formula (8) we derive a recurrence relation for the solution of the Cauchy problem (1)-(2). For convenience, we'll rewrite it first

$$y(x, \varepsilon, f) = \sum_{m=1}^{\infty} \frac{(Sf, \varphi_m)}{\lambda_m} \varphi_m(x). \quad (8)$$

Using integration by parts, we transform the coefficients of this series.

$$\begin{aligned} (Sf, \varphi_m) &= (Sf, \lambda_m B^{-1} S \varphi_m - \varepsilon B^{-1} \varphi_m^{(n)}) = \lambda_m (Sf, B^{-1} S \varphi_m) - \\ &- \varepsilon (Sf, B^{-1} \varphi_m^{(n)}) = \lambda_m ((B^{-1})^* Sf, S \varphi_m) - \varepsilon ((B^{-1})^* Sf, \varphi_m^{(n)}) = \\ &| (B^{-1})^* S = S(\overline{B^{-1}}) | = \lambda_m (S\overline{B^{-1}} f, S \varphi_m) - \varepsilon (S\overline{B^{-1}} f, \varphi_m^{(n)}) = \\ &= \lambda_m (B^{-1} f, \varphi_m) - \varepsilon (S(\overline{B})^{-1} f, \varphi_m^{(n)}); \\ (S\overline{B^{-1}} f, \varphi_m^{(n)}) &= \sum_{k=0}^{n-1} (-1)^k (S\overline{B^{-1}} f)^{(k)} \varphi_m^{(n-1-k)} \Big|_0^1 + \\ &+ (-1)^n \int_0^1 (S\overline{B^{-1}} f)^{(n)} \varphi_m(x) dx = \sum_{k=0}^{n-1} S(\overline{B^{-1}} f)^{(k)} \varphi_m^{(n-1-k)}(x) \Big|_0^1 + \\ &+ (S(\overline{B^{-1}} f)^{(n)}, \varphi_m) = (\overline{B^{-1}} f)^{(n-1)}(0) \varphi_m(1) + (S(\overline{B^{-1}} f)^{(n)}, \varphi_m), \Rightarrow \\ (Sf, \varphi_m) &= \lambda_m (\overline{B^{-1}} f, \varphi_m) - \varepsilon (\overline{B^{-1}} f)^{(n-1)}(0) \varphi_m(1) - \varepsilon (S(\overline{B^{-1}} f)^{(n)}, \varphi_m); \end{aligned}$$

Therefore,

$$\begin{aligned} y(x, \varepsilon, f) &= \overline{B^{-1}} f - (\overline{B^{-1}} f)^{(n-1)}(0) \sum_{m=1}^{\infty} \frac{\varepsilon \varphi_m(1)}{\lambda_m} - \varepsilon y\left(x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f\right) = \\ &= \overline{B^{-1}} f(x) - (\overline{B^{-1}} f)^{(n-1)}(0) \psi(x) - \varepsilon y\left(x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f\right). \quad (13) \end{aligned}$$

Note that the function  $\overline{B^{-1}} f$  as a strong solution to the Cauchy problem belongs to the space  $W_2^{n-1}[0,1]$ ; therefore, for the validity of the formula obtained, it suffices to require that  $f(x) \in W_2^1[0,1]$ . Denoting,

$$D^0 = I, D = \frac{d^n}{dx^n} \overline{B^{-1}},$$

rewrite the formula as:

$$y(x, \varepsilon, f) = B^{-1}D^0 f(x) - (B^{-1}D^0 f)^{(n-1)}(0)\psi(0) - \varepsilon y(x, \varepsilon, Df).$$

Further,

$$y(x, \varepsilon, Df) = B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x) - \varepsilon y(x, \varepsilon, D^2 f), \Rightarrow$$

$$\begin{aligned} y(x, \varepsilon, f) &= B^{-1}D^0 f(x) - (B^{-1}D^0 f)^{(n-1)}(0)\psi(x) - \\ &- \varepsilon [B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x) - y(x, \varepsilon, D^2 f)] = B^{-1}D^0 f(x) - \\ &- (B^{-1}D^0 f)^{(n-1)}(0)\psi(x) - \varepsilon [B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x)] + \\ &+ \varepsilon^2 y(x, \varepsilon, D^2 f). \end{aligned}$$

Continuing this process by the method of mathematical induction, we obtain

$$\begin{aligned} y(x, \varepsilon, f) &= \sum_{k=0}^{n-1} (-1)^k [B^{-1}D^k f(x) - (B^{-1}D^k f)^{(n-1)}(0)\psi(x)] \varepsilon^k + \\ &+ (-1)^n \varepsilon^n y(x, \varepsilon, D^n f), \end{aligned}$$

where  $\|y(x, \varepsilon, D^n f)\| \leq \frac{\|D^n f\|}{\alpha}$ . From this formula it is seen that, if  $f(x) \in W_2^n[0,1]$ , then  $D^n f \in L^2(0,1)$  and  $y(x, \varepsilon, D^n f) \in W_2^n[0,1]$ ; the function  $\psi(x)$ , at least  $n$  - times continuously differentiable.  $D^k f(x) \in W_2^{n-k}[0,1]$ ,  $B^{-1}D^k f(x) \in W_2^{2n-k-1}$ . For  $k = n - 1$ ,  $B^{-1}D^{n-1}f(x) \in W_2^n[0,1]$ . We formulate the result.

**Theorem 3.4.** If the  $k$  - th coefficient of equation (1)  $n - k$  times is continuously differentiable on the interval  $[0,1]$  and satisfies the following conditions:

- (a)  $a_1(x) \geq \alpha > 0, \forall x \in [0,1]$ ;
- (b)  $\left(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$ ;
- (c)  $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0, 1, 2, \dots, n-1$ , (14)

where  $c_k^m$  - are binomial coefficients and the right part belongs to the space  $W_2^n[0,1]$ , then the solution of the Cauchy problem (1) - (2) also belongs to the space  $W_2^n[0,1]$  and admits an asymptotic expansion of the form:

$$\begin{aligned} y(x, \varepsilon, f) &= \sum_{k=0}^{n-1} (-1)^k [B^{-1}D^k f(x) - (B^{-1}D^k f)^{(n-1)}(0)\psi(x)] \varepsilon^k + \\ &+ (-1)^n \varepsilon^n y(x, \varepsilon, D^n f), \end{aligned}$$

where

$$\|y(x, \varepsilon, D^n f)\| \leq \frac{\|D^n f\|}{\alpha}, D^0 = I, D = \frac{d^n}{dx^n} \overline{B^{-1}}.$$

$$\begin{cases} Bz(x) = a_1(x)z^{(n-1)}(x) + a_2(x)z^{(n-2)}(x) + \dots + a_n(x)z(x), \\ z(0) = 0, z'(0) = 0, \dots, z^{(n-2)}(0) = 0; \end{cases}$$

$$\begin{cases} \varepsilon\psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \dots + a_n(x)\psi(x) = 0, \\ \psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1. \end{cases}$$

**4. Discussions.**

**Remark 4.1.** Whether it is possible to prove the recurrent formula (13) directly, by direct computations, if possible, we would get rid of the conditions (14) that are needed for the self-conjugacy of the operator  $SC_\varepsilon$ .

By virtue of the known formula, we have

$$\begin{aligned} \varepsilon y(x, \varepsilon, (\overline{B^{-1}f})^{(n)}) &= \varepsilon \int_0^x K(x, t) (\overline{B^{-1}f}(t))^{(n)} dt = \\ &= \varepsilon \left[ \sum_{m=0}^{n-1} (-1)^m \frac{\partial^m K}{\partial t^m} (\overline{B^{-1}f})^{n-1-m} \Big|_0^x + (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}f}(t) dt \right]; \end{aligned}$$

If  $f \in L^2(0,1)$ , then  $\overline{B^{-1}f} \in W_2^{n-1}[0,1]$ , i.e. This function has absolutely continuous derivatives up to the  $n - 2$  - order and a derivative of the  $n - 1$  - order that is summable with square. Thus, all derivatives  $(\overline{B^{-1}f})^{(k)}$  up to  $n - 2$  - nd order inclusive are continuous and formulas  $(\overline{B^{-1}f})^{(k)}(0)$  make sense and, moreover, they all vanish.

A little earlier we proved the formula  $(L_\varepsilon^*)^{-1} = (L_\varepsilon^{-1})^*$ , now we use this formula.

$$\begin{aligned} L_\varepsilon^{-1}f(x) &= \int_0^x K(x, t)f(t)dt = \int_0^1 \theta(x - t) K(x, t)f(t)dt, \\ (L_\varepsilon^{-1})^*g(x) &= \int_0^1 K^*(x, t)g(t)dt = \int_0^1 \theta(x - t)K(t, x)g(t)dt = \int_x^1 K(t, x)g(t)dt; \end{aligned}$$

We act by the operator  $L_\varepsilon^*$  on both sides of this equality, in the end we get

$$g(x) = L_\varepsilon^* \cdot \int_x^1 K(t, x)g(t)dt.$$

Now let's calculate the right side, for convenience

$$z(x) = \int_x^1 K(t, x)g(t)dt$$

we have

$$\begin{aligned} z'(x) &= -K(t, x)g(t)|_{t=x} + \int_x^1 \frac{\partial K}{\partial x}(t, x)g(t)dt = \int_x^1 \frac{\partial K}{\partial x}(t, x)g(t)dt, \\ z''(x) &= -\frac{\partial K}{\partial x}(t, x)g(t)|_{t=x} + \int_x^1 \frac{\partial^2 K}{\partial x^2}(t, x)g(t)dt = \int_x^1 \frac{\partial^2 K}{\partial x^2}(t, x)g(t)dt, \dots, \\ z^{(n-1)}(x) &= \int_x^1 \frac{\partial^{n-1} K}{\partial x^{n-1}}(t, x)g(t)dt, \\ z^{(n)}(x) &= -\frac{\partial^{n-1} K}{\partial x^{n-1}}(t, x)g(t)|_{t=x} + \int_x^1 \frac{\partial^n K}{\partial x^n}(t, x)g(t)dt; \end{aligned}$$

Therefore

$$L_{\varepsilon}^* z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x)z(x)]^{(k)};$$

$$[a_{n-k}(x)z(x)]^{(k)} = \sum_{j=0}^k a_{n-k}^{(k-j)}(x) z^{(j)} c_k^j = \sum_{j=0}^k a_{n-k}^{(k-j)}(x) c_k^j \int_x^1 \frac{\partial^j K}{\partial x^j}(t, x) g(t) dt =$$

$$= \int_x^1 \sum_{j=0}^k a_{n-k}^{(k-j)}(x) c_k^j \frac{\partial^j K}{\partial x^j} g(t) dt = \int_x^1 [a_{n-k}(x)K(t, x)]^{(k)} g(t) dt, 1 \leq k \leq n-1;$$

Therefore,

$$L_{\varepsilon}^* z = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x)z(x)]^{(k)} =$$

$$(-1)^n \varepsilon \int_x^1 \frac{\partial^n K}{\partial x^n}(t, x) g(t) dt - (-1)^n \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t, x) g(t) \Big|_{t=x} + \dots =$$

$$= \int_x^1 \left[ (-1)^n \varepsilon \frac{\partial^n K}{\partial x^n}(t, x) + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x)K(t, x)]^{(k)} \right] g(t) dt -$$

$$- (-1)^n \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t, x) g(t) \Big|_{t=x} = g(x).$$

Due to the arbitrariness of  $g(x)$ , we conclude that

$$(-1)^{n-1} \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t, x) = 1$$

and

$$L_{\varepsilon}^+ K(t, x) = 0,$$

i.e. the first variable, the Cauchy kernel is the solution of the homogeneous equation  $L_{\varepsilon} K(x, t) = 0$  and the second variable is a solution of the homogeneous equation  $L^+ K(t, x) = 0$ .

Now using the formula

$$\int_0^x v(x) u^{(n)}(t) dt = \sum_{m=0}^{n-1} (-1)^m v^{(m)} u^{(n-1-m)}(t) \Big|_0^x + (-1)^n \int_0^x v^{(n)}(t) u(t) dt,$$

convert expressions:  $\varepsilon y \left( x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f \right)$ .

$$\varepsilon y \left( x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f \right) = \varepsilon \int_0^x K(x, t) \frac{d^n}{dx^n} \overline{B^{-1}} f(t) dt =$$

$$= \varepsilon \left[ \sum_{m=0}^{n-1} (-1)^m \frac{\partial^m K}{\partial t^m} (\overline{B^{-1}} f)^{(n-1-m)}(t) \Big|_0^x + (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt \right] =$$

$$= \varepsilon \left[ (-1)^{n-1} \frac{\partial^{n-1} K}{\partial t^{n-1}} (\overline{B^{-1}} f)(t) \Big|_{t=x} - K(x, t) (\overline{B^{-1}} f)^{(n-1)}(0) \right] +$$

$$+ \varepsilon (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt;$$



From the equation

$$(-1)^n \varepsilon \frac{\partial^n K}{\partial t^n} + B^+ K(x, t) = 0$$

we have

$$(-1)^n \varepsilon \frac{\partial^n K}{\partial t^n} = -B^+ K(x, t).$$

so

$$\begin{aligned} \varepsilon (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt &= - \int_0^x B^+ K(x, t) \overline{B^{-1}} f(t) dt = \\ &= - \left[ \int_0^x K(x, t) B \overline{B^{-1}} f(t) dt + [K, \overline{B^{-1}} f] \right] = - \underbrace{[K, \overline{B^{-1}} f]}_0 - \int_0^x K(x, t) f(t) dt = \\ &= - \int_0^x K(x, t) f(t) dt; \\ \int_0^x B^+ K \overline{B^{-1}} f(t) dt &= \int_0^x \overline{B^{-1}} f(t) B^+ K dt = (B^{-1} f, B^+ K); \\ (Bu, v) &= [u, v] + (u, B^+ v); \end{aligned}$$

Gentle  $u = \overline{B^{-1}} f, v = K$ , we get

$$\begin{aligned} (B \overline{B^{-1}} f, K) &= [\overline{B^{-1}} f, K] + (B^{-1} f, B^+ K), \\ [u, v] &= \sum_{k=1}^n \sum_{m=0}^{n-k-1} (a_k v)^{(m)} (-1)^m u^{(n-k-1-m)}(x), \\ [\overline{B^{-1}} f, K] &= \sum_{k=1}^n \sum_{m=0}^{n-k-1} (a_k K)^{(m)} (-1)^m (\overline{B^{-1}} f)^{(n-k-1-m)} \Big|_0^x = 0. \end{aligned}$$

So

$$\begin{aligned} K(x, t)|_{t=x} = 0, \frac{\partial K}{\partial t} \Big|_{t=x} = 0, \dots, \frac{\partial^{(n-2)} K}{\partial t^{(n-2)}} \Big|_{t=x} = 0, \\ (\overline{B^{-1}} f)(0) = 0, (\overline{B^{-1}} f)'(0) = 0, \dots, (\overline{B^{-1}} f)^{(n-2)}(0) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon y \left( x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f \right) &= \varepsilon \left[ (-1)^{n-1} \frac{\partial^{n-1} K}{\partial t^{n-1}} \Big|_{t=x} \overline{B^{-1}} f - K(x, 0) (\overline{B^{-1}} f)^{(n-1)}(0) \right] - \\ &- \int_0^x K(x, t) f(t) dt = \overline{B^{-1}} f(x) - (\overline{B^{-1}} f)^{(n-1)}(0) \psi(x) - y(x, \varepsilon, f), \end{aligned}$$

where  $\psi(x)$  is a solution to the Cauchy's problem

$$\begin{aligned} \varepsilon \psi^{(n)}(x) + a_1(x) \psi^{(n-1)}(x) + \dots + a_n(x) \psi(x) &= 0, \\ \psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) &= 1. \end{aligned}$$

Theorem 3.4 we can now reformulate.

**Theorem 4.1.** If the  $k$  - th coefficient of equation (1) is continuously differentiable  $n - k$  ( $k = 0, 1, 2, \dots, n$ ) times on the interval  $[0, 1]$  and satisfies the following conditions:

(a)  $a_1(x) \geq \alpha > 0, \forall x \in [0, 1];$

(b)  $(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)) \geq 0, \forall y \in D(L_\varepsilon)$

and the right part  $f(x) \in W_2^n[0, 1]$ , the solution of the Cauchy problem(1)-(2) also belongs to the space  $W_2^n[0, 1]$  and admits an asymptotic expansion of the form:

$$y(x, \varepsilon, f) = \sum_{k=0}^{n-1} (-1)^k \left[ B^{-1} D^k f(x) - (B^{-1} D^k f)^{(n-1)}(0) \psi(x) \right] \varepsilon^k + (-1)^n \varepsilon^n y(x, \varepsilon, D^n f),$$

where

$$\|y(x, \varepsilon, D^n f)\| \leq \frac{\|D^n f\|}{\alpha}, \quad D^0 = I, \quad D = \frac{d^n}{dx^n} \overline{B^{-1}},$$

$$\begin{cases} Bz(x) = a_1(x)z^{(n-1)}(x) + a_2(x)z^{(n-2)}(x) + \dots + a_n(x)z(x), \\ z(0) = 0, z'(0) = 0, \dots, z^{(n-2)}(0) = 0; \end{cases}$$

$$\begin{cases} \varepsilon \psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \dots + a_n(x)\psi(x) = 0, \\ \psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1. \end{cases}$$

**5. Summary.** The spectral theory of equations with a deviating argument can be successfully applied in the study of singularly perturbed problems. The residual term formula can be used to control current errors in the numerical solution of such problems.

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### ЖОҒАРҒЫ РЕТТІ КӘДІМГІ ДИФФЕРЕНЦИАЛДЫҚ ТЕНДЕУДІҢ СИНГУЛЯР ӘСЕРЛЕНГЕН КОШИ ЕСЕБІН ШЕШУДІҢ ОПЕРАТОРЛЫҚ ӘДІСІ ТУРАЛЫ

**Аннотация.** Бұл еңбекте аргументін ауытқыту әдісімен кәдімгі  $n$  –ретті дифференциалдық теңдеудің сингуляр әсерленген Коши есебінің шешімі асимптотикалық қатарға таратылды. Ал қатардың қалдыға теңдеудің оң жағындағы бос мүшесі арқылы бағаланды. Бұл саланың көптеген еңбектері кәделі саналады және олар техникалық мәселелермен тығыз байланысты, мүмкін, сондықтан болар, алынған бағамдар  $O$  - үлкен немесе  $o$  - кіші шамалары арқылы өрнектелген, сондықтан олар тек теориялық мазмұнға ие, сондықтан нақты кәдеге жарамайды, соған қарамастан мұндай жұмыстар жетіп артылады. Ұсынылып отырған еңбектің негізгі артықшылығы, алгоритмінің қарапайымдылығы мен қалдықтың формуласы болса керек, ол теңдеудің бос мүшесі арқылы өрнектелген және нақты бағаланған.

**Түйін сөздер:** Сингуляр әсерленген, спектралді таралым, ауытқыған аргумент, қалдық мүшенің бағамы, жалқы оператор, Гилберт пен Шмидтің теоремасы, әсіре үзіксіз оператор, Фридрихстың леммасы, Кошидің есебі, асимптотикалық таралым, мардымсыз параметр.

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## ОБ ОДНОМ ОПЕРАТОРНОМ МЕТОДЕ РЕШЕНИЯ СИНГУЛЯРНО ВОЗМУЩЕННОЙ ЗАДАЧИ КОШИ ДЛЯ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $n$ -ГО ПОРЯДКА

**Аннотация.** В настоящей работе, методом отклоняющегося аргумента, получено асимптотическое разложение решения задачи Коши для обыкновенного дифференциального уравнения  $n$  – го порядка с переменными коэффициентами, с оценкой остаточного члена через правую часть уравнения. Многие работы посвященные к этой теме носят прикладной характер, и полученные им оценки остаточного члена выражены в терминах  $O$  –большое, или  $o$  –малое, поэтому имеют теоретическое значение, нежели прикладное, как они утверждают. Основным достоинством предлагаемого нами метода является простота его алгоритма, и формула остаточного члена, явно выраженная через правую часть уравнения, и его оценка.

**Ключевые слова:** Сингулярное возмущение, спектральное разложение, отклоняющийся аргумент, оценка остаточного члена, самосопряженный оператор, теорема Гилберта - Шмидта, вполне непрерывный оператор, лемма Фридрикса, задача Коши, асимптотическое разложение, малый параметр.

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<sup>3</sup>School-Lyceum №250 named after T.Komekbayev, Karmakchi area, Kyzylorda region, Kazakhstan[\\*sarsembaeva.a@kaznu.kz](mailto:sarsembaeva.a@kaznu.kz); [†odsuren@seas.num.edu.mn](mailto:odsuren@seas.num.edu.mn)**MAY 5, 2015 SOLAR FLARE DATA ANALYSIS IN SUNPY**

**Abstract.** In this paper was monitored solar flare registered on May 5, 2015. This flare, which peaked at 6:11 pm EDT from a sunspot called Active Region 2339 (AR2339), is classified as an X2.7-class flare. We have performed solar data analysis using the Python/SunPy tool. SunPy was chosen as the principle data analysis environment since it provides easy to use interfaces to the Virtual Solar Observatory (VSO).

**Keywords:** solar flares, emission measure, reconnection rate, SunPy.

**INTRODUCTION**

Solar flares are one of the most powerful energetic events in the solar atmosphere. Given their role in the energy balance of the solar corona and their role played in driving space weather, many studies investigated the energy build-up and initiation of flares, concentrating on the events preceding the onset of a flare [1].

Our study uses X-ray flare databases. The flare was collected using the dataset provided by the Geostationary Operational Environmental Satellite (GOES) [2]. GOES flares are classified as A, B, C, M, or X-class, according to their peak flux ( $\text{W m}^{-2}$ ) observed in the 0.1 to 0.8 nm wavelength range. We selected the X-class flares corresponding to a flux in excess of  $10^{-4} \text{ W m}^{-2}$  at Earth, respectively. The GOES flare lists are available at NGDC/NOAA [3].

This research has made use of SunPy, an open-source and free community-developed solar data analysis package written in Python [4]. Python/SunPy chosen as the principle data analysis environment since it provides easy to use interfaces to the Virtual Solar Observatory (VSO). SunPy is a data analysis toolkit, which provides the necessary software for analyzing solar and heliospheric datasets in Python. SunPy aims to provide a free and open-source alternative to the current standard, an IDL based solar data analysis environment known as SolarSoft (SSW) [5-9].

In this work, we have observed solar flare occurred on May 5, 2015. This flare, which peaked at 6:11 pm EDT on May 5, 2015 from a sunspot called Active Region 2339 (AR2339), is classified as an X2.7-class flare.

**SOLAR DATA VISUALIZATION**

To find and overplot the location of the brightest pixel, we first created the Map using the FITS data and imported the coordinate functionality. In the Figure 1 shown the brightest pixel location in different wavelengths obtained by Python/SunPy.

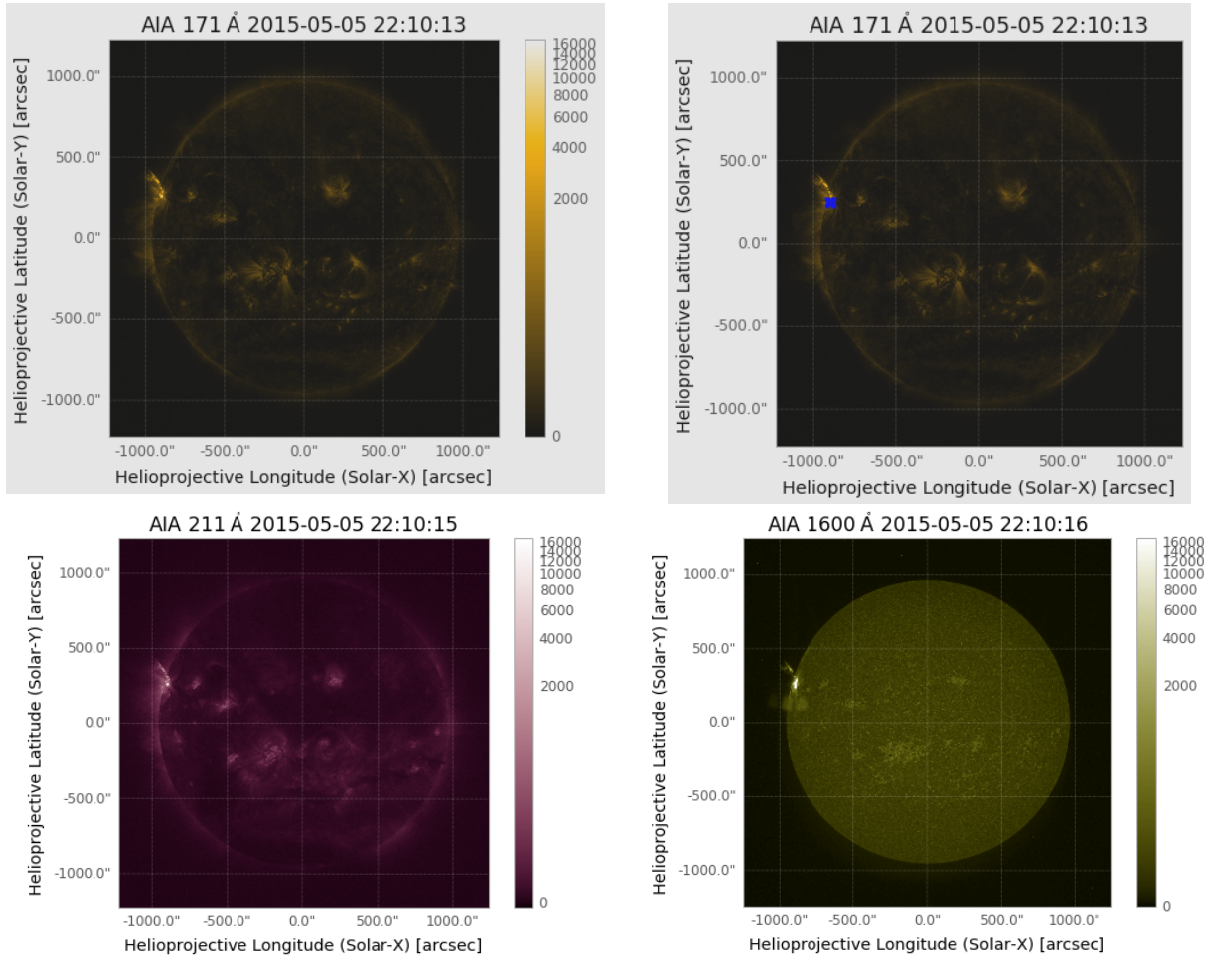


Figure 1 - The brightest pixel location (Active area 2339 in AIA 171 Å, AIA 211 Å and AIA 1600 Å)

To obtain the GOES flare intensity, we first grab GOES XRS data for a particular time of interest that is May 5, 2015. Then the data loaded into a TimeSeries. Next we grab the HEK data for this time from the NOAA Space Weather Prediction Center (SWPC). The Figure 2 shows the total flux of X-rays, which was registered on May 5, 2015.

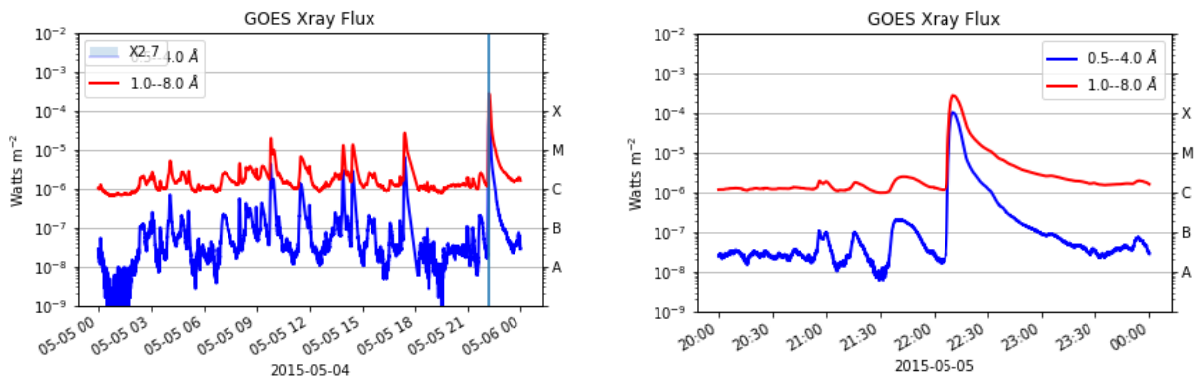


Figure 2 - Total X-ray flux obtained in GOES 13 and GOES 15

To enhance emission above the limb, we first created the Map using the FITS data. Next we build two arrays which include all of the x and y pixel indices. Then we converted this to helioprojective coordinates and created a new array which contains the normalized radial position for each pixel. Next we plot it along with a fit to the data. We fit the logarithm of the intensity since the intensity drops of very quickly as a function of distance from the limb [10-11].

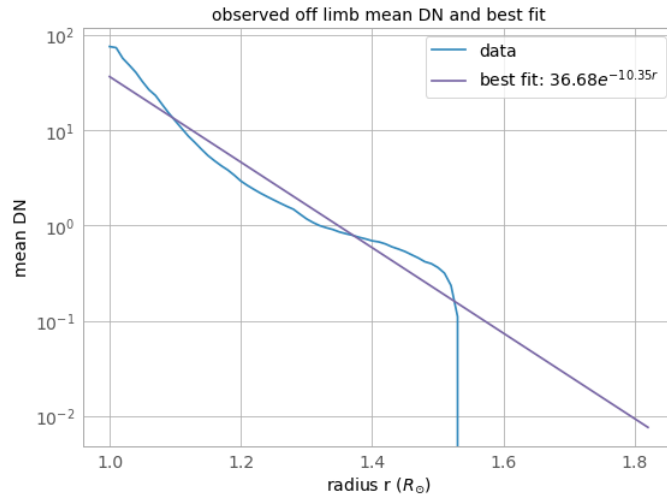


Figure 3 - Observed off limb mean DN and best fit

The National Oceanic and Atmospheric Administration (NOAA) launches and maintains a set of satellites called Geostationary Operational Environmental Satellites (GOES), carrying weather monitoring instruments. Each GOES satellite also carries a solar X-ray package (the “X-Ray Sensor”, or XRS) consisting of a collimator that feeds a pair of ion chambers. These ion chambers measure the Sun’s spatially integrated soft X-ray flux in two wavelength bands, 0.5–4 and 1–8Å, with a 3-s cadence. The GOES soft X-ray detectors have provided an essentially uninterrupted monitor of the Sun’s activity for 30 years, and are a valuable resource for the study of past solar activity and the prediction of space weather [12-16].

For quantitative physical understanding of processes in the Sun’s atmosphere, the X-ray fluxes themselves are of limited use. However, they reflect the temperature and emission measure of the plasma that produces the soft X-rays, and these physical quantities are of great importance: from them, the energetics of solar flares and other energy releases can be deduced [17-18].

Corresponding volume emission measure of the solar soft X-ray emitting plasma observed by the GOES/XRS is shown in Fig.4. The volume emission measure were obtained in SunPY using the methods of White et al. [17] who used the CHIANTI atomic physics database to model the response of the ratio of the short (0.5-4 angstrom) to long (1-8 angstrom) channels of the XRSs onboard various GOES satellites.

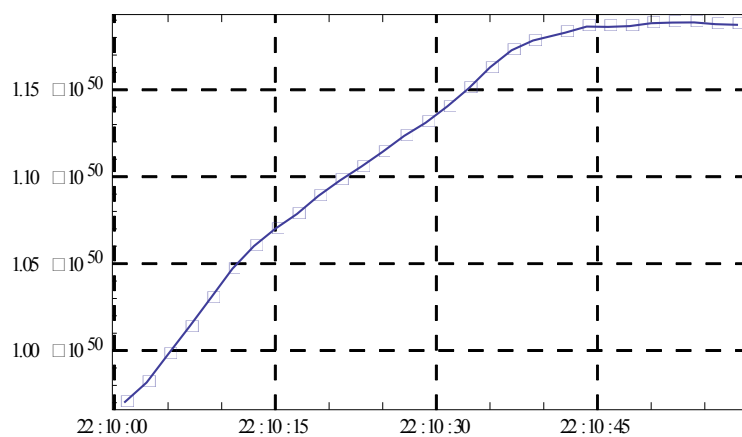


Figure 4 - Obtained emission measure of the solar soft X-ray emitting plasma observed by the GOES/XRS

**CONCLUSION**

The scientific python community is much more established in other disciplines than it is in solar physics. SunPy is making use of existing scientific python projects, with deeper integration with projects like Astropy and scikit-image possible in the future. Using SunPy package we have obtained the values of

temperature and emission measure from a GOESLightCurve. This function calculates the isothermal temperature and corresponding volume emission measure of the solar soft X-ray emitting plasma observed by the GOES/XRS.

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### 5 МАМЫР, 2015 КҮН ЖАРҚЫЛЫН SUNPY АРҚЫЛЫ ТАЛДАУ

**Аннотация.** Осы мақалада 2015 жылдың 5 мамырында тіркелген күн жарқылдарының бақылауы жүргізілді. 2339 (AR2339) активті аймақта ШБС сағат бойынша сағат 6:11-де максимум мәнінде X2.7-класс күн жарқылы болып тіркелді. Python/SunPy құралы арқылы күн деректері талданды. SunPy виртуалды күн обсерваториясына (VSO) интерфейсін пайдалануды жеңілдететіндіктен негізгі деректерді талдау құралы ретінде таңдалды.

**Түйін сөздер:** күн жарқылы, шығарындылар өлшемі, қайта ұштасу жылдамдығы, SunPy.

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### АНАЛИЗ СОЛНЕЧНЫХ ВСПЫШКОВ, ЗАРЕГИСТРИРОВАННЫХ 5 МАЯ 2015 В SUNPY

**Аннотация.** В этой статье отслеживалась солнечная вспышка, зарегистрированная 5 мая 2015 года. Солнечная вспышка, которая достигла максимума в 6:11 вечера по восточному поясному времени от солнечного пятна, называемого активным регионом 2339 (AR2339), классифицируется как вспышка класса X2.7. Мы проводили анализ солнечных данных с помощью инструмента Python/SunPy. SunPy была выбрана в качестве основной среды анализа данных, поскольку она предоставляет простые в использовании интерфейсы для Виртуальной солнечной обсерватории (VSO).

**Ключевые слова:** солнечные вспышки, мера эмиссии, скорость пересоединения, SunPy.

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## 26 JANUARY, 2019 SOLAR FLARES DIAGNOSTICS BASED ON THE SOFT X-RAY EMISSION MEASURES

**Abstract.** In this paper, we recently analyzed solar flares registered on January 26, 2019. We have identified several physical quantities of solar flares and estimated reconnection rate of solar flares. To determine the physical parameters we used images taken with the AIA instrument on board SDO satellite at wavelengths AIA 193 Å, 131 Å, AIA 174 Å, SXT - pictures, HMI Magnetogram, GOES XRT-data.

**Keywords:** solar flares, emission measure, reconnection rate.

### INTRODUCTION

A solar flare occurs when magnetic energy that has built up in the solar atmosphere is suddenly released. Large flares can emit up to  $10^{32}$  ergs of energy. This energy is ten million times greater than the energy released from a volcanic explosion.

The National Oceanic and Atmospheric Administration (NOAA) launches and maintains a set of satellites called Geostationary Operational Environmental Satellites (GOES), carrying weather monitoring instruments. Each GOES satellite also carries a solar X-ray package (the “X-Ray Sensor”, or XRS) consisting of a collimator that feeds a pair of ion chambers. These ion chambers measure the Sun’s spatially integrated soft X-ray flux in two wavelength bands, 0.5–4 and 1–8Å, with a 3-s cadence. The GOES soft X-ray detectors have provided an essentially uninterrupted monitor of the Sun’s activity for 30 years, and are a valuable resource for the study of past solar activity and the prediction of space weather [1-4].

Our study uses X-ray flare databases. The flare was collected using the dataset provided by the Geostationary Operational Environmental Satellite (GOES) [5-6]. GOES flares are classified as A, B, C, M, or X-class, according to their peak flux ( $\text{W m}^{-2}$ ) observed in the 0.1 to 0.8 nm wavelength range. We selected the B and C-class flares corresponding to a flux in excess of  $10^{-7} \text{ W/m}^{-2}$  and  $10^{-6} \text{ W/m}^{-2}$  at Earth, respectively. The GOES flare lists are available at NGDC/NOAA [7].

This research has made use of SunPy, an open-source and free community-developed solar data analysis package written in Python [8]. Python/SunPy was chosen to analyze the emission measure (EM) obtained from GOES soft X-ray data since it provides easy to use interfaces to the Virtual Solar Observatory (VSO). SunPy aims to provide a free and open-source alternative to the current standard, an IDL based solar data analysis environment known as SolarSoft (SSW).

In this work, we have observed solar flares occurred on January 26, 2019. The sun emitted a significant solar flares, peaking at 06:47 a.m., 13:22 p.m., 15:55 p.m., 16:28 p.m. (EDT). This flares are classified as the B2.6, C5.0, B3.2, B8.6-class flares. Solar flares can be classified according to their brightness in the x-ray wavelengths. There are three categories: X-class flares are big; they are major events that can trigger radio blackouts around the whole world and long-lasting radiation storms in the

upper atmosphere. M-class flares are medium-sized; they generally cause brief radio blackouts that affect Earth's polar regions. Minor radiation storms sometimes follow an M-class flare. Compared to X- and M-class events, B and C-class flares are small with few noticeable consequences here on Earth. Solar flares are different to 'coronal mass ejections' (CMEs), which were once thought to be initiated by solar flares. CMEs are huge bubbles of gas threaded with magnetic field lines that are ejected from the Sun over the course of several hours. If a CME collides with the Earth, it can excite a geomagnetic storm [9-13]. In Fig. 1 shown the images obtained on the board of GHN satellite in AIA 193 Å, 131 Å, HMI, AIA 174 Å wavelengths [6]. To determine the length of the loops, we used SXT images. From the SXT data, we get values for the length of the loops.

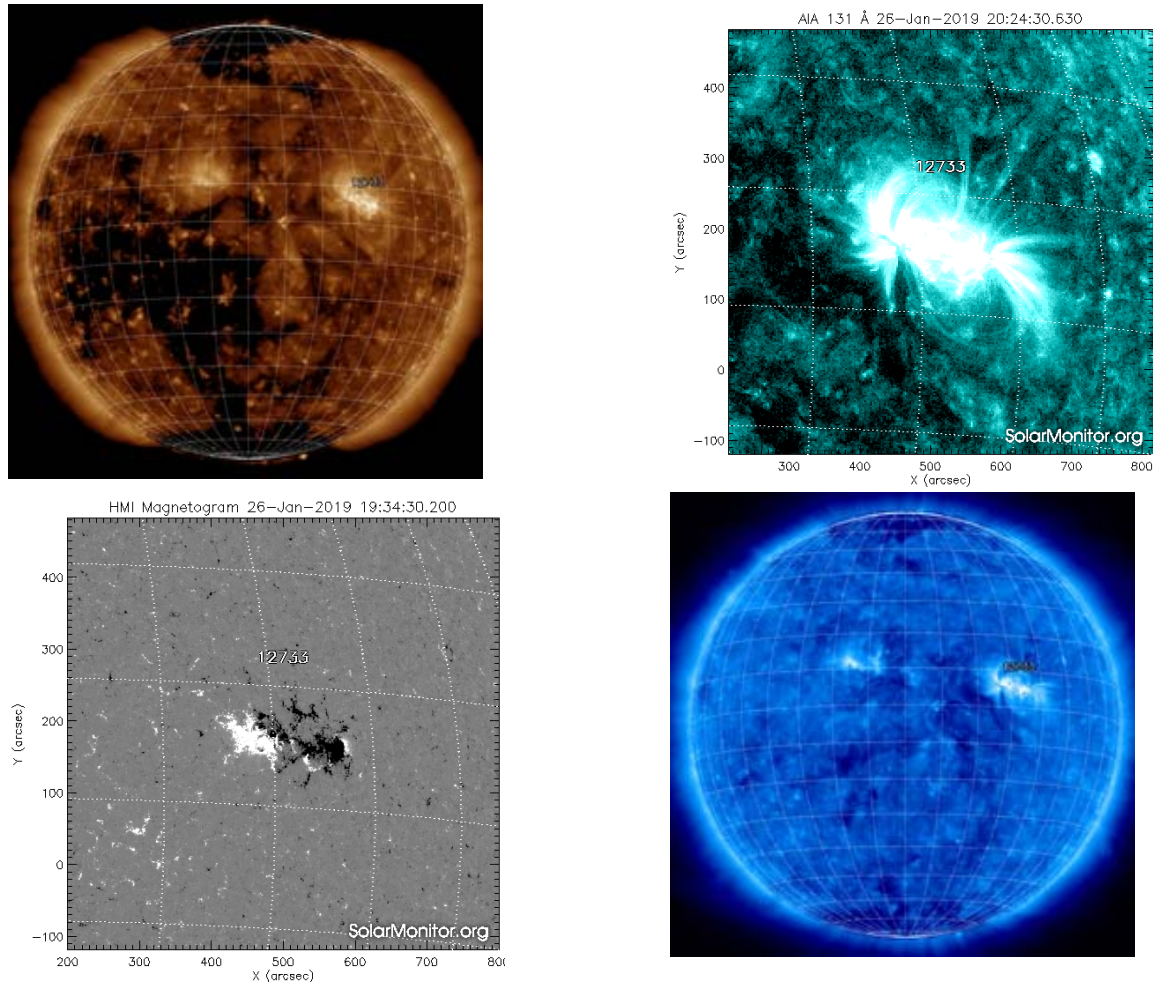


Figure 1 - Active area 2733 in AIA 193 Å, 131 Å, HMI, AIA 174 Å

### DATA ANALYSIS

Using SunPy package we have obtained the values of temperature and emission measure from a GOESLightCurve. This function calculates the isothermal temperature and corresponding volume emission measure of the solar soft X-ray emitting plasma observed by the GOES/XRS. This was done using the observed flux ratio of the short (0.5-4 angstrom) to long (1-8 angstrom) channels [14-16]. The results are returned in a new LightCurve object which contains metadata and flux data of the input LightCurve object in addition to the newly found temperature and emission measure values. The temperature and volume emission measure are calculated using the methods of White et al. (2005) who used the CHIANTI atomic physics database to model the response of the ratio of the short (0.5-4 angstrom) to long (1-8 angstrom) channels of the XRSs onboard various GOES satellites [17]. Corresponding volume emission measure of the solar soft X-ray emitting plasma observed by the GOES/XRS are shown in Fig.2.

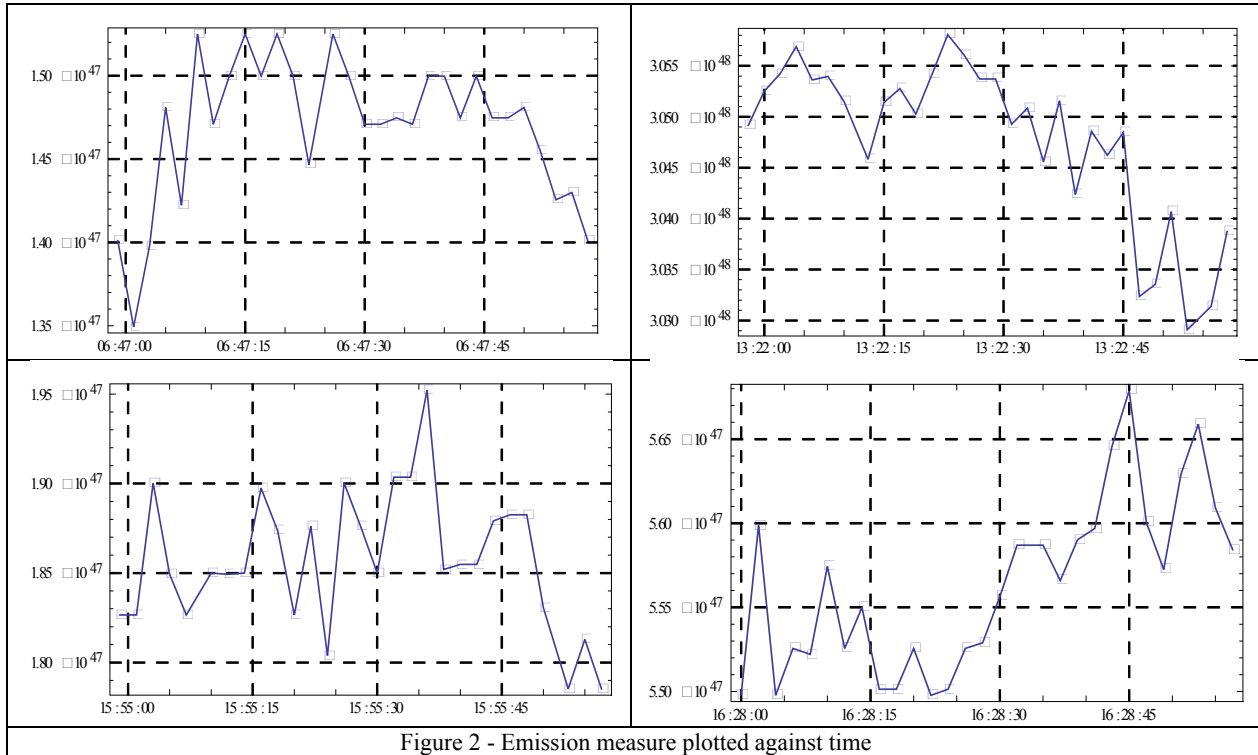


Figure 2 - Emission measure plotted against time

For quantitative physical understanding of processes in the Sun’s atmosphere, the X-ray fluxes themselves are of limited use. However, they reflect the temperature and emission measure of the plasma that produces the soft X-rays, and these physical quantities are of great importance: from them, the energetics of solar flares and other energy releases can be deduced.

The volume emission measure were obtained in SunPY using the methods of White et al. [18] who used the CHIANTI atomic physics database to model the response of the ratio of the short (0.5-4 angstrom) to long (1-8 angstrom) channels of the XRSs onboard various GOES satellites [21-25].

**RESULTS**

Using the method described in [19-20], we analyzed solar flare that have been registered on January 26, 2019. Examined the dependence of the reconnection rate from GOES class of solar flares. Figure 3 shows the dependence of the reconnection rate from temperature and time scale.

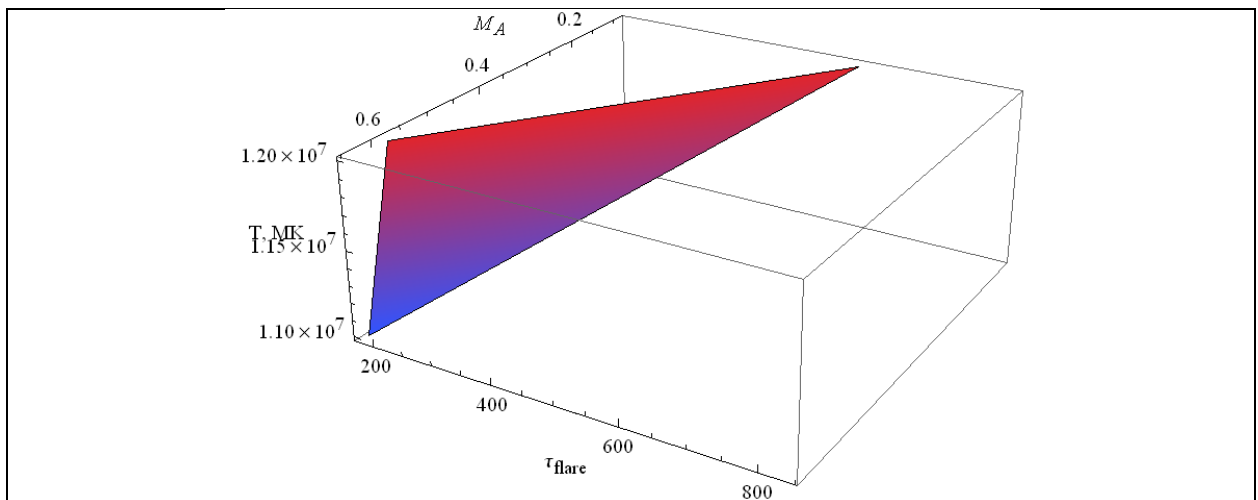


Figure 3 - Reconnection rate  $M_A$  plotted against time scale

## CONCLUSION

The values of reconnection rate are distributed in the range from  $10^{-2}$ - $10^{-1}$ . Here, the value of the reconnection rate decreases as the GOES class increases. The value of the reconnection rate obtained in this study is within 2 order of magnitude from the predicted maximum value of the Petschek model.

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### 26 ҚАҢТАР, 2019 ЖҰМСАҚ РЕНТГЕН СӘУЛЕСІНІҢ НЕГІЗІНДЕГІ КҮН ЖАРҚЫЛЫНЫҢ ДИАГНОСТИКАСЫ

**Аннотация.** Осы мақалада 2019 жылдың 26 қаңтарында тіркелген күн жарқылдарының статистикалық зерттеулері жүргізілді. Біз күн жарқылдарының физикалық мәндері мен қайта ұштасу жылдамдығын бағаладық. Физикалық параметрді анықтау үшін SDO спутникінің бортында AIA инструментінің 193 Å, 131 Å, AIA 174 Å толқын ұзындығында алынған және SXT суреті, HMI Magnetogram, SOLIS Chromospheric Magnetogram, GOES XRT- деректері пайдаланылды.

**Түйін сөздер:** күн жарқылы, шығарындылар өлшемі, қайта ұштасу жылдамдығы.

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### 26 ЯНВАРЯ, 2019 Г. ДИАГНОСТИКА СОЛНЕЧНЫХ ВСПЫШЕК НА ОСНОВЕ РЕЗУЛЬТАТОВ РЕГИСТРАЦИИ МЯГКОГО РЕНТГЕНОВСКОГО ИЗЛУЧЕНИЯ

**Аннотация.** В этой статье нами проведены статистические исследования вспышек, зарегистрированных 26 января 2019 г. Мы определили несколько физических величин вспышек и оценили скорость пересоединения солнечных вспышек. Для определения физических параметров мы использовали снимки, полученные с инструмента AIA на борту спутника SDO на длинах волн 193 Å, 131 Å, AIA 174 Å, SXT - снимки, HMI Magnetogram, SOLIS Chromospheric Magnetogram, GOES XRT-данные.

**Ключевые слова:** солнечные вспышки, мера эмиссии, скорость пересоединения.

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**COMPUTATIONAL EXPERIMENTS FOR RESEARCH OF  
FLOW AERODYNAMICS AND TURBULENT CHARACTERISTICS  
OF SOLID FUEL COMBUSTION PROCESS**

**Abstract.** Some of the most interesting and useful from the point of view of practical application are the issues of heat and mass transfer modeling in the presence of physicochemical processes in real geometry areas. Such areas are the combustion chambers of various heat and power plants, and internal combustion engines.

**Key Words:** combustion chamber, boiler, burners, solid fuel, high-ash coal, numerical simulation, computational experiment.

**Introduction**

Currently, there is an increased interest in the study of heat and mass transfer processes in high-temperature environments in the presence of combustion therein. These processes run under conditions of strong non-isothermality and flow turbulence, multiphase environment, significant influence of nonlinear effects of thermal radiation, interphase interaction, and multiple stages of chemical reactions occurring at that time. Such phenomena are widespread, and they play an important part in thermophysical processes and their study is an urgent task of macrokinetics, combustion and explosion physics and present-day thermophysics.

Turbulent high-temperature and chemically-reactive media are exceptional in terms of their physicochemical properties, hands-on capabilities and engineering applications. Research on heat and mass transfer in such media is relevant in the creation of new physicochemical technologies, in the design of aviation and rocket technology, in the development of new furnace units, gas turbines and internal combustion engines.

Of particular relevance is the study of heat and mass transfer in high-temperature reactive media and simulation of physicochemical processes occurring during combustion of pulverized coal fuel to solve the problems of modern power engineering and ecology. Consideration of these issues is relevant in connection with the concept of "energy security" of the country, on the one hand, and the development of "clean" fuel combustion processes in compliance with strict standards for the environmental emissions of harmful substances, on the other hand.

There is an increased interest in the study of heat and mass transfer processes in the presence of physicochemical transformations, currently observed. Examination of the patterns of such flows is of fundamental importance in constructing a theory of physics of combustion and explosion and an enormous practical orientation in the creation of new physicochemical technologies and in the development of technological processes and systems with the rational utilization of energy resources.

To study heat and mass transfer processes, the most commonly used methods are well-known methods of the turbulent jets theory [1-9], when researchers use pre-selected velocity or temperature profiles, integral laws of conservation of momentum, heat content, etc. And, the selected profiles with sufficient approximation approximate the experimental profiles.

### Mathematical Model and Basic Equations

The problem of modeling is very complex, as it involves the interaction of turbulent combustion of many chemical components with multiphase processes (particles of gaseous or solid fuel and carbon in the flow field) and with radiation transport. Modeling in varying degrees includes particle size distribution (calculated in finite size ranges at all points in the region), flow or zone characteristics of radiation transfer, and soot distribution data (soot is formed as a result of thermal decomposition of hydrocarbons and is eliminated by oxidation; both processes represent a complex chemical kinetics problem).

For three-dimensional fluid motion with variable physical properties, the field of velocity, temperature and concentration is described by a differential equation system (1-4)

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho u_i), \quad (1)$$

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial}{\partial x_j}(\tau_{i,j}) - \frac{\partial \rho}{\partial x_j} + \rho f_i, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho h_i) = -\frac{\partial}{\partial x_i}(\rho u_i h) - \frac{\partial q_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} + \tau_{ij} \frac{\partial u_j}{\partial x_i} + S_q, \quad (3)$$

$$\frac{\partial}{\partial t}(\rho c_\beta) = -\frac{\partial}{\partial x_i}(\rho c_\beta u_i) + \frac{\partial}{\partial x_i} R_\beta, \quad (4)$$

where  $I = 1, 2, 3; j = 1, 2, 3; \beta = 1, 2, 3, \dots, N$ .

To simulate turbulent viscosity, the well-known  $k-\varepsilon$  turbulence model was used, comprised of the equation for the conservation of kinetic energy of turbulence -  $k$ , its dissipation rate  $\varepsilon$ , and the model relation for turbulent viscosity. The  $k-\varepsilon$  turbulence model is the standard model for a flow with forced and natural convection.

In flows with high solids content, solid media has a significant effect on convective and diffusive transport. However, the presence of solids in carbon monoxide from coal dust combustion units is so insignificant (with the exception of near-burners area) that the second phase effect is neglected in the calculations [10-13]. Then the process of burning solid fuel in the combustion chambers can be represented as follows: the flame is a two-phase gas-dispersed system, and the effect of the solid phase on the flow aerodynamics is insignificant [14].

In determining the integral absorption coefficients  $K_{abs}$ , it is necessary to take into account the mechanisms of emission of gas and solid particles. If there is a thermodynamic equilibrium between gas and solid particles, the radiation of suspension is described by adding dust and gas emissions. Thus, the share contributed by gas and solid particles is described by the following sum [15-16]:

$$K_{abs} = K_{abs,G} + \sum K_{abs,P,k} \quad (5)$$

The values of the mass coefficient and the absorption coefficient of the radiation effect of water vapor and carbon dioxide used in this work are presented in Table 1.

$$K_{abs,G} = a_{CO_2} \cdot k_{CO_2}^* \cdot p_{CO_2} + a_{H_2O} \cdot k_{H_2O}^* \cdot p_{H_2O} \quad (6)$$

### Building a physical and geometric model of the problem of pulverized coal torch burning in the combustion chamber of the BKZ-160 boiler

The computational experiment in the work was performed for the combustion chamber of the BKZ-160 boiler at Almaty Thermal Power Plant (Kazakhstan), steam generating capacity is 160 t/h, at a

pressure of 9,8 MPa and a steam superheat temperature of 540°C. Thermal performance for steam  $Q = 119,5$  MW (97,8 Gcal/h), thermal power of the furnace  $N = 124,4$  MW (107 Gcal/h). The boiler is designed to burn coal.

Table 1 - Mass coefficient and absorption coefficient of gas radiation

<i>B</i> Component	$a_p$ [1]	$k_p^*$ , [1/(m·bar)]
CO <sub>2</sub>	$0.275-8.4 \cdot 10^{-5} \cdot T_G$	$85.0 \cdot T_G^{-0,33}$
H <sub>2</sub> O	$7.2 \cdot T_G^{-0,4}$	$1100 T_G^{-0,82}$

On either side of the furnace chamber there are 4 units of direct-flow slot burners (2 burners per unit), directed tangentially to a circle with a diameter of 60x4 with a step of 64 mm. The screens are divided into 12 independent circulation loops. Pipes of the front and rear screens in the lower part form a cold funnel, and in the upper part of the rear screen pipes are bent into the combustion chamber, forming an "aerodynamic" nose. After the "aerodynamic" nose of the rear screen pipes are gathered into chambers, where a steam-water mixture is sent to the boiler drum through the festoon.

Performance of one burner in terms of fuel is 4 tons/h. The secondary air flow rate through the burner is  $V = 6000$  nm<sup>3</sup>/h with an excess air coefficient of  $\alpha=0,38$ . The secondary air heating temperature is  $t=380^\circ\text{C}$ . The cross-sectional area of the secondary air channels at the burner outlet is  $F=0,2\text{m}^2$ , which ensures the level of velocities of the secondary air at the burner outlet being  $W=40$  m/s.

Hot air consumption per mill is 12000 nm<sup>3</sup>/h. After the mill the exhaust air is fed into the furnace through 4 dump burners located from the back and the front of the boiler.

Based on the air balance with the excess air coefficient at the furnace outlet being  $\alpha_r = 1,27$ , false air inflows into the furnace and dust systems make up about 40%, which impairs the efficiency of the boiler.

In the burners' area, where the ignition takes place, the torch is essentially non-uniform. However, at a distance from the burners, the concentrations of dust, oxygen and combustion products are equalized, as well as the temperature over the cross section of the flame [16-26].

Almaty TPP-3 is equipped with six BKZ-160 boilers with a steam capacity of 160 t/h each.

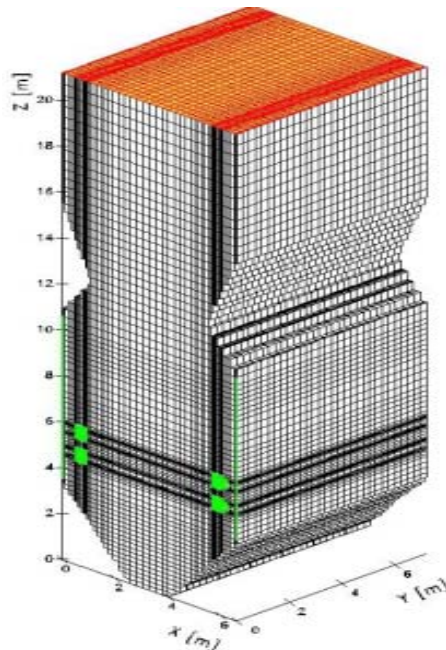


Figure 1 - General view of the boiler BKZ-160 combustion chamber and its breakdown into control volumes

### Results of the Computational Experiment

Below are the results of a 3-D simulation of solid fuel (coal) combustion processes in the combustion chamber of the constructed model.



The fields in the figures are shown as arrow-vectors, the length of which represents the full velocity value, their direction is related to the full velocity direction at the selected point of the combustion chamber. The flow aerodynamics in the furnace chamber, shown in Figures 3-5, built on the calculated velocity data, fully coincides with the description of the nature of the flow in tangential furnaces, available in the literature.

The volumetric pattern of vectors' disposition clearly shows the flow pattern: places of tangential fuel supply (coal) and oxidizer (air) at different velocities through burners located on the front and rear walls of the combustion chamber, formation of a conditional circle in the center of the combustion chamber and flow symmetry (Figure 2-5).

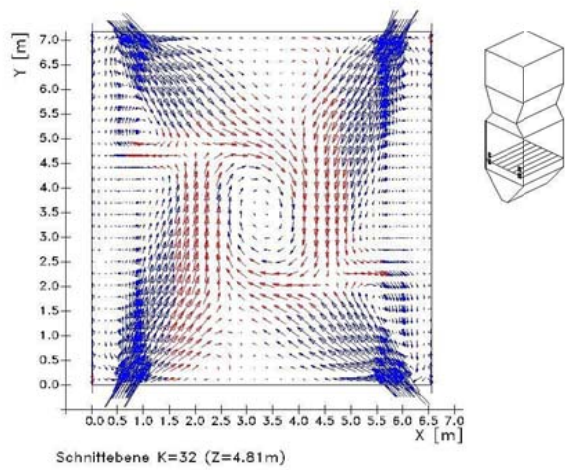


Figure 2 - Vector field of full velocity  $V$  in the cross section of combustion chamber in the lower burner tier ( $h = 4.81\text{m}$ )

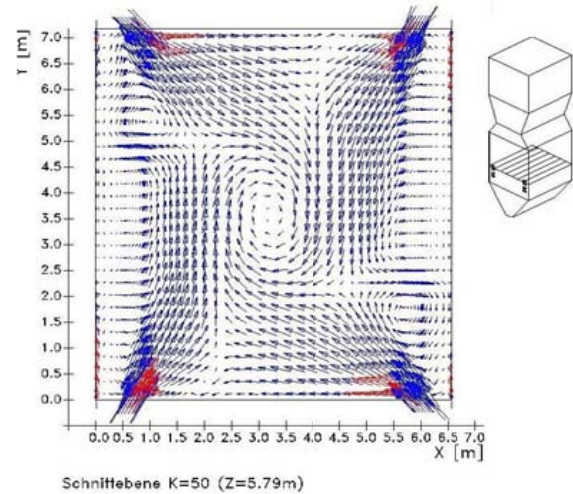


Figure 3 – Vector field of full velocity in the cross section of combustion chamber in the upper burner tier ( $h=5.79\text{m}$ )

The flows of pulverized coal, secondary and tertiary air entering the furnace space create a volume vortex-type flow in the center of combustion chamber, which undoubtedly improves the mixing process and increases the intensity of heat and mass transfer.

This, in turn, results in an increase in the residence time of coal particles in the combustion chamber and to a decrease in chemical and unburned carbon loss due to their more complete burnout.

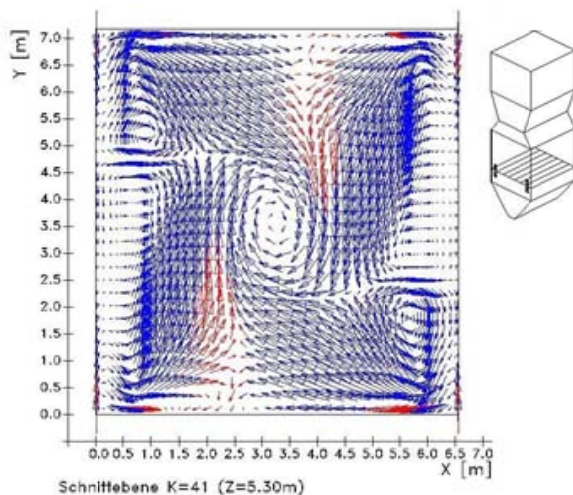


Figure 4 – Full velocity vector field  $V$  in the cross section of combustion chamber in the area between the burner tiers ( $h=5.30\text{m}$ )

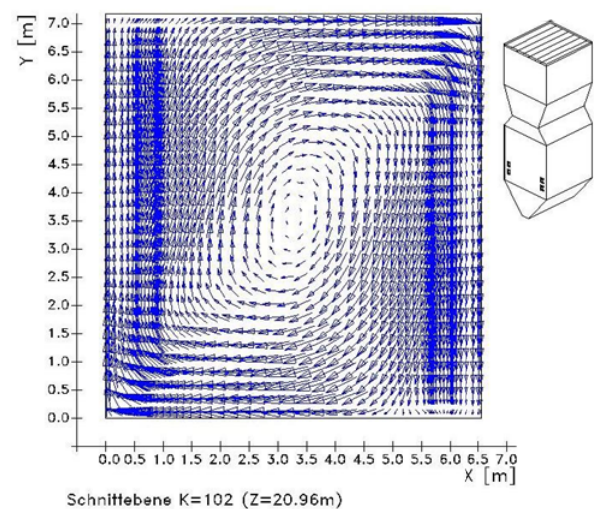


Figure 5 – Full velocity vector field  $V$  at combustion chamber outlet ( $h=20.96\text{m}$ )

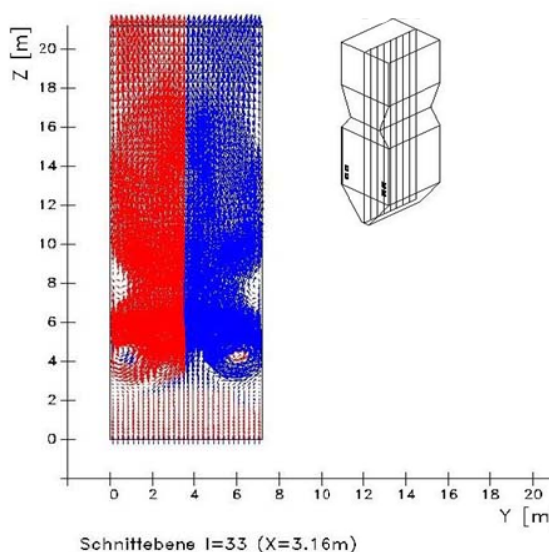


Figure 6 – Full velocity vector field in the longitudinal cross-section of combustion chamber at (x=3.16m)

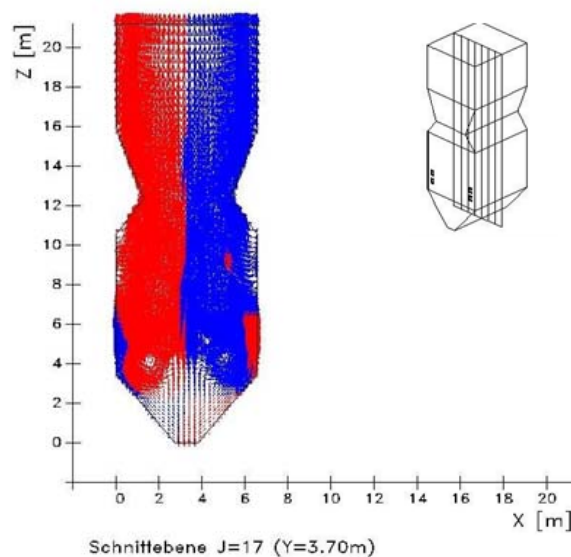


Figure 7 – Full velocity vector field V in the longitudinal cross-section of combustion chamber (y=3.70m)

Counter dust and gas flows from opposing burners hitting the walls of the combustion chamber create a reverse flow (Figure 4), and part of the flow goes down to the funnel, forming two symmetrical vortices below the burners (Figures 6 and 7). In the area below the burner belt ( $k < 32$ ,  $h < 4.61$  m), the formation of a reverse flow (Figure 6 and 7) can be observed, which is typical for all types of combustion chambers and is associated with air inflows from the bottom of the chamber, made in a funnel form.

The central vortex motion of a pulverized coal flow results in a uniform heating of combustion chamber walls, in a reduction in the slagging of thermal screens and heat losses. Even at the combustion chamber outlet ( $k=102$ ,  $h=20.96$  m), the velocity field levels off, no large velocity gradients are observed, the vortex nature of the flow goes down, and a uniform symmetrical flow is observed relative to the center of the chamber (Figure 5).

This character of the flow results in the fact that the most intense combustion occurs in the central zone of the combustion chamber, in the region of the burner belt. It is here that all the thermophysical and concentration characteristics of the process occurring in the combustion chamber reach their extreme values, as indicated by the analysis of the temperature and concentration fields presented below.

### Conclusion

Computational experiments were carried out on the 3-D modeling of the combustion process of a pulverized coal torch in the combustion chamber of the utility boiler BKZ-160, and main characteristics of the flow aerodynamics were determined: full-velocity field  $V = \sqrt{u^2 + v^2 + w^2}$ .

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**ҚАТТЫ ОТЫННЫҢ ЖАНУ ПРОЦЕСІНІҢ ТУРБУЛЕНТТІК СИПАТТАМАЛАРЫ  
ЖӘНЕ АҒЫС АЭРОДИНАМИКАСЫН ЗЕРТТЕУ БОЙЫНША ЕСЕПТЕУ ТӘЖІРИБЕЛЕРІ**

**Аннотация.** Тәжірибеге қолдану тұрғысынан ең қызықты және пайдалы болып, нақты геометрия саласында физика-химиялық процестер болған кезде жылумассаалмасуды модельдеу мәселелері бар. Мұндай салалар әр түрлі жылу энергетикалық қондырғылардың жану камералары, іштен жану қозғалтқыштары болып табылады

**Түйін сөздер:** Жану камерасы, қазандық, жанарғы, күлділігі жоғары көмір, қатты отын, сандық моделдеу, сандық тәжірибе.

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### **ВЫЧИСЛИТЕЛЬНЫЕ ЭКСПЕРИМЕНТЫ ПО ИССЛЕДОВАНИЮ АЭРОДИНАМИКИ ТЕЧЕНИЯ И ТУРБУЛЕНТНЫХ ХАРАКТЕРИСТИК ПРОЦЕССА ГОРЕНИЯ ТВЕРДОГО ТОПЛИВА**

**Аннотация.** Одними из интереснейших и полезных с точки зрения практического применения являются вопросы моделирования тепломассопереноса при наличии физико-химических процессов в областях реальной геометрии. Такими областями являются камеры сгорания различных теплоэнергетических установок, двигатели внутреннего сгорания.

**Ключевые слова:** Топочная камера, котел, горелки, твердое топливо, высокозольный уголь, численное моделирование, вычислительный эксперимент.

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## NEWS

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**INTERACTIVE VIRTUALIZATION IN THE ENVIRONMENT  
OF FLASH-CC, JAVA SCRIPT OF ALGORITHMS THE PHENOMENON  
OF THERMOPHYSICS AND MOLECULAR PHYSICS,  
AS ACHIEVEMENT OF HIGHLY EFFECTIVE TRAINING**

**Abstract.** The training program assumes not only studying of theoretical material, but also performance of laboratory works. They are necessary for development of skills of measurement of physical quantities, performance of physical experiences, ability to draw the correct conclusions from the observations. Modern computer technologies allow to add this traditional scheme of training. The training programs modeling physical processes and the phenomena which not always manage to be shown "in live" in the training conditions can provide to pupil's substantial assistance. Processes of the phenomenon of molecular physics and thermophysics are visualized and online virtualized by means of the computer program environments Adobe Flash-CC, Java script. The made, laboratory work on a research of processes of the phenomenon of molecular physics and thermophysics is very effective at development of this course, and the technology of creation of the virtual and interactive laboratory described in this article is very relevant for creation of the similar virtual and interactive laboratories (VIL) in other objects. This virtual interactive laboratory development is introduced in educational process of the Eurasian technological university and is successfully applied in training.

**Keywords:** Algorithm, virtual interaktivization, heat conductivity, virtual and interactive laboratory (VIL), Flash technology, thermophysics and molecular physics.

**Introduction.** Physics - science experimental. Therefore, the training program assumes not only studying of theoretical material, but also performance of laboratory works. They are necessary for development of skills of measurement of physical quantities, performance of physical experiences, ability to draw the correct conclusions from the observations.

Modern computer technologies allow to add this traditional scheme of training. The training programs modeling physical processes and the phenomena which not always manage to be shown "in live" in classroom conditions can provide to pupil's substantial assistance. The value of such programs, however, should not be exaggerated. It is necessary to remember - nothing can replace a real physical experiment in which a part a natural phenomenon, but not the copy of "virtual reality" simulated by the person talks to the researcher. The program «Heatphysical Laboratory» can be used in the course of training differently.

*First*, with its help it is possible to show demonstrations during an explanation of new material. Such demonstrations are very evident, improve perception and awaken interest in a subject. Advantage of such demonstrations, in comparison with usual videos, also that the teacher can quickly, changing parameters in the program, to answer questions of pupils: "And what will be if ...?" The answer is simple - let's make and we will look.

*Secondly*, the program can be used as a "laboratory" practical work. It is the best of all to give these classes in a computer class. In this case each pupil will be able to perform the laboratory work

independently. Preliminary preparation is usually necessary for such occupation - when the theory of the phenomenon is already known, then intelligent work with experimental installations can bring benefit in deeper understanding of a subject. A set of laboratory works and demonstrations offered by us far does not exhaust all opportunities of the program, and the teacher can always complement this list with own developments.

In the third, on the basis of the program it is possible to perform research works and projects. Within the opportunities, the program allows to conduct researches enough difficult heat physical processes and the phenomena which theory is beyond the program of training. Inquisitive pupils could open, for example, for themselves the law of a heat gives of Newton if to think over the scheme of an experiment and to take the corresponding measurements.

Below examples of laboratory works and demonstrations which can be used in the program are given. It must be kept in mind that the calculations which are carried out by our program for model of real gas have rather qualitative character, and do not apply for exact quantitative compliance. The list of laboratory works offered for performance on the computer on thermophysics and molecular physics.

Laboratory work No. 1 Studying of heat exchange of two bodies.

Laboratory work No. 2 Determination of specific heat of substance.

Laboratory work No. 3 – Melting ice.

Laboratory work No. 4 Boiling water.

Laboratory work No. 5-Isothermal rating of gas.

Laboratory work No. 6 Definition of a universal gas constant.

Laboratory work No. 7 Isoprocesses in gases.

Laboratory work No. 8 Studying of heat conductivity process.

### **Realization virtual- interaktivization stages of works in the program environments Adobe Flash-CC, Java script.**

The present requirement of time for transfer big volume information on the Internet, including animation images in movements and work with them demands the large volume of memory. And for reduction of capacity of use of memory use for virtual – interaktivization of Flash technologies is very effective. [4,5]. From the basic vector – graphical format of Flash technologies was created Shore wave Flash (SWF) – a branch. But, it is not the first vector format, it is the Web broadcast mechanism – pages to SWF as finding of the graphic representation, the coordinating link of the tool equipment and the graphic representation. Advantage of SWF-of the application it easily shipping on other Wednesday, i.e. this format is used in different is information – the program platform (in the Mac OS Macintosh operating system, in OS - Windows OS). One more feature of SWF – the constructed main images not only accept animation but also in addition, an opportunity to create interactive elements and audio of installation. Besides, mathematical formula communications of physical processes can be turned into interactive elements, management of their changes give the chance, to carry out on the computer, interactive virtual researches. For example, as shown in the drawing that the mathematical dependences found Snellius for reflection and light refraction and to form interactive virtual laboratory, very conveniently the formats SWF, CC of them – the program Flash environment. For transfer on distances of interactive multimedia additions are carried out on the known SWF format – in the Web application for the Internet.

Why, to emergence of this application of this format in Macromedia, for browsers of two main networks of the Plug in component, and to distribution to Internet Explorer and Netscape Communicator the worldwide computer network affected. One more reason popularity of SWF – a format this very easy and convenient application instruments for other platforms development of Macromedia. For example: - use the program device to creation of the multimedia presentations – Macromedia Director Shockwave Studio, and are used the program device to creation of graphic images – Macromedia Author wave, Macromedia Course Builder. Therefore, among Web – the publication the most recognizable and easily applied publication is Macromedia Flash Web – gives the chance to decorate each website with animation and to collect the full page. Action Script Tools - allows to collect Web addition effectively and its modern languages similarly probably on the scenario Java Script, Action Script and by means of the editor of Devigger is the solution of often applied elements. When there is a work of Flash – Mx you can construct the collected clip or import graphics, later in process of work will be able to process and by

means of an assembly ruler use effect of revival (Time line) [5]. Such clip or the movie can be interactive, i.e. certain images can be changed at discretion and to influence events in the clip. You export it in the Flash format, adding the page, and transfer to the page as the Web – the server. Each clip or the movie collected by tools in the Flash system can change depending on a type of the carried-out tasks and it can be seen via the browser of the Internet.

Practically, for interactive visualization and management, on the computer, experimental installation and devices, set the object to carry out by means of computer programs in the environment of Flash. For example, for performance following operations, the computer program is written in the environment of Flash – CC:

General view a window of the program

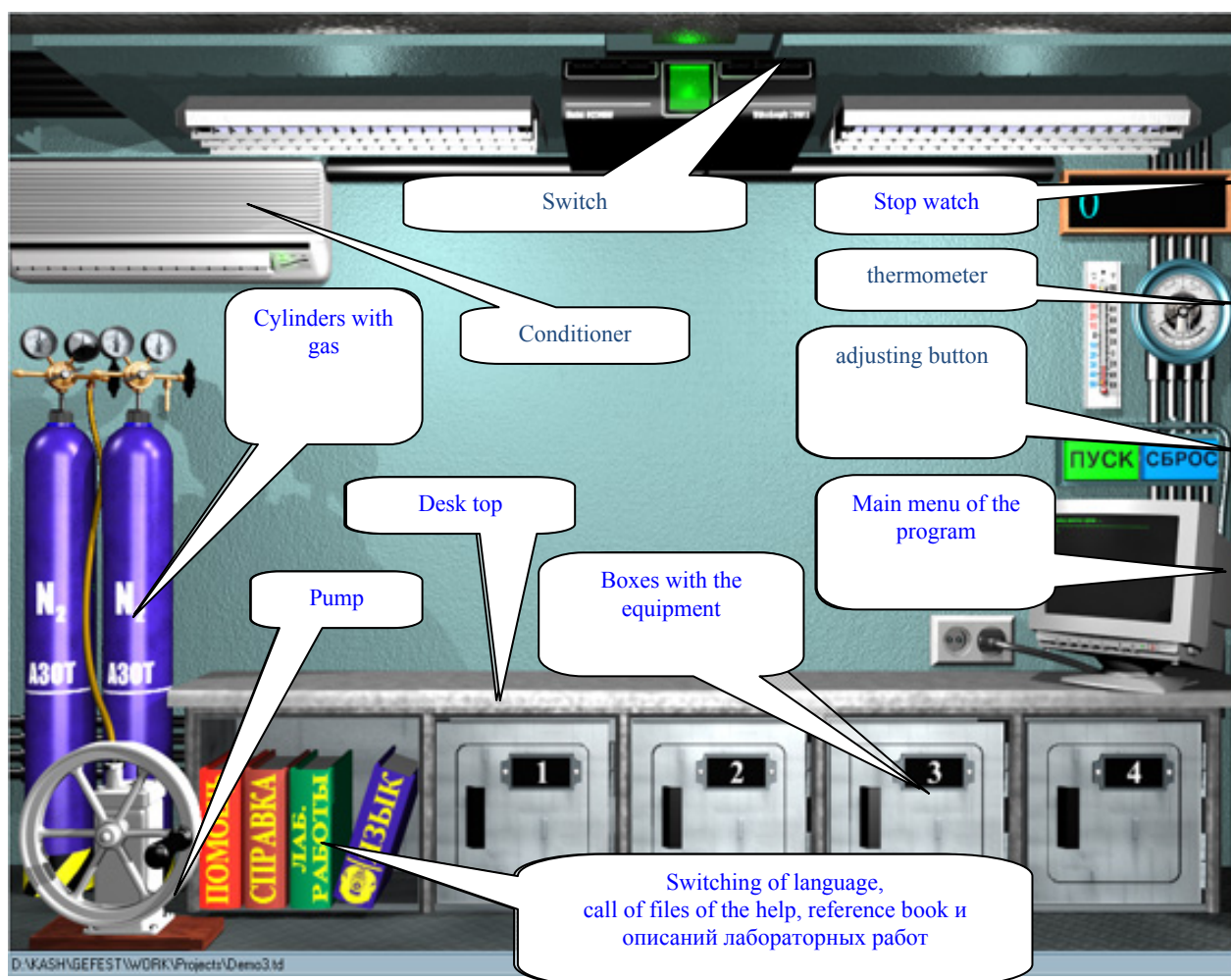


Figure 1 - General view a window of the program

On active elements the cursor of a mouse takes the hand form. Pressing of the left or right button of a mouse opens the corresponding window of parameters of an element or makes action.

#### **Work with the equipment**

The equipment is stored in desktop boxes.

In the first box there are vessels for gases. Their two views - a usual vessel and the heat-insulated vessel of the Dewar. As shown in figure 2, all vessels same, as in life:



Figure 2 - A vessel of the Dewar and cylinders for gases

In the second box there are a thermoelement and a chemical glass.

In the third box are located: support, core and candle.

Are in the fourth box: a container with substance, bar of ice (blue), bar of copper or gold (golden), bar of other metals (gray).

#### Equipment installation on a desktop

1. The index of a mouse is brought on the necessary box of a table, and pressing of the left button of a mouse the corresponding door opens.

2. Further the index of a mouse is brought on the necessary device, the left button is pressed and keeps, and the device is transferred by the movement of a mouse above a desktop surface. After an otpuskaniye of the button of a mouse, the device will be installed on a desktop. It is also possible to transfer the devices which are already located on a desktop.

3. In certain cases one of devices is installed on another, for example a glass, on a heating element. It is possible to collect the whole pile of containers which will exchange heat among themselves if stand the friend on the friend.

4. If to put a vessel for gases on a desktop near cylinders of storage of gases, then he will unite a hose to the pump. In such situation (and only in it) is possible the rating and pumping of gases in a vessel.

5. The thermoelement established near the socket will be connected to it, and in this case becomes active, i.e. will heat or cool a container or a glass with substance which on it are installed.

6. The candle is intended only for heating of the core fixed in a support. Cores with various heat conductivity are fixed in a support. Distributions of temperatures on length of cores are automatically shown on graphics received experimentally, as shown in figure 3.

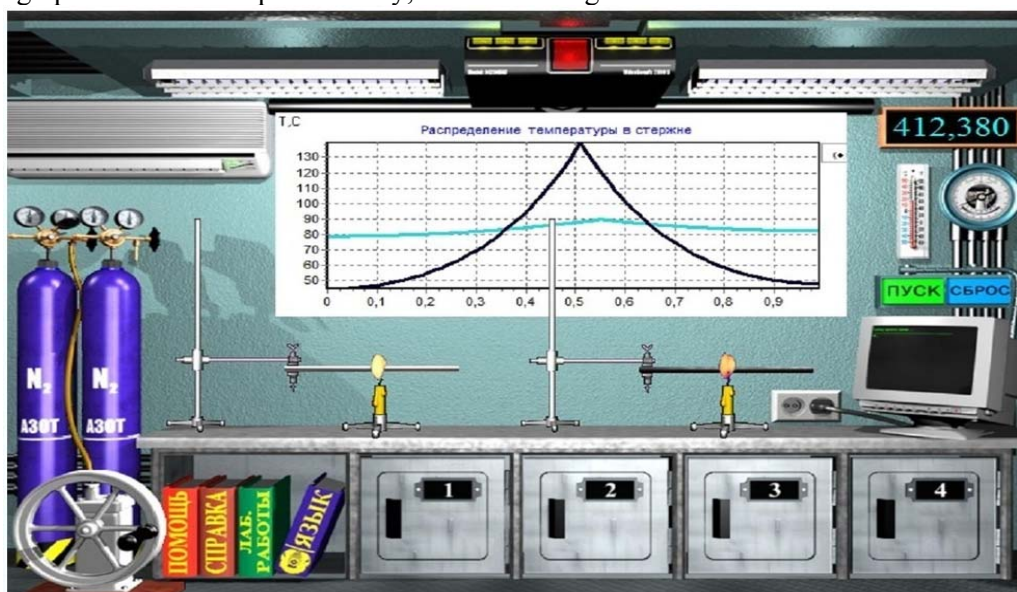


Figure 3 - Laboratory work on experimental distribution of temperatures in cores with various heat conductivity



### Choice of language and reference

Having left-clicked a mouse on images of books, you can:

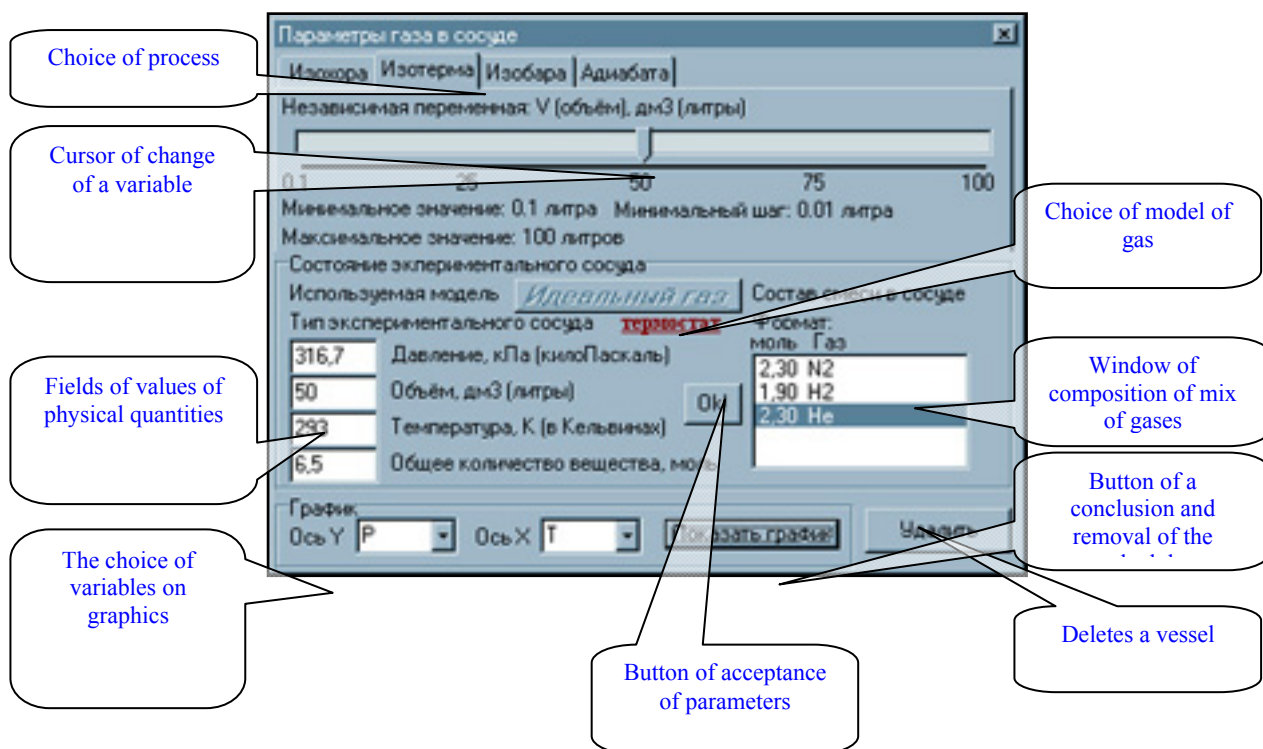
- To choose language (Russian or Kazakh);
- To open the file of the help in the corresponding language. In this file the description of working methods with the program, its opportunities and restrictions is given;
- To open the reference book on thermophysics with the short description of the theory of some phenomena which can be modelled in the program;
- To open the file containing descriptions of laboratory works and demonstrations.

It must be kept in mind that these files are written in the editor of Microsoft Word therefore they can be supplemented with the developments of laboratory works and the description of new demonstrations. However, in this case authors of the program do not bear maintenance responsibility of these additions by transfer them together with the program to the third parties.

### Panels of parameters of details

Each detail of installation has the panel of parameters. This panel opens if to right-click a mouse on the image of a detail. All numerical values of parameters are given in the International System of Units of SI. Let's consider the main panels of parameters shown in figure 4.

### Vessel with the ideal gas



## Vessel with the real gas

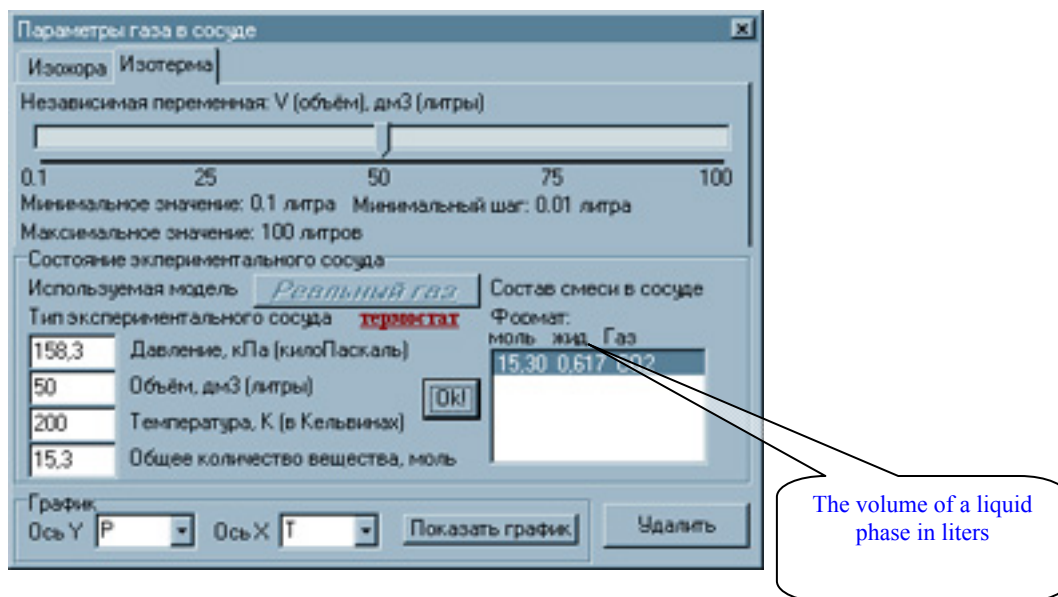


Figure 4 - Main panels of parameters of ideal and real gases.

### Conclusion

As a part of information technologies the new industry develops - it is a virtual interaktivization and visualization of the hardly understood subjects of physics, chemistry, biology and other objects [5]. And creation is virtual – interactive laboratories in the called objects meet the operated measuring devices very seldom. Therefore, the technology of creation of the virtually-interactive laboratory (VIL) for the section of physics given in this work. "The thermophysics and molecular physics" will be very relevant to creators similar to VIL – at higher step in other objects of knowledge. Such VIL – on the computer are very effective for development of a certain course of knowledge and develop independent research skills and awaken to creative search of methods of a research. Given VIL on thermophysics and molecular physics, due to visually and interactive intervention in change process an experiment condition, it is very useful to fast development of a subject of physics by students and to development of skills a research. Brought VIL – on thermophysics and molecular physics are introduced in educational process of the Eurasian Technological University and are successfully applied.

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### ОҚЫТУДАҒЫ ЖОҒАРЫ ТИІМДІЛІК ЖЕТІСТІКТЕРІ РЕТІНДЕ, ЖЫЛУ ФИЗИКАСЫ ЖӘНЕ МОЛЕКУЛАЛЫҚ ФИЗИКА ҚҰБЫЛЫСТАРЫНЫҢ АЛГОРИТМДЕРІН FLASH-CC, JAVA SCRIPT ОРТАЛАРЫНДА ИНТЕРАКТИВТІ ТҮРДЕ ВИРТУАЛДАУ

**Аннотация.** Ұсынылған оқыту-үйрету бағдарламасы тек теориялық материалдарды оқытумен шектелмейді, сонымен бірге мұнда, жылу физикасы және молекулалық физика бөлімінен зертханалық жұмыстарды орындауға болады. Олар физикалық шамаларды өлшеуге дағдылануға, физикалық тәжірибе жұмыстарын орындауға, өзінің зерттеу байқауларынан дұрыс қорытынды шығара білуге қажет. Қазіргі компьютерлік технологиялар осындай дәстүрлі оқыту схемасын толықтыруға мүмкіндік береді. Компьютерлік оқыту бағдарламалары, физикалық процесстерді және құбылыстарды модельдеп, «тірілей» көзге көрсетуге мүмкіндік болмаған кезде, компьютерлік көрсету арқылы, оқушыларға орасан зор көмегін

тигізеді. Молекулалық физика және жылу физикасының процесстері мен құбылыстары Adobe Flash-CC, Java script компьютерлік бағдарламалар ортасында интерактивті түрде, виртуалды зертхана жұмыстарына айналдырылды. Жасалған, жылу физикасы мен молекулалық физика құбылыстары мен процесстерін зерттейтін лабораториялық жұмыстар, физиканың осы аталған бөлігін меңгеруде өте тиімді, ал осы мақалада көрсетілгендей, виртуалды-интерактивті лабораторияларды жасап шығару технологиялары, басқа пәндерден болашақта жасалынатын виртуалды-интерактивті лабораторияларға көмек ретінде өте актуалды. Аталған виртуалды-интерактивті лабораториялық құрылым Евразия технологиялық университетінің оқу процессіне енгізіліп, оқыту барысында жетістікпен қолданыста.

**Түйін сөздер:** Алгоритм, виртуалды интерактивтендіру, жылу өткізгіштік, виртуалды-интерактивті лаборатория(ВИЛ), Flash технологиясы, жылу физикасы және молекулалық физика.

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### **ИНТЕРАКТИВНАЯ ВИРТУАЛИЗАЦИЯ В СРЕДЕ FLASH-CC, JAVA SCRIPT АЛГОРИТМОВ ЯВЛЕНИИ ТЕПЛОФИЗИКИ И МОЛЕКУЛЯРНОЙ ФИЗИКИ, КАК ДОСТИЖЕНИЕ ВЫСОКОЭФФЕКТИВНОГО ОБУЧЕНИЯ**

**Аннотация.** Обучающая программа предполагает не только изучение теоретического материала, но и выполнение лабораторных работ по разделу теплофизика и молекулярная физика. Они необходимы для выработки навыков измерения физических величин, выполнения физических опытов, умения делать правильные выводы из своих наблюдений. Современные компьютерные технологии позволяют дополнить эту традиционную схему обучения. Обучающие программы, моделирующие физические процессы и явления, которые не всегда удается показать "в живую" в обучающих условиях, могут оказать учащимся существенную помощь. Процессы явления молекулярной физики и теплофизики визуализированы и интерактивно виртуализированы с помощью компьютерных программных сред Adobe Flash-CC, Java script. Сделанная, лабораторная работа по исследованию процессов явления молекулярной физики и теплофизики очень эффективна при освоении данного курса, а технология создания виртуально-интерактивной лаборатории описанной в данной статье, очень актуальна для создания аналогичных виртуально-интерактивных лабораторий(ВИЛ) по другим предметам. Данная виртуально-интерактивная лабораторная разработка внедрена в учебный процесс Евразийского технологического университета и успешно применяется в обучении.

**Ключевые слова:** Алгоритм, виртуальная интерактивизация, теплопроводность, виртуально-интерактивная лаборатория (ВИЛ), технология Flash, теплофизика и молекулярная физика.

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## **CALCULATION AND VISUALIZATION OF QUANTUM-MECHANICAL TUNNEL EFFECT**

**Abstract.** The article presents calculations and visualization of wave packet tunneling through the potential barrier of various widths done by using the MATLAB software. The graph of distribution of the probability density along barrier width is obtained in two and three dimensions. The article contains materials about quantum-mechanical tunnel effect such as the reflection of the part of the wave packet from the potential barrier and passing through the barrier of the other part. Calculations and visualization of the tunneling process are performed for various widths of the barrier. Figures given in the article show the probability of finding the particle at a certain time, i.e. the square of the absolute value of the state vector  $|\psi\rangle^2$ . The part of the total number of particles tunnels through barrier, the other part reflects from it. The probability of tunneling increases with decrease of the barrier width. The detailed analysis of the graphs reveals that the wave function exponentially decreases inside the potential barrier therefore the observation of this effect requires a rather small width of the barrier. Students are offered to develop programs for various widths of the potential barrier. Results of experiments are used during the study of the quantum mechanics.

**Key words:** wave packet, reflected wave, transmitted wave, potential barrier, barrier width, probability.

Nowadays all educational institutions of Kazakhstan are provided with computer hardware and software, interactive boards and internet. Almost all teachers have completed language and computer courses for professional development. Hence the educational institutions have all conditions for using computer training programs and models for performing computer laboratory works. In recent years the new computer system for carrying out mathematical calculations MATLAB is being widely used in many universities and engineering institutions throughout the world [1-7]. Unfortunately, the numerical calculations carried out by students are often done by means of the calculator. Modern computers are frequently used only for presentation of the work. Actually students should be able not only to solve these or other engineering problems, but also do them by using modern methods, that is, using personal computers.

Students of the physics specialties 5B060400 and 5B011000 successfully master the discipline “Computer modeling of physical phenomena” which is the logical continuation of the disciplines “Information technologies in teaching physics” and “Use of electronic textbooks in teaching physics”. The goal of this discipline is to study and learn the MATLAB program language, acquaintance with its huge opportunities for modeling and visualization of physical processes.

In our early works [8-26] we used the MATLAB system for modeling and visualization of physical processes related with mechanics, molecular physics, electromagnetism and quantum physics. This software has enabled us to solve ordinary differential equations (ODE), visualize equipotential lines of

charged conductors system, describe the motion of charged particles in electric, magnetic and gravitational fields and etc.

The present article is devoted to calculation and visualization of the quantum-mechanical tunnel effect by using the package of MATLAB applied programs.

**Formulation of the problem.** Let us consider a well-known example – tunneling of the particle through the potential barrier. Let the particle with mass  $m$  to move with speed  $v$  towards the potential barrier of height  $x$  (fig.1).

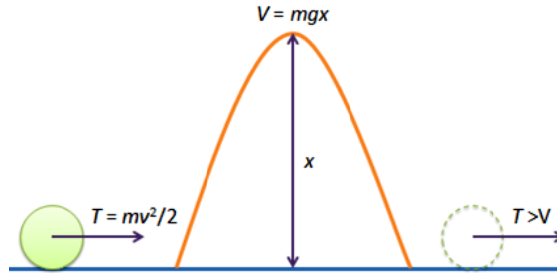


Fig.1 - Tunneling of the particle through the potential barrier

According to the classical mechanics it can surmount this barrier if only its kinetic energy  $mv^2/2$  is greater than the potential energy  $mgx$  necessary for surmounting the barrier. But quantum mechanics states that there is a nonzero probability that the particle will surmount the "potential barrier" even its energy is insufficient from the point of view of classical mechanics. It is said that the particle tunnels through the potential barrier.

For calculation of this effect it is necessary to introduce into the Hamiltonian the potential  $V$  as a barrier and find how the wave packet will behave itself.

The apparent expression of the time evolution operator is:

$$\tilde{U}(t) = \exp\left(-\frac{i}{\hbar} \hat{H}(t)\right) \tag{1}$$

here  $\hat{H}$  is the Hamiltonian operator.

In dynamic problems the state vector is usually chosen in the form of the wave packet localized in small area of space. Let's take the state vector at the initial time to be the following function:

$$\psi(x) = (\pi d^2)^{-1/4} \exp\left(ipx - \frac{x^2}{2d^2}\right) \tag{2}$$

here  $p$  is the momentum of the particle,  $d$  is the "width" of the wave packet. The mass of the particle and Planck's constant are taken to be equal to one. The real part of the function is shown in the fig.2.

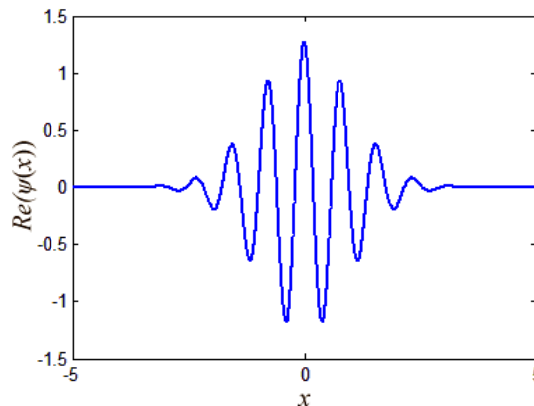


Fig.2 - The wave function at the initial time

This function doesn't depend upon the time. To find the function at the other moment of time it is necessary to act on it with the evolution operator (1).

The dynamics of this action can be observed by using the following Matlab program:

```

a) barrier's width d = 1;
clear % cleaning of the screen
%% Initial data
n = 200; % the number of points (depends upon the power of PC)
% wave packet
p = -4; % packet momentum
d = 1; % barrier's width
x = linspace(-10,10,n);
offset = -7;
wave = ((exp(1i*p*(x-offset)))*(exp(-((x-offset).^2)/(2*d^2))))/((pi*d^2)^1/4);
wave = wave';
%%
x = linspace(-6, 6, n)';
h = (x(2)-x(1));
Dsquared = (diag(ones(1,n-1),1) - 2*diag(ones(1,n)) + diag(ones(1,n-1), -1))/h^2; % the second
derivative
V = ((x > 0).*(x<0.5))*60; % potential barrier
%%
H = -Dsquared + diag(V); % Hamiltonian
t = 0.01; % the initial moment of time
figure1 = figure;
axes1 = axes('Parent',figure1);
for i = 1:55 % the cycle for results output
%%
U = expm(-1i*t*H); % time evolution operator
%%
wave1 = U*wave; % application of the evolution operator to the wave packet
t = t+0.01;
% plot(x, real(wave1),'Color',[1 0 0]) % the real part of the wave function
plot(x, abs(wave1).^2,'Color',[1 0 0]) % visualization of the probability
hold % resolution of the drawing
plot(x,V) % visualization
ylim(axes1,[-0.5 2]);
hold % resolution of the drawing
pause(0.2); % in dependence upon the power of PC
end

```

The result is shown in the fig.3 (the tunneling picture).

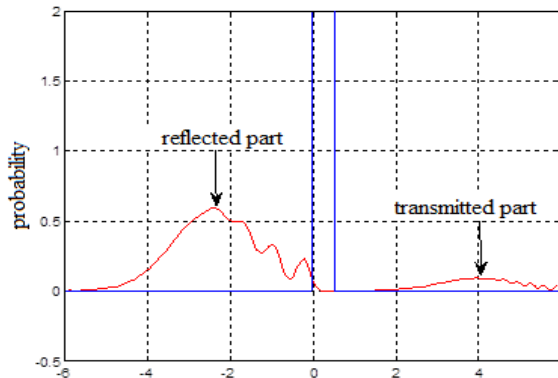


Fig.3 - Tunneling through the potential barrier with the width d=1

```

b) barrier width  $d = 1$ ;
clear % cleaning of the screen
%% Initial data
n = 200; % the number of points (depends upon the power of PC)
% wave packet
p = -4; % packet momentum
>> d=1.25; % barrier's width
>> x = linspace (-10, 10, n);
offset = -7;
wave = ((exp(1i*p*(x-offset))).*(exp(-(x-offset).^2/(2*d^2))))./((pi*d^2)^1/4);
wave = wave';
%%
x = linspace (-6, 6, n)';
h = (x (2)-x(1));
Dsquared = (diag(ones(1,n-1),1) - 2*diag(ones(1,n)) + diag(ones(1,n-1), -1))/h^2; % the second
derivative
V = ((x > 0).*(x<0.5))*60; % potential barrier
%%
H = -Dsquared + diag (V); % Hamiltonian
t = 0.01; % the initial moment of time
figure1 = figure;
axes1 = axes ('Parent',figure1);
for i = 1:55 % the cycle for results output
U = expm(-1i*t*H); % time evolution operator
wave1 = U*wave; % application of the evolution operator to the wave packet
t = t+0.01;
% plot(x, real (wave1),'Color',[1 0 0]) % the real part of the wave function
plot(x, abs (wave1).^2,'Color',[1 0 0]) % visualization of the probability
hold % resolution of the drawing
plot(x,V) % visualization
ylim(axes1,[-0.5 2]);
hold % resolution of the drawing
pause(0.2); % in dependence upon the power of PC
end

```

The result is shown in the fig.4

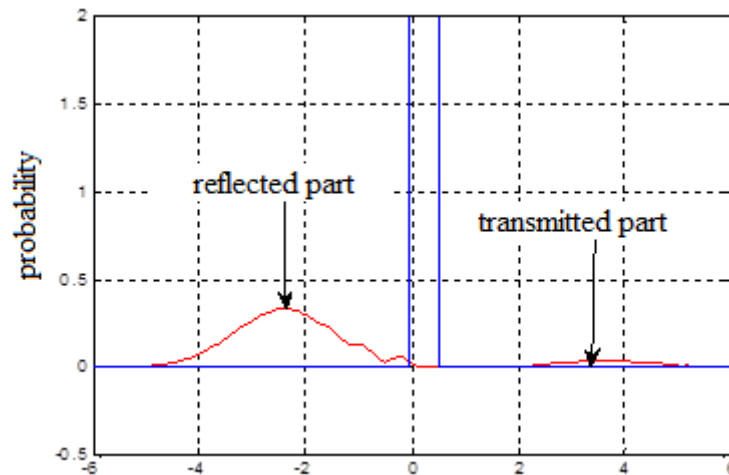


Fig.4 - Tunneling through the potential barrier with the width  $d = 1.25$

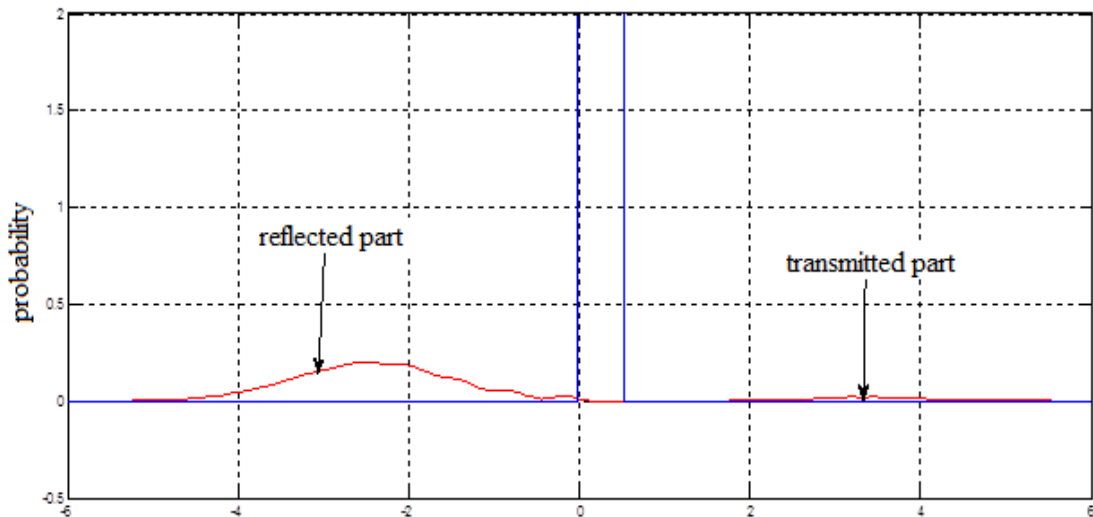
```
c) barrier's width d=1.5;
% cleaning of the screen
%% Initial data
>>n = 200; % the number of points (depends upon the power of PC)
% wave packet
>>p = -4; % packet momentum
>> d=1.5; % barrier's width
>> x = linspace (-10, 10, n);
>>offset = -7;
>>wave = ((exp (1i*p*(x-offset))).*(exp(-((x-ffset).^2)/(2*d^2))))./((pi*d^2)^1/4);
>>wave = wave';
%%
>>x = linspace(-6, 6, n)';
>>h = (x(2)-x(1));
>>Dsquared = (diag(ones(1,n-1),1) - 2*diag(ones(1,n)) + diag(ones(1,n-1), -1))/h^2; % the second
derivative
V = ((x > 0).*(x<0.5))*60; % potential barrier
%%
>>H = -Dsquared + diag(V); % Hamiltonian
%%
>>t = 0.01; % the initial moment of time
>>figure1 = figure;
>>axes1 = axes('Parent',figure1);
>>for i = 1:55 % the cycle for results output
%%
>>U = expm(-1i*t*H); % time evolution operator
%%
>>wave1 = U*wave; % application of the evolution operator to the wave packet
>>t = t+0.01;
% plot(x, real(wave1),'Color',[1 0 0]) % the real part of the wave function
>>plot(x, abs(wave1).^2,'Color',[1 0 0]) % visualization of the probability
>>hold % resolution of the drawing
>>plot(x,V) % visualization
>>ylim(axes1,[-0.5 2]);
>>hold % resolution of the drawing
>>pause(0.2); % in dependence upon the power of PC
>>end
```

The result is shown in the fig.5

The program for visualization of the particle tunneling through potential barrier in three dimensions when  $d=1$  is given as the following:

```
clear % cleaning of the screen
%% Initial data
>>n = 70; % the number of points (depends upon the power of PC)
% wave packet
>>px = 7; % x –component of the momentum
>>d = 1;
>>x = linspace(-10,10,n);
>>offset = 5;
waveX=((exp(1i*px*(x-offset))).*(exp(-((x-offset).^2)/(2*d^2))))./((pi*d^2)^1/4);
>>waveX = waveX'; % x –component of the wave packet
```



Fig.5 - Tunneling through the potential barrier with the width  $d = 1.5$ 

```

>>py = 0; % y –component of the momentum
>>d = 1;
>>y = linspace(-10,10,n);
>>offset = 0;
waveY=((exp(1i*py*(y-offset))).*(exp(-((y-offset).^2)/(2*d^2))))./(pi*d^2)^1/4);
>>waveY = waveY'; % y –component of the wave packet
>>wave = waveY*waveX'; % two dimensional wave packet
%% components of Hamilton operator
>>h = x(2)-x(1);
Dsquared = (diag(ones(1,n-1),1) - 2*diag(ones(1,n)) + diag(ones(1,n-1), -1))/h^2; %second derivative
>>V = ((x > 0).*(x<0.3))*60; % x-component of the potential barrier
>>Hx = -Dsquared + diag(V); % x-component of Hamiltonian
>>Hy = -Dsquared; % y-component of Hamiltonian
%%
>>t = 0; %the initial moment of time
>>figure1 = figure;
>>axes1 = axes('Parent',figure1);
>>for tt=1:43 % The main cycle of the results output
>>Ux = expm(-1i*tt*Hx); % x-component of the time evolution operator
>>Uy = expm(-1i*tt*Hy); % y-component of the time evolution operator
%% application of the x-component of the time evolution operator to the wave packet
for i=1:n
>>wave(i,:) = (Ux*wave(i,:))';
>>end
%% application of the y-component of the time evolution operator to the wave packet
for i=1:n
>>wave(:,i) = Uy*wave(:,i);
>>end
%% drawing the graphs
>> [X,Y] = meshgrid(x,y);
>>mesh(X,Y,abs(wave))
>>zlim(axes1,[0 1.5]);
>>t=t+0.002;
>>pause(0.5); % in dependence upon the power of PC
>>end

```

The result is shown in the fig.6

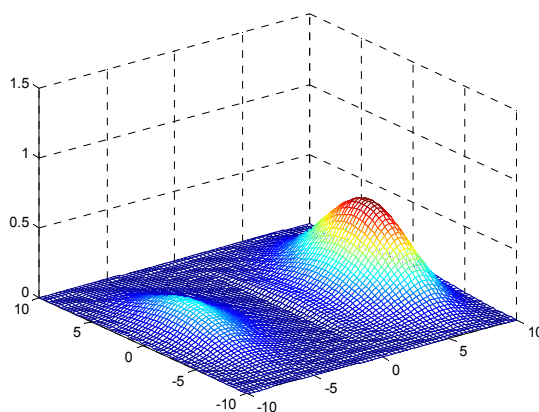


Fig.6 - Tunneling through the potential barrier with the width  $d=1$  in three dimensions

### Conclusion

The article presents the program for calculation and visualization of tunneling of a wave packet through a potential barrier with various widths. The graphs of the probability density versus barrier width are drawn in two- and three-dimensions. Brief data from the theory of quantum mechanical tunnel effect states that the one part of the wave packet is reflected from a potential barrier and the other part passes through it. Calculations and visualization of the particle tunneling are given for various width of the barrier. The obtained figures show the squares of absolute values of the state vector, i.e. the probability to find the particle at a given moment of time. The part of the total number of particles tunnels through the barrier, and the other part of them is reflected. The probability of passing through the barrier increases with the decrease of the barrier's width. The detailed analysis of the graphs reveals that the wave function exponentially decreases inside the potential barrier therefore the observation of this effect requires a rather small width of the barrier. Students are offered to develop programs for various widths of the potential barrier. Results of experiments are used during the study of the quantum mechanics.

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### **МАТЛАВ ЖҮЙЕСІНДЕ «КВАНТТЫҚ МЕХНИКАЛЫҚ ТУННЕЛДІК ЭФФЕКТИНІ ЕСЕПТЕУ МЕН БЕЙНЕЛЕУ»**

**Аннотация.** Толқындық пакеттің потенциалдық тосқауылдан туннелденіп өту процесін есептеу мен бейнелеу ұсынылады және ықтималдық тығыздығының тосқауыл ені бойында өзгерісінің екі өлшемдік және үшөлшемдік кеңістікте графиктерін салу ұсынылады. Кванттық-механикалық туннелдік эффектінің қысқаша теориясы берілген: тосқауылға соққан толқындық пакеттің бір бөлігі тосқауылдан шағылады, ал қалған бөлігі тосқауылдан өтеді. Тосқауылдың енінің әртүрлі шамасына байланысты туннелдену есептелген және бейнеленген. Суреттерде күй векторының абсолют шамасының квадраты салынған, яғни берілген уақыттағы бөлшектің табылуының ықтималдығы салынған. Тосқауылға соққан бөлшектердің жалпы санының бір бөлігі туннелденеді де, ал қалған бөлігі тосқауылдан шағылады. Тосқауылдан өту ықтималдығы оның ені кішіреуімен артады. Процесті талдаудан келесіні байқайтынымыз: тосқауыл ішінде толқындық функция экспоненциалды өшеді, сондықтан оны бақылау үшін тосқауылдың енін мүмкіндігінше кішірейту керек. Студенттерге өз бетінше бірнеше тәжірибелер жасау ұсынылады. Тәжірибелер нәтижелері кванттық механиканы меңгеруде қолданылады.

**Түйін сөздер.** Толқындық пакет, шағылған толқын, өткен толқын, потенциалдық тосқауыл, тосқауыл ені, ықтималдық.

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### «РАСЧЕТ И ВИЗУАЛИЗАЦИЯ КВАНТОВОМЕХАНИЧЕСКОГО ТУННЕЛЬНОГО ЭФФЕКТА» В СИСТЕМЕ MATLAB

**Аннотация.** Предлагается программа расчета и визуализации процесса туннелирования волнового пакета сквозь потенциальный барьер с построением графика зависимости плотности вероятности вдоль ширины барьера в двумерном и трехмерном представлениях. Приводятся краткие сведения из теории квантовомеханического туннельного эффекта: часть волнового пакета отражается от потенциального барьера, но другая часть проходит его. Приведены расчеты и визуализация туннелирования при различной ширине барьера. На рисунках показаны квадрат абсолютного значения вектора состояния  $|\psi|^2$ , то есть вероятность обнаружить частицу в данный момент времени. Из общего количества частиц какая-то их часть туннелирует сквозь барьер, а какая-то отражается. Вероятность прохождения увеличивается с уменьшением толщины барьера. Детальный анализ показывает, что волновая функция экспоненциально затухает внутри потенциального барьера, поэтому для наблюдения данного эффекта ширина барьера должна быть достаточно мала. Студентам предлагается самостоятельно поэкспериментировать. Результаты экспериментов используются при изучении и освоении квантовой механики.

**Ключевые слова.** Волновой пакет, отраженная волна, прошедшая волна, потенциальный барьер, ширина барьера, вероятность.

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## CALCULATION AND VISUALIZATION OF SMALL OSCILLATIONS OF A DOUBLE PLANE PENDULUM

**Abstract.** The article considers the calculation and visualization of small oscillations of a double plane simple pendulum. It contains the brief derivation of the motion equation and its solutions; the mathematical model of the motion in the form of the system of nonlinear differential equations. Experiments with the double pendulum are performed at various ratios of bodies' masses and initial angles, in particular for three values of  $\mu = m_1 / m_2$ :  $\mu_1 = 0.1$ ,  $\mu_2 = 0.2$ ,  $\mu_3 = 0.3$  when the lengths of the pendulum are  $l = l_1 = l_2 = 0.25$  m and  $g = 9.8$  m/s<sup>2</sup>. The angles of pendulum deviation are given in radians. The graphs of small oscillations of the pendulum show that beats occur in the system during which energy cyclically passes from one pendulum to another. When one pendulum almost stops, the other pendulum is at its maximum amplitude. After a while pendulums "exchange their states" and so on. Oscillations with a bigger frequency  $\omega_1$  are modulated by lower frequency oscillations with a frequency  $\omega_2$ .

**Key words.** Double pendulum, small oscillations, beats, energy exchange, natural frequencies, natural mode.

**Introduction.** Nowadays all educational institutions of Kazakhstan are provided with computer hardware and software, interactive boards and internet. Almost all teachers have completed language and computer courses for professional development. Hence the educational institutions have all conditions for using computer training programs and models for performing computer laboratory works. In recent years the new computer system for carrying out mathematical calculations MATLAB is being widely used in many universities and engineering institutions throughout the world [1-7]. Unfortunately, the numerical calculations carried out by students are often done by means of the calculator. Modern computers are frequently used only for presentation of the work. Actually students should be able not only to solve these or other engineering problems, but also do them by using modern methods, that is, using personal computers.

Students of the physics specialties 5B060400 and 5B011000 successfully master the discipline "Computer modeling of physical phenomena" which is the logical continuation of the disciplines "Information technologies in teaching physics" and "Use of electronic textbooks in teaching physics". The goal of this discipline is to study and learn the MATLAB program language, acquaintance with its huge opportunities for modeling and visualization of physical processes.

In our early works [8-26] we used the MATLAB system for modeling and visualization of physical processes related with mechanics, molecular physics, electromagnetism and quantum physics. This software has enabled us to solve ordinary differential equations (ODE), visualize equipotential lines of charged conductors system, describe the motion of charged particles in electric, magnetic and gravitational fields and etc.

The present article is devoted to simulation of the double plane pendulum by using the package of MATLAB applied programs.

Double pendulum is, undoubtedly, a real miracle of the nature. The jump of complexity which is observed upon transition from a simple pendulum to a double one is amazing. Oscillations of the simple pendulum are of periodic nature. At small deviations from equilibrium such oscillations are harmonic and obey the sine or cosine law. In case of nonlinear oscillations the period depends on amplitude, but the periodicity of motion is conserved. In other words, the approximation of small oscillations quite perfectly reflects the essential properties of the simple pendulum.

The double pendulum "behaves" itself absolutely differently. At the mode of small oscillations the double pendulum moves in a beat pattern, i.e. with variation of resultant amplitude with time. At the increase of energy the nature of oscillations of the pendulum changes dramatically – oscillations become chaotic. In spite of the fact that the double pendulum can be described by the system of several ordinary differential equations, which is quite deterministic model, the emergence of chaos seems very unusual. This situation reminds the Lorentz system where the deterministic model of three equations also reveals chaotic behavior. Let's perform an experiment with the appendix given below and observe the motion of the double pendulum at various ratios of bodies' masses and initial angles.

At first we will develop the mathematical model of the double pendulum in the form of the system of nonlinear differential equations. Let us begin with the derivation of Lagrange equations.

**Lagrange equations.** In Lagrangian mechanics the system is described by using the generalized coordinates and the generalized velocities. In the considered problem the angles of deviation of the pendulums  $\alpha_1, \alpha_2$  and angular velocities  $\dot{\alpha}_1, \dot{\alpha}_2$  are taken as such variables. Using the specified variables, let us to derive the Lagrangian of the double pendulum and write down differential equations of Lagrange.

The simplified model of the double pendulum is shown in figure 1. Let's consider rods to be weightless. Their lengths are equal to  $l_1$  and  $l_2$ . The masses of small bodies (they are presented by balls of finite radius) are  $m_1$  and  $m_2$ . It is assumed that there is no friction in suspension points.

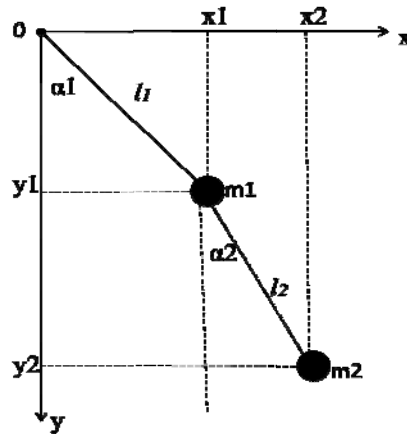


Fig.1 - The simplified model of the double plane pendulum

The masses' coordinates:

$$\begin{aligned} x_1 &= l_1 \sin \alpha_1, & x_2 &= l_1 \sin \alpha_1 + l_2 \sin \alpha_2, \\ y_1 &= -l_1 \cos \alpha_1, & y_2 &= -l_1 \cos \alpha_1 - l_2 \cos \alpha_2 \end{aligned} \quad (1)$$

The Lagrange function which is equal to the difference between kinetic and potential energies of the pendulum (respectively T and V) is expressed by the formula:

$$\begin{aligned}
 L = T - V &= \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - V_1 - V_2 = \\
 &= \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} - m_1 g y_1 - m_2 g y_2
 \end{aligned} \tag{2}$$

By taking into account the formula (1) we receive the following expression of the Lagrange function:

$$\begin{aligned}
 L &= \left( \frac{m_1}{2} + \frac{m_2}{2} \right) l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + \\
 &+ m_1 g l_1 \cos \alpha_1 + m_2 g l_1 \cos \alpha_1 + m_2 g l_2 \cos \alpha_2
 \end{aligned} \tag{3}$$

Since we consider small oscillations the angles  $\alpha_1, \alpha_2$  are small so the  $\cos(\alpha_1 - \alpha_2)$  can be expanded into Maclaurin series:

$$\cos(\alpha_1 - \alpha_2) = 1 - \frac{(\alpha_1 - \alpha_2)^2}{2!} + \frac{(\alpha_1 - \alpha_2)^4}{4!} - \dots$$

We approximate the function  $\cos(\alpha_1 - \alpha_2)$  by taking the first term of a Maclaurin series since the subsequent terms are negligibly small. With this approximation the equation (3) becomes

$$\begin{aligned}
 L &= \left( \frac{m_1}{2} + \frac{m_2}{2} \right) l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 + \\
 &+ m_1 g l_1 \cos \alpha_1 + m_2 g l_1 \cos \alpha_1 + m_2 g l_2 \cos \alpha_2
 \end{aligned} \tag{4}$$

Now we can write down the Lagrange equation:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_i} \right) - \frac{\partial L}{\partial \alpha_i} = 0; \quad i = 1, 2 \tag{5}$$

Do the derivatives:

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1 + m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2;$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1;$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_1} \right) = (m_1 + m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2;$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_2} \right) = m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1;$$

$$\frac{\partial L}{\partial \alpha_1} = -m_1 g l_1 \sin \alpha_1 - m_2 g l_1 \sin \alpha_1 = |\sin \alpha \approx \alpha| = -m_1 g l_1 \alpha_1 - m_2 g l_1 \alpha_1;$$

$$\frac{\partial L}{\partial \alpha_2} = -m_2 g l_2 \sin \alpha_2 = |\sin \alpha \approx \alpha| = -m_2 g l_2 \alpha_2;$$

Put it all together to get two Lagrange's differential equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_1} \right) - \frac{\partial L}{\partial \alpha_1} = (m_1 + m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 + (m_1 + m_2) g l_1 \alpha_1 = 0$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\alpha}_2} \right) - \frac{\partial L}{\partial \alpha_2} = m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 + m_2 g l_2 \alpha_2 = 0$$

This system of differential equations can be written in a compact matrix form. Firstly, we introduce the following matrixes:

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix}, \quad M = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & l_2^2 m_2 \end{pmatrix},$$

$$K = \begin{pmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then the system of differential equations is expressed as

$$M \ddot{\alpha} + K \alpha = 0$$

When there is the oscillation of one body such equation describes free undamped oscillation with a particular frequency. In case of the double pendulum oscillation the solution of this equation (as you will see below) will contain two characteristic frequencies which are called normal or natural modes. Natural modes represent the real part of a complex vector function

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \text{Re} \begin{bmatrix} (H_1) e^{i\omega t} \\ (H_2) e^{i\omega t} \end{bmatrix}$$

here  $H_1, H_2$  are the eigenvectors,  $\omega$  is the real value of the frequency. The values of natural frequencies  $\omega_{1,2}$  are determined by solving the characteristic equations

$$\det(K - \omega^2 M) = 0$$

$$\begin{bmatrix} (m_1 + m_2) g l_1 - \omega^2 (m_1 + m_2) l_1^2 & -\omega^2 m_2 l_1 l_2 \\ -\omega^2 m_2 l_1 l_2 & m_2 g l_2 - \omega^2 m_2 l_2^2 \end{bmatrix}$$

$$(m_1 + m_2) g^2 - \omega^2 (m_1 + m_2) (l_1 + l_2) g + \omega^4 m_1 l_1 l_2 = 0$$

We obtained the biquadratic equation for frequencies  $\omega$ . Let us calculate the discriminant:

$$\begin{aligned} D &= (m_1 + m_2)^2 g^2 (l_1 + l_2)^2 - 4 m_1 (m_1 + m_2) g^2 l_1 l_2 = \\ &= g^2 (m_1 + m_2) \left[ (m_1 + m_2) (l_1 + l_2)^2 - 4 m_1 l_1 l_2 \right]; \end{aligned}$$

Thus the square of the natural frequencies  $\omega_{1,2}$  is equal to

$$\omega_{1,2}^2 = \frac{g}{2 m_1 l_1 l_2} \left\{ (m_1 + m_2) (l_1 + l_2) \pm \sqrt{(m_1 + m_2) \left[ (m_1 + m_2) (l_1 + l_2)^2 - 4 m_1 l_1 l_2 \right]} \right\}$$



This expression is quite lengthy. Therefore we assume that the lengths of both pendulums are the same:  $l_1 = l_2 = l$ . Then the natural frequencies will be determined by more compact formula

$$\omega_{1,2}^2 = \frac{g}{l} \left[ (1 + \mu) \pm \sqrt{(1 + \mu)\mu} \right], \text{ where } \mu = \frac{m_2}{m_1}$$

This formula shows that natural frequencies  $\omega_{1,2}$  depend only upon the parameter  $\mu$  (when  $g/l = 1$ ). If masses are identical  $m_1 = m_2 = m$ , i.e.  $\mu = 1$  then the natural frequencies are equal to

$$\omega_{1,2} = \sqrt{\frac{g}{l}(2 \pm \sqrt{2})}.$$

Here is the MatLab program for calculation and visualization of the double plane pendulum's oscillation:

```
>> mye=0.1;
>> g=9.8;
>> l=0.25;
>> A=mye+((1+mye)*mye).^0.5;
>> w1=(g/l).^0.5*(1+A).^0.5;
>> w2=(g/l).^0.5*(1-A).^0.5;
>> t=0:0.1:5;
>> alfa1=(-pi/12)*(mye./(1+mye)).^0.5*(cos(w1.*t))+... (pi/12)*(mye./(1+mye)).^0.5*(cos(w2.*t));
>> plot(t,alfa1)
>> grid on
>> hold on
>> alfa2=(pi/12)*cos(w1.*t)+(pi/12)*cos(w2.*t);
>> plot(t,alfa2,'ro-')
>> gtext('alfa2')
>> gtext('alfa1')
>> gtext('mye=0.1')
```

The result is presented in the fig.2

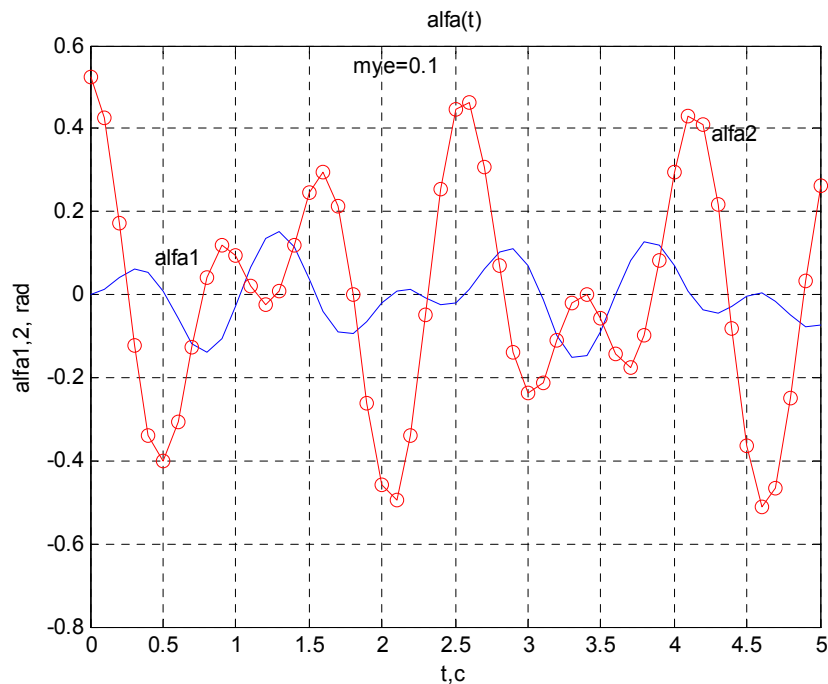


Fig.2 - The graphs of small oscillations of the double pendulum at  $\mu_1 = 0.1$

```

>> mye=0.2;
>> g=9.8;
>> l=0.25;
>> A=mye+((1+mye)*mye).^0.5;
>> w1=(g/l).^0.5*(1+A).^0.5;
>> w2=(g/l).^0.5*(1-A).^0.5;
>> t=0:0.1:5;
>> alfa1=(-pi/12)*(mye./(1+mye)).^0.5*(cos(w1.*t))+...
(pi/12)*(mye./(1+mye)).^0.5*(cos(w2.*t));
>> plot(t,alfa1)
>> grid on
>> hold on
>> alfa2=(pi/12)*cos(w1.*t)+(pi/12)*cos(w2.*t);
>> plot(t,alfa2,'ro-')
>> gtext('alfa2')
>> gtext('alfa1')
>> gtext('mye=0.2')

```

The result is presented in the fig.3

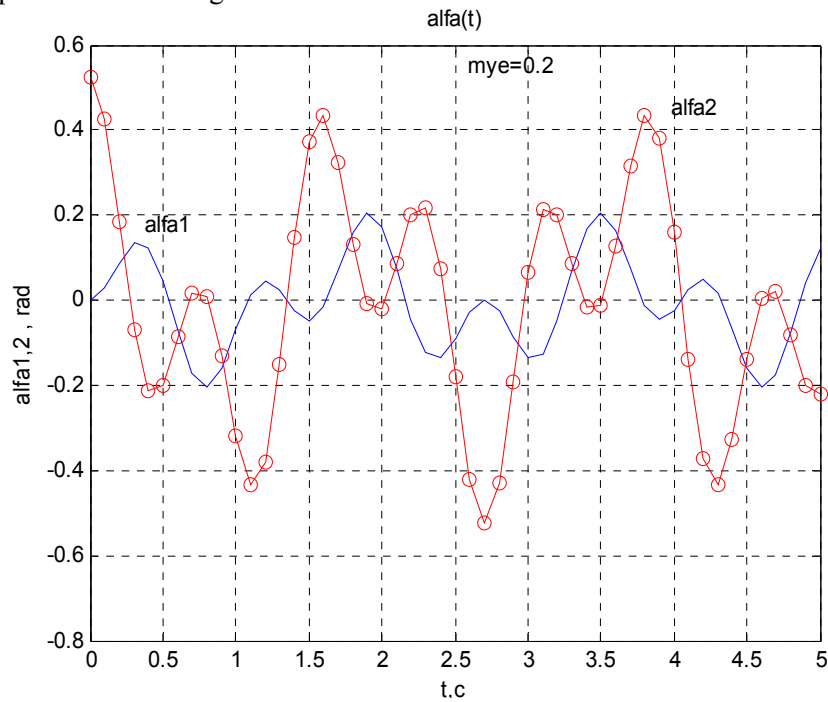


Fig.3 - The graphs of small oscillations of the double pendulum at  $\mu_2 = 0.2$

```

>> mye=0.3;
>> g=9.8;
>> l=0.25;
>> A=mye+((1+mye)*mye).^0.5;
>> w1=(g/l).^0.5*(1+A).^0.5;
>> w2=(g/l).^0.5*(1-A).^0.5;
>> t=0:0.1:5;
>> alfa1=(-pi/12)*(mye./(1+mye)).^0.5*(cos(w1.*t))+...
(pi/12)*(mye./(1+mye)).^0.5*(cos(w2.*t));
>> plot(t,alfa1)
>> grid on
>> hold on

```

```

>>alfa2=(pi/12)*cos(w1.*t)+(pi/12)*cos(w2.*t);
>>plot(t,alfa2,'ro-')
>>gtext('alfa2')
>> gtext('alfa1')
>> gtext('mye=0.3')

```

The result is presented in the fig.4

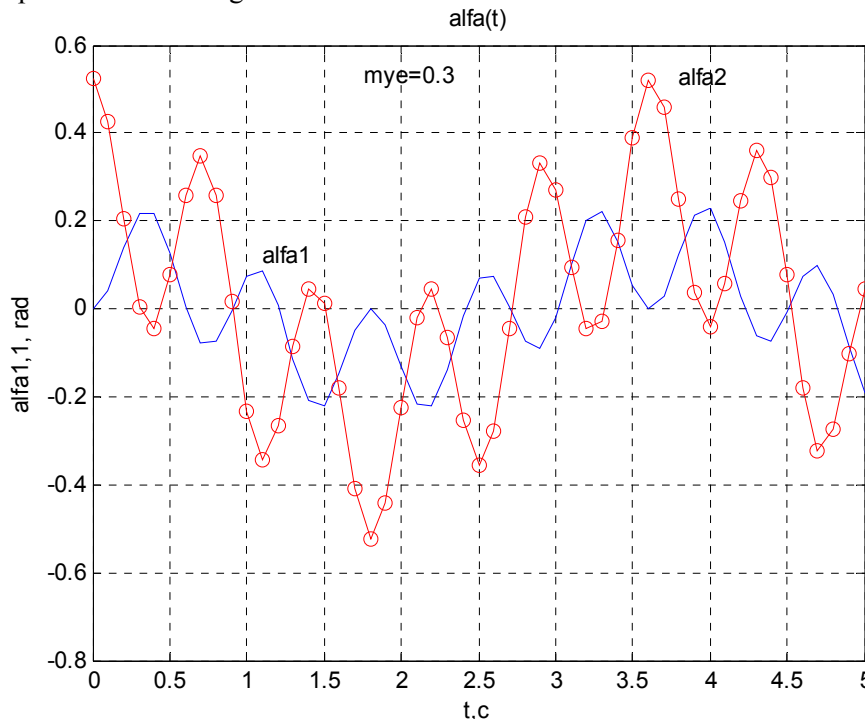


Fig.4 - The graphs of small oscillations of the double pendulum at  $\mu_3 = 0.3$

Here angles  $\alpha_1(t)$ ,  $\alpha_2(t)$  are expressed in radians, and  $t$  time is taken in seconds. Figures 2-4 demonstrate the graphs of small oscillations of pendulums for three values of  $\mu$ :  $\mu_1=0.1$ ,  $\mu_2=0.2$ ,  $\mu_3=0.3$  when  $l_1 = l_2 = l = 0.25m$ ,  $g = 9.8 m/s^2$ . The angles of displacement of pendulums from equilibrium position are given in radians. It is seen from the graphs that beats occur during double pendulum small oscillations at which energy is cyclically transmitted from one pendulum to the other one. When one pendulum almost stops, the other one is at its maximum amplitude. After a while pendulums "change roles" and so on. Oscillations with a bigger frequency  $\omega_1$  are modulated by lower-frequency oscillations with frequency  $\omega_2$ .

**Conclusion.** The article considers the calculation and visualization of small oscillations of a double plane simple pendulum. It contains the brief derivation of the motion equation and its solutions; the mathematical model of the motion in the form of the system of nonlinear differential equations. Experiments with the double pendulum are performed at various ratios of bodies' masses and initial angles, in particular for three values of  $\mu = m_1 / m_2$ :  $\mu_1= 0.1$ ,  $\mu_2= 0.2$ ,  $\mu_3= 0.3$  when the lengths of the pendulum are  $l = l_1 = l_2 = 0.25m$  and  $g = 9.8 m/s^2$ . The angles of pendulum deviation are given in radians. The graphs of small oscillations of the pendulum show that beats occur in the system during which energy cyclically passes from one pendulum to another. When one pendulum almost stops, the other pendulum is at its maximum amplitude. After a while pendulums "exchange their states" and so on. Oscillations with a bigger frequency  $\omega_1$  are modulated by lower frequency oscillations with a frequency  $\omega_2$ . The results of calculation and visualization of small oscillations of a double plane simple pendulum can be used in the theoretical mechanics.

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## ҚОС ЖАЗЫҚ МАЯТНИКТИҢ АУЫТҚУЫ КІШІ ТЕРБЕЛІСТЕРІН ЕСЕПТЕУ МЕН БЕЙНЕЛЕУ

**Аннотация.** Қос жазық математикалық маятниктің ауытқуы кіші тербелістерін есептеу мен бейнелеу ұсынылады. Қозғалыс теңдеулерін қорытып шығарылуы мен шешімі келтірілген, сызықтық емес дифференциал теңдеулер бойынша математикалық моделі құрастырылған. Денелердің массаларының әртүрлі қатынастары мен бастапқы ауытқу бұрыштары әртүрлі, атап айтқанда,  $g = 9.8\text{м/с}^2$ ,  $l=l_1=l_2=0.25\text{м}$  шартында, массалар қатынасының үш түрі шамасында  $\mu = m_1 / m_2$ :  $\mu_1=0.1$ ,  $\mu_2=0.2$ ,  $\mu_3=0.3$  жағдайлары үшін қос маятниктің қозғалысын бақылауға арналған эксперименттер жүргізілген. Маятниктердің кіші тербелістерінің графиктері келтірілген. Ауытқу бұрыштары радианда берілген.

Графиктерден жүйеде маятниктердің арасында циклді энергия алмасуы нәтижесінде соққы құбылысы пайда болатыны байқалады. Маятниктің біреуінің ауытқуы тоқтағанда екіншісі максимал амплитудада тербеле бастайды. Әлдебір уақыттан кейін маятниктер рөлі ауысады және осылайша қайталанатын. Тербеліс жиілігі  $\omega_1$  үлкені жиілігі кіші  $\omega_2$  тербеліспен модуляцияланады.

**Түйін сөздер.** Қос маятник, ауытқуы кіші тербелістер, соққы, энергиямен алмасу, меншікті жиіліктер, нормаль (табиғи) модалар.

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## РАСЧЕТ И ВИЗУАЛИЗАЦИЯ МАЛЫХ КОЛЕБАНИЙ ДВОЙНОГО ПЛОСКОГО МАЯТНИКА

**Аннотация.** Предлагается расчет и визуализация малых колебаний двойного плоского математического маятника. Приведен краткий вывод уравнения движения и их решения, построена математическая модель в виде системы нелинейных дифференциальных уравнений. Проведены эксперименты по наблюдению за движением двойного маятника при различных отношениях масс тел и начальных углах, в частности для трех значений  $\mu = m_1 / m_2$ :  $\mu_1=0.1$ ,  $\mu_2=0.2$ ,  $\mu_3=0.3$ , при условии  $l=l_1=l_2=0.25\text{м}$ ,  $g = 9.8\text{м/с}^2$ . Приведены графики малых колебаний маятников. Углы отклонения маятников приведены в радианах. Из графиков видно, что в системе происходят биения, при которых энергия циклически переходит от одного маятника к другому. Когда один маятник почти останавливается, другой раскачивается с максимальной амплитудой. Через некоторое время маятники "меняются ролями" и так далее. Колебания с большей частотой  $\omega_1$  модулируются низкочастотными колебаниями с частотой  $\omega_2$ .

**Ключевые слова.** Двойной маятник, малые колебания, биения, обмен энергиями, характерные частоты, нормальные мода.

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## ON PROJECTIONAL ORTHOGONAL BASIS OF A LINEAR NON-SELF -ADJOINT OPERATOR

**Abstract.** In this paper we study spectral properties of a linear non-self-adjoint operator with an internal symmetry of the form:

$$PL = L^*P, \quad LQ = QL^*;$$

where  $P^* = P$ ,  $Q^* = Q$  are orthogonal projections,  $L^*$  is an operator, adjoint to the operator  $L$  in the Hilbert space  $H$ . It is shown that a spectrum of such operator is real. In the case of a discrete operator, with a complete system of eigenvectors and associated vectors, the projections of eigenvalues and associated vectors of the operator  $L$  and its adjoint operator form an orthonormal basis. A class of Sturm-Liouville operators with such symmetry is found, moreover, it is found that the characteristic function of such an operator factorizes. An illustrative example is provided.

**Keywords:** Linear non-self-adjoint operator, real spectrum, basis, root vectors, completeness, theory of electric signals, plasma theory, discrete operator, invariant subspaces, root subspaces, completely continuous operator, eigenvectors and associated vectors, internal symmetry, projection, resolvent.

**1. Introduction.** The aim of the paper is to study the spectral properties of a certain class of linear non-self-adjoint operators  $L$  with a real spectrum that have the following internal symmetry

2.

$$PL = L^*P, \quad LQ = QL^*; P^* = P, \quad Q^* = Q,$$

where  $L$  is a linear operator with a domain  $D(A)$ , that belongs to the Hilbert space  $H$ , and  $P$  and  $Q$  are orthogonal projections, defined in this space.

We will assume that the Hilbert space  $H$  is separable.

We consider a completely continuous operator  $T$ , acting in the separable Hilbert space  $H$ . We denote by  $R_\lambda$  the following operator

$$(T - \lambda I)^{-1}. \tag{1.1}$$

The set of points of the plane  $\lambda$ , for which the operator (1.1) is everywhere defined and bounded, is called the resolvent set, and its complement is called the spectrum of the operator  $T$ . It is known that the spectrum of the completely continuous operator  $T$  consists of at most a countable number of points

$$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n, \dots,$$

which can have a limit point only at zero. If the space  $H$  is infinite-dimensional, then zero is always a point of the spectrum of the completely continuous operator. Each non-zero point of the spectrum  $\lambda_s$  of the completely continuous operator  $T$  corresponds to a finite-dimensional invariant subspace  $K_s$ , which is defined as the set of elements of  $f$ , canceled by some power operator  $T - \lambda_s I$ :

$$(T - \lambda_s I)^n f = 0 \tag{1.2}$$

The subspace  $K_s$  is called root subspace. Dimension of the root subspace, corresponding to the point of the spectrum  $\lambda_s$ , we will denote by  $m_s$ .

In each of the root subspace  $K_s$ , since it is finite dimensional, it is possible to choose a basis, in which the transformation is written by a Jordan matrix. This basis consists of several chains of equalities:

$$f_{11}, f_{21}, \dots, f_{R_1 1}; f_{12}, f_{22}, \dots, f_{R_2 2}, \dots; f_{1q}, f_{2q}, \dots, f_{R_q q}. \tag{1.3}$$

Each chain forms a basis in one of invariant subspaces on which this root subspace  $K_s$  splits. For elements of each chain the following equalities hold:

$$Tf_{1\tau} = \lambda_s f_{1\tau}, Tf_{2\tau} = \lambda_s f_{2\tau} + f_{1\tau}, \dots, Tf_{R_\tau \tau} = \lambda_s f_{R_\tau \tau} + f_{R_\tau \tau}$$

The first element of the chain is an eigen element of the operator  $T$ , corresponding to the eigenvalue  $\lambda_s$ , and the others are so-called associated elements.

In the paper we consider linear non-self-adjoint operators, acting in the separable Hilbert space  $H$  and with a discrete spectrum. The last one means that all points of a spectrum of the operator  $A$  (with the possible exception of one) are isolated, and the corresponding them subspaces are finite-dimensional. A finite-dimensional invariant subspace of the operator  $A$ , concerning to a certain point of the spectrum  $\lambda_s$ , is usually called the root subspace. We will denote it by  $K_s$ .

A root subspace  $K_s$  can be characterized as a collection of elements  $f$ , which satisfy the following equation at some integer  $m \geq 1$

$$(A - \lambda_s I)^m f = 0. \tag{0.1}$$

As is well known, completely continuous operators, as well as unbounded (for example, differential) operators that have completely continuous inverse, has a discrete spectrum.

The main problem of the paper is to study the conditions under which a system of finite-dimensional invariant (root) subspaces of an operator turns out to be a basis in  $H$  or in the range of the operator.

**Definition 1.1.** A system of elements  $\{e_n\}, n = 1, 2, \dots$  forms a basis in the space  $H$ , if any element  $x \in H$  can be uniquely represented in the form of the convergent series

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

Every basis is a complete uniformly minimal system. However, a complete minimal system may not form a basis in space. For example, the trigonometric system  $e_0(t) = 1, e_{2n-1}(t) = \sin nt, e_{2n}(t) = \cos nt, (n = 1, 2, \dots)$  is complete and uniformly minimal system in the space  $C(-\pi, \pi)$ , but does not form basis there.

**Definition 1.2.** System  $\{e_i\}, i = 1, 2, \dots$  is called an unconditional basis in the space  $H$  if it remains a basis for any permutation of its elements.

Let  $T$  be a linear bounded operator, acting in the space  $H$  and have a bounded inverse. If the system  $\{e_i\}$  is a basis, then the system  $\{Te_i\}$  is also a basis. If  $\{e_i\}$  is unconditional basis, then and  $\{Te_i\}$  is unconditional basis.

In the Hilbert space  $H$ , any orthogonal basis is unconditional. It turns out that any unconditional basis in the Hilbert space can be represented in the form  $\{Te_i\}, \{e_i\}$  is an orthogonal normed basis. Such bases were called Riesz bases. They can be characterized by the following properties: there exist positive numbers  $m$  and  $M$  such that for any  $x \in H$

$$m \cdot \sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 \leq M \cdot \sum_{i=1}^{\infty} |(x, e_i)|^2.$$



Completeness is a necessary condition to have a basis. We clarify that a system of finite-dimensional invariant subspaces of a certain operator is called complete in a Hilbert space  $H$ , if any element  $h \in H$  can be approximated with predetermined accuracy by the norm of a finite linear combination of elements, each of which belongs to one of the invariant subspaces. It is well known that if some completely continuous operator is self-adjoint, then the system of its finite-dimensional invariant subspaces is complete in the range of values of the operator, moreover it forms an orthonormal (after normalization) basis (in this case the root subspaces turn out to be proper).

In the case of a general completely continuous operator, the completeness may not occur. The simplest example of this kind is the integration operator

$$Af = \int_0^x f(t)dt, \quad 0 \leq x \leq 1, \quad (0.2)$$

which acts in the Hilbert system of functions, that are Lebesgue integrable square in the interval  $[0,1]$ . Further we will denote this space as  $L_2(0,1)$ . It is easy to verify that the operator (0.2), being completely continuous, has only a single point of the spectrum — zero and does not have any eigenvector. Consequently, it has no finite-dimensional invariant subspaces at all.

In the theory of non-self-adjoint operators in a Hilbert space, questions on completeness and basicity of systems of root vectors play an important role. For many classes of non-self-adjoint operators, completeness of system of root vectors has now been studied quite fully. Important results in this direction are contained in [2] - [11] and monographs [1].

Problems of basicity of root vector systems are investigated much less thoroughly than the questions of completeness. The basis condition of the root vector system was studied for dissipative operators by B.R. Mukminov [12], I.M. Glazman [13] and A.S. Markus [14], and for weakly perturbed self-adjoint and normal operators by A.S. Markus [15] and by Visitey and A.S. Markus [16]. The methods, developed in [12] - [17], make it possible to establish that a system of root vectors of an operator belongs only to the class of Bari bases [18]. The class of Bari bases is very narrow, and basis property of the system of root vectors has been established in [12] - [17] with rather strict restrictions on operator.

In [19], a completely new analytical method was proposed for questions on basicity, based on systematic use of theorems on interpolation by analytic functions. In this case, L. Carleson theorem [20] on interpolation by bounded analytic functions was used. In this paper, series of theorems are established, these theorems contain conditions, sufficient and, in some cases necessary, for a system of root subspaces of an operator in a Hilbert space to be Riesz basis in its closed linear hull. Connection of operator-theoretic and differential-theoretic considerations was made on basis of the well-known J. von Neumann theorem [21] - [22]. Ideas and methods of this paper were continued in the monograph [23].

V.P. Mikhailov [24] and G.M. Keselman [25] proved the Riesz basis property of systems of eigenfunctions and associated functions of an ordinary  $n$ -th order differential operator with strongly regular boundary value conditions in  $L_2$ . In the case of irregular boundary value conditions, a system of eigenfunctions and associated functions of the problem does not form even usual basis in  $L_2$ .

Basicity problem was completely solved only for the Sturm-Liouville model operator

$$\begin{aligned} Ly &= -y''(x) = \lambda^2 y(x), \quad x \in (0,1), \\ a_{11}y(0) + a_{12}y'(0) + a_{13}y(1) + a_{14}y'(1) &= 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(1) + a_{24}y'(1) &= 0, \end{aligned}$$

in [26].

In the case when the boundary value conditions are regular, but not strongly regular, the question on basis property of systems of eigenfunctions and associated functions has not yet been completely solved, thus, very active research is being conducted in this direction [27] - [31].

Theory of bases has, besides the theoretical value, purely practical value, and is used in the theory of electrical signals [32] and plasma [33].

## 2. Research Methods.

**Lemma 2.1.** If a system of vectors  $\{y_n\}$ ,  $n = 1, 2, \dots$  is complete in the space  $H$ , then systems  $\{Py_n\}$  and  $\{Qy_n\}$  are complete in the subspaces  $H_1 = PH$  and  $H_2 = QH$ , respectively, where  $P$  and  $Q$  are orthogonal projections given by the formulas

$$P = \frac{I + S}{2}, \quad Q = \frac{I - S}{2},$$

and the operator  $S$  has the form  $Su(x) = u(1 - x)$ ,  $\forall u(x) \in L^2(0,1) = H$ .

**Proof.** Let for some element  $g \in H_1$  of the subspace  $H_1$  the following equality holds:

$$(g, Py_n) = 0, \quad n = 1, 2, \dots$$

where  $(\cdot, \cdot)$  is a dot product in the space  $H$ , then there exists an element of the space  $H$ , such that  $g = Pf$ , consequently,

$$(Pf, Py_n) = 0, \quad (P^2f, y_n) = (Pf, y_n) = (g, y_n) = 0, \quad n = 1, 2, \dots$$

due to completeness of the system  $\{y_n\}$  in the space  $H$  we have  $g = 0$ , that is required to prove.

### Example.

It is known that the system  $\{e^{inx}\}$ ,  $n = 0, \pm 1, \pm 2, \dots$  is complete in the space  $H = L^2(-\pi, \pi)$ . We put that  $Su(x) = u(1 - x)$  and

$$P = \frac{I + S}{2}, \quad Q = \frac{I - S}{2},$$

then the system  $u_n(x) = Pe^{inx} = \frac{e^{inx} + e^{-inx}}{2} = \cos nx$  is complete in the space of even functions  $H_1 = PH$ ; The system  $v_n(x) = ae^{inx} = i \sin nx$  is complete in the subspace of odd functions  $H_2 = QH$ .

Let an operator  $L$  be densely defined and have a completely continuous inverse operator  $L^{-1}$ . Then a spectrum of the operator  $L$  is discrete and consists of only eigenvalues. Suppose that the following formulas hold:

$$PL = L^*P, \quad LQ = QL^*,$$

where  $P$  and  $Q$  are orthogonal projections, i.e.

$$P^* = P, \quad Q^* = Q.$$

**Lemma 2.2.** If  $y_n$  is an eigenvector, and  $\widetilde{y}_n$  is an associated vector of the operator  $L$ , corresponding to the eigenvalue  $\lambda_n$  and the equality  $Py_n = 0$  holds, then  $P\widetilde{y}_n \neq 0$  and the vector  $P\widetilde{y}_n$  is eigenvector for the adjoint operator  $L^*$ .

**Proof.** By assumption of the theorem, we have  $L\widetilde{y}_n - \lambda_n\widetilde{y}_n = K_n \cdot y_n$ , where  $K_n$  is some nonzero constant, then

$$PL\widetilde{y}_n - \lambda_n P\widetilde{y}_n = K_n Py_n = 0, \Rightarrow L^*P\widetilde{y}_n - \lambda_n P\widetilde{y}_n = 0;$$

$$\text{If } P\widetilde{y}_n = 0, \text{ then } \widetilde{y}_n = (P + Q)\widetilde{y}_n = Q\widetilde{y}_n, \Rightarrow LQ\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot y_n.$$

Acting by the operator  $Q$  to the both sides of this equality, and taking into account  $Qy_n \neq 0$  (otherwise we have  $y_n = 0$ ), we get

$$QLQ\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot Qy_n \neq 0.$$

Due to the formula  $LQ = QL^*$ , we obtain

$$Q^2L^*\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot Qy_n \neq 0, \quad QL^*\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot Qy_n,$$

$$LQ\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot Qy_n \neq 0.$$

Thus, due to the formula,  $\widetilde{y}_n = Q\widetilde{y}_n$ , we get

$$LQ\widetilde{y}_n - \lambda_n Q\widetilde{y}_n = K_n \cdot Qy_n \neq 0.$$

We obtained a contradiction, since the self-adjoint operator  $LQ$  has no adjoint vectors, therefore  $P\widetilde{y}_n \neq 0$  and  $L^*P\widetilde{y}_n = \lambda_n P\widetilde{y}_n$ .

**Remark.** From the equality  $Ly_n = \lambda_n y_n$ , when  $Py_n = 0$ , we have

$$L\left(\begin{array}{c} Py_n \\ 0 \end{array} + Qy_n\right) = \lambda_n \left(\begin{array}{c} Py_n \\ 0 \end{array} + Qy_n\right), \Rightarrow LQy_n = \lambda_n Qy_n, \quad LQ(Qy_n) = \lambda_n Qy_n \neq 0;$$

i.e.  $Qy_n$  is an eigenvector of the operator  $LQ$ .

**Lemma 2.3.** if to the eigen function  $y_n$  there corresponds a nonzero associated function  $\widetilde{y}_n$ , then  $Py_n = 0$  and  $P\widetilde{y}_n \neq 0$ .

**Proof.** By condition of the theorem, we have

$$Ly_n = \lambda_n y_n, \quad y_n \neq 0, \quad L\widetilde{y}_n - \lambda_n \widetilde{y}_n = K_n \cdot y_n, \quad K_n \neq 0.$$

Then acting by the operator  $P$  to both sides of the last formula, and using the formula  $PL = L^*P$ , we have

$$PL\widetilde{y}_n - \lambda_n P\widetilde{y}_n = K_n Py_n = 0, \Rightarrow L^*P\widetilde{y}_n - \lambda_n P\widetilde{y}_n = K_n Py_n, \Rightarrow$$

$$L^*P(P\widetilde{y}_n) - \lambda_n P\widetilde{y}_n = K_n Py_n.$$

If  $Py_n \neq 0$ , then  $P\widetilde{y}_n \neq 0$ , and the self-adjoint operator  $L^*P$  has an associated vector, which is impossible, consequently, if to the eigenvector  $y_n$  there corresponds an associated vector  $\widetilde{y}_n$ , then  $Py_n = 0$ , and from the previous lemma it follows that  $P\widetilde{y}_n \neq 0$  and the vector  $P\widetilde{y}_n$  is eigenvector for the operator  $L^*$ .

These three lemmas form the basis of our method.

### 3. Research results.

**Theorem 3.1.** If root vectors of the operators  $L$  and  $L^*$  are complete in the space  $H$ , and

$$1) PL = L^*P;$$

$$2) LQ = QL^*;$$

$$3) P^2 = P, P^* = P; Q^2 = Q, Q^* = Q,$$

then spectrum of the operator  $L$  is real, and normed projections of root vectors of the operators  $L$  and  $L^*$  form an orthonormal basis in  $H$ , i.e.

$$Pf = \sum_{n=1}^{\infty} (Pf, P\varphi_n) \frac{P\varphi_n}{\|P\varphi_n\|}, \quad Qf = \sum_{n=1}^{\infty} (Qf, Q\psi_n) \frac{Q\psi_n}{\|Q\psi_n\|},$$

where  $\{\varphi_n\}$ ,  $n = 1, 2, \dots$  are root vectors of the operator  $L$ ,  $\{\psi_n\}$ ,  $n = 1, 2, \dots$  are root vectors of the operator  $L^*$ .

**Proof.** From the formulas 1) and 2), we have

$$(PL)^* = L^*P^* = L^*P = PL; \quad (LQ)^* = Q^*L^* = QL^* = LQ;$$

consequently, operators  $PL$  and  $LQ$  are self-adjoint in the space  $H$ .

$$\text{If } Ly_n = \lambda_n y_n, \text{ then } PLy_n = \lambda_n Py_n, L^*Py_n = \lambda_n Py_n, L^*P(Py_n) = \lambda_n Py_n,$$

consequently, as  $Py_n \neq 0$ ,  $\lambda_n$  is an eigenvalue of the self-adjoint operator, therefore it is real.

$$\text{If } Ly_n = \lambda_n y_n \text{ and } Py_n = 0, \text{ then } PLy_n = \lambda_n Py_n = 0, L(Py_n + Qy_n) = \lambda_n(P + Q)y_n,$$

$$LQy_n = \lambda_n Qy_n \text{ and } Qy_n \neq 0,$$

hence, and in this case  $\lambda_n$  is an eigenvalue of the self-adjoint operator  $LQ$ , thus it is a real value.

By our assumption system of eigen and associated functions  $\{\varphi_n\}, n = 1, 2, \dots$ , of the operator  $L$  is complete in the space  $H$ , then the system  $\{P\varphi_n\}, n = 1, 2, \dots$ , is complete in the subspace  $H_1 = PH$  (see Lemma 2.1). Since all eigenvalues  $\lambda_n$  ( $n = 1, 2, \dots$ ) of the operator  $L$  are real, then spectra of the operators  $L$  and  $L^*$  are the same.

If  $L\varphi_n = \lambda_n \varphi_n$ , then due to the formula  $PL = L^*P$  we have  $PL\varphi_n = \lambda_n P\varphi_n, L^*P\varphi_n = \lambda_n P\varphi_n$ , consequently, the vector  $P\varphi_n$  is eigenvector for the self-adjoint operator  $L^*P$ .

If  $P\varphi_n \neq 0$ , then due to Lemma 2.3, there is no associated function.

If  $P\varphi_n = 0$ , then there may be an attached vector  $\widetilde{\varphi}_n$ , such that

$$L\widetilde{\varphi}_n - \lambda_n \widetilde{\varphi}_n = K_n \varphi_n, \quad K_n \neq 0.$$

Then  $PL\widetilde{\varphi}_n - \lambda_n P\widetilde{\varphi}_n = K_n P\varphi_n = 0$ , moreover, due to Lemma 2.2, we have  $P\widetilde{\varphi}_n \neq 0$ . Operator  $L$  does not have associated vectors higher than first order. Indeed, if

$$L\widetilde{\widetilde{\varphi}}_n - \lambda_n \widetilde{\widetilde{\varphi}}_n = K_n \widetilde{\varphi}_n, \quad K_n \neq 0,$$

then

$$P\varphi_n = 0, \quad P\widetilde{\varphi}_n \neq 0,$$

thus

$$\begin{aligned} PL\widetilde{\widetilde{\varphi}}_n - \lambda_n P\widetilde{\widetilde{\varphi}}_n &= K_n P\widetilde{\varphi}_n \neq 0, \Rightarrow L^*P\widetilde{\widetilde{\varphi}}_n - \lambda_n P\widetilde{\widetilde{\varphi}}_n = K_n P\widetilde{\varphi}_n \neq 0, \Rightarrow \\ &\Rightarrow L^*P(P\widetilde{\widetilde{\varphi}}_n) - \lambda_n P\widetilde{\widetilde{\varphi}}_n = K_n P\widetilde{\varphi}_n \neq 0. \end{aligned}$$

Consequently,  $P\widetilde{\widetilde{\varphi}}_n \neq 0$ , and this contradicts self-adjointness of the operator  $L^*P$ .

Therefore, if the sequence  $\{\varphi_n\}, n = 1, 2, \dots$ , consists of eigen and associated functions of the operator  $L$ , then the sequence  $\{P\varphi_n\}, n = 1, 2, \dots$  consists of eigenvectors of the self-adjoint operator  $L^*P$ , hence it is a complete and orthogonal system. Rejecting zero elements, if there is any of them, we get a complete orthogonal system  $\{P\varphi_n\}, n = 1, 2, \dots$  (cleaned system). Consequently, the system  $\{P\varphi_n/\|P\varphi_n\|\}, n = 1, 2, \dots$  is an orthonormal basis of the space  $H_1 = PH$ , i.e. for any vector  $f$  from  $H$  the following decomposition holds

$$Pf = \sum_{n=1}^{\infty} (Pf, P\varphi_n) \frac{P\varphi_n}{\|P\varphi_n\|};$$

2) Let a system of eigen and associated functions  $\{\psi_n\}, n = 1, 2, \dots$ , of the operator  $L^*$  be complete in the space  $H$ , then the system  $\{Q\psi_n\}, n = 1, 2, \dots$ , is complete in the subspace  $H_2 = QH$ .

$$\text{If } L^*\psi_n = \lambda_n \psi_n, \text{ then } QL^*\psi_n = \lambda_n Q\psi_n,$$

$$LQ\psi_n = \lambda_n Q\psi_n, LQ(Q\psi_n) = \lambda_n Q\psi_n;$$

If  $Q\psi_n \neq 0$ , then due to Lemma 2.3, there is not any associated vector. If  $Q\psi_n = 0$ , then maybe there is an attached vector  $\widetilde{\psi}_n$ , i.e.

$$L^*\widetilde{\psi}_n - \lambda_n \widetilde{\psi}_n = K_n \psi_n, \quad K_n \neq 0,$$

then  $QL^*\widetilde{\psi}_n - \lambda_n Q\widetilde{\psi}_n = K_n Q\psi_n = 0, \Rightarrow$

$$LQ\widetilde{\psi}_n - \lambda_n Q\widetilde{\psi}_n = 0, \Rightarrow LQ(Q\widetilde{\psi}_n) - \lambda_n Q\widetilde{\psi}_n = 0,$$

moreover, due to Lemma 2.2, we have  $Q\widetilde{\psi}_n \neq 0$ .

Consequently, in any case the vector  $Q\widetilde{\psi}_n$  is eigenvector for the operator  $LQ$ . Due to self-adjointness of the operator  $LQ$ , eigenvectors  $\{Q\psi_n\}$  are mutually orthogonal, and according to our assumption and Lemma 2.1, are complete in the subspace  $H_2 = QH$ , consequently, the system  $Q\psi_n/\|Q\psi_n\|$ ,  $n = 1, 2, \dots$  forms a orthonormal basis in the subspace  $H_2$ , i.e.

$$Qf = \sum_{n=1}^{\infty} (Qf, Q\psi_n) \frac{Q\psi_n}{\|Q\psi_n\|}.$$

#### 4. Discussion.

We consider the model Sturm - Liouville operator in the space  $L^2(0,1)$ .

$$Ly = -y''(x), \quad x \in (0,1), \quad (4.1)$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(1) + a_{14}y'(1) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(1) + a_{24}y'(1) = 0, \end{cases} \quad (4.2)$$

where  $a_{ij}$  ( $i = 1, 2; j = 1, 2, 3, 4$ ) – are arbitrary complex numbers. By  $\Delta_{ij}$  we denote minors of the boundary matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Therefore,

$$\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}, \quad (i, j = 1, 2, 3, 4). \quad (4.3)$$

If the following inequality holds

$$\Delta = \Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{32} + \Delta_{34} \neq 0,$$

then the operator (4.1) - (4.2) is invertible and its inverse operator  $L^{-1}$  is completely continuous.

**Theorem 4.1.** Invertible Sturm-Liouville operator satisfies the following equalities

$$PL = L^+P, \quad LQ = QL^+, \quad (4.4)$$

if and only if it has the form

$$Ly = -y''(x), \quad x \in (0,1), \quad (4.1)$$

$$\begin{cases} y(0) + ky'(0) + y(1) - ky'(1) = 0, \\ (1 - 2l)y(0) - (2\alpha + k)y'(0) - (1 - 2l)y(1) - (2l - 2\alpha - k)y'(1) = 0, \end{cases} \quad \text{we denote as (4.5),}$$

its adjoint has the form:

$$Lz = -z''(x), \quad x \in (0,1) \quad (4.2)^+$$

$$\begin{cases} (1 - 2l)z(0) - lz'(0) - (1 - 2l)z(1) - lz'(1) = 0, \\ (1 - k - 2\bar{\alpha})z(0) - \bar{\alpha}z'(0) + (k + 2\bar{\alpha})z(1) - (k + \bar{\alpha})z'(1) = 0, \end{cases} \quad (4.5)^+$$

where  $k = \bar{k}$ ,  $l = \bar{l}$  – is a real number,  $\alpha$  – is an arbitrary complex number,  $P$  and  $Q$  are orthogonal projections given by the formulas:

$$P = \frac{l+S}{2}, \quad Q = \frac{l-S}{2}, \quad (4.6)$$

where  $I$  – is a unit operator, operator  $S$  is defined by the formula

$$Su(x) = u(1 - x), \quad \forall u(x) \in L^2(0,1). \quad (4.7)$$

**Theorem 4.2.** If for the invertible Sturm-Liouville operator (4.1) - (4.2) the following formulas hold

$$\begin{aligned} \text{a) } PL &= L^+P, \\ \text{b) } LQ &= QL^+; \end{aligned} \quad (4.4)$$

where

$$P = \frac{I+S}{2}, \quad Q = \frac{I-S}{2}, \quad (4.6)$$

$$Su(x) = u(1 - x), \quad \forall u(x) \in L^2(0,1), \quad (4.7)$$

then the characteristic function of this operator is factorized as follows:

a) If  $kl(1 - 2l) \neq 0$ , then

$$\Delta(\lambda) = 2l \left( \frac{2l-1}{l} \cdot \frac{\sin \frac{\lambda}{2}}{\lambda} - \cos \frac{\lambda}{2} \right) \cdot \left( kl \sin \frac{\lambda}{2} + \cos \frac{\lambda}{2} \right);$$

b) If  $l = \frac{1}{2}$ , then

$$\Delta(\lambda) = -\frac{\cos \frac{\lambda}{2}}{2} \left( \lambda k \cos \frac{\lambda}{2} + \cos \frac{\lambda}{2} \right);$$

c) If  $l = 0$ ,  $k \neq 0$ , then

$$\Delta(\lambda) = -2 \sin \frac{\lambda}{2} \left( k \sin \frac{\lambda}{2} + \frac{\cos \frac{\lambda}{2}}{2} \right);$$

d) If  $l = 0$ ,  $k = 0$ , then

$$\Delta\Delta(\lambda) = -\frac{2 \sin \frac{\lambda}{2} \cos \frac{\lambda}{2}}{2};$$

where  $k$ ,  $l$  – are real coefficients of the boundary conditions (4.5), and the characteristic function  $\Delta(\lambda)$  has the form:

$$\Delta(\lambda) = \Delta_{12} + \Delta_{34} + \Delta_{13} \frac{\sin \lambda}{\lambda} + (\Delta_{14} + \Delta_{32}) \cos \lambda + \Delta_{24} \lambda \sin \lambda,$$

where  $\Delta_{\square\square}$  are minors from (4.3).

**Theorem 4.3.** Eigenvalues and eigenfunctions of the boundary value problem

$$Ly = -y''(x), \quad x \in (0,1),$$

$$\begin{cases} y(0) + y(1) = 0, \\ y(0) - y(1) - 2\alpha[y'(0) - y'(1)] = 0, \end{cases}$$

consist of two series:

$$\text{a) } \lambda_n^{(1)} = 2n\pi, \quad y_n^{(1)} = K_n \sin 2n\pi x, \quad n = 1, 2, \dots;$$

$$b) \lambda_n^{(2)} = (2n + 1)\pi, \quad y_n^{(2)} = B_n \left[ 2\alpha \cos(2n\pi + \pi)x + \frac{\sin(2n+1)\pi x}{(2n+1)\pi} \right], \quad n = 1, 2, \dots;$$

where  $\alpha$  – is an arbitrary complex number,  $K_n, B_n$  – are arbitrary constants.

In this case, the normalized system

$$\{ P y_n^{(2)}, Q z_n^{(2)} \},$$

forms an orthonormal basis of the space  $L^2(0,1)$ , where

$$P y_n^{(2)} = \frac{B_n}{(2n+1)\pi} \sin(2n\pi + \pi)x; \quad n = 0, 1, 2, \dots;$$

$$Q z_n^{(2)} = (-1)^n K_n \sin 2n\pi x, \quad n = 1, 2, \dots$$

**Theorem 4.3<sup>+</sup>**. Eigenvalues and eigenfunctions of the boundary value problem

$$L^+ z = -z''(x) = \mu^2 z(x), \quad x \in (0,1),$$

$$\begin{cases} z(0) - z(1) = 0, \\ z(0) - \bar{\alpha}[z'(0) + z'(1)] = 0; \end{cases}$$

consist of two series:

$$a) \mu_n^{(1)} = 2n\pi + \pi,$$

$$z_n^{(1)}(x) = A_n \cos(2n + 1) \pi \left( \frac{1}{2} - x \right);$$

$$b) \mu_n^{(2)} = 2n\pi, \quad n = 1, 2, \dots$$

$$z_n^{(2)}(x) = K_n \left[ 4\bar{\alpha}n\pi \cos 2n\pi \left( \frac{1}{2} - x \right) - \sin 2n\pi \left( \frac{1}{2} - x \right) \right],$$

where  $\alpha$  – is an arbitrary complex number, and  $K_n$  – are arbitrary constants.

Moreover,  $Q z_n^{(1)} = 0, Q z_n^{(2)} = (-1)^n K_n \sin 2n\pi x$ , which confirms results of the main Theorem 3.1.

We note that Sturm-Liouville operators of the class, that we studied, are reconstructed in a single spectrum [34].

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### СЫЗЫҚТЫҚ СЫҢАРЛЫ ОПЕРАТОРДЫҢ ОРТОГОНӘЛДІ ПРОЕКЦИЯЛЫҚ БАЗИСІ ТУРАЛЫ

**Аннотация.** Бұл еңбекте, мынадай

$$PL = L^*P, \quad LQ = QL^*$$

ішкі симметриясы бар сызықтық сыңарлы оператордың спектралдік қасиеттері зерттелді, мұндағы  $P^* = P, Q^* = Q$  – ортогоналді проекторлар, ал  $L^*$  – сыңар оператор. Әңгіме Гилберттің сепарабелді  $H$  кеңістігінде болып отыр.

Мұндай операторлардың спектрі нақты сандар өсінде жататыны көрсетілді, сондай-ақ меншікті және олармен еншілес векторлар системасы кеңістікте толық дискретті операторлардың түпкі векторларының проекциялары мен оған сыңар оператордың түпкі векторларының проекциялары бірігіп,  $H$  кеңістігінде ортогонал базис құрайтыны көрсетілді, әрине, ортанормаланған соң.

Штурм-Лиувилл операторларының ішінен осындай симметриясы бар операторлар класы бөліп алынды. Мұндай операторлардың характеристикалық функциялары көбейткіштерге жіктелетіні дәлелденді. Теореманың мәні мысал арқылы айқындала түсті.

**Түйін сөздер:** Сзықтық сыңарлы оператор, нақты спектр, түпкі векторлар, толымдылық, электр сигналдарының теориясы, плазманың теориясы, дискретті оператор, ішкікеңістіктер, инвариантты кеңістіктер, түпкі кеңістіктер, меншікті және еншілес векторлар, ішкі симметрия, проектор, резольвента.

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## О ПРОЕКЦИОННО ОРТОГОНАЛЬНОМ БАЗИСЕ ЛИНЕЙНОГО НЕСАМОСOPЯЖЕННОГО ОПЕРАТОРА

**Аннотация.** В настоящей работе исследованы спектральные свойства линейного несамосопряженного оператора обладающего внутренней симметрией вида

$$L = L^*P, \quad LQ = QL^*;$$

где  $P^* = P$ ,  $Q^* = Q$  – ортогональные проекторы,  $L^*$  – оператор, сопряженный к оператору  $L$  в гильбертовом пространстве  $H$ . Показан, что спектр такого оператора вещественный. В случае дискретного оператора, с полной системой собственных и присоединенных векторов, проекций собственных и присоединенных векторов оператора  $L$  и его сопряженного образуют ортонормированный базис. Найден класс операторов Штурма – Лиувилля, обладающий такой симметрией, при этом обнаружено, что характеристическая функция такого оператора факторизуется. Приведен иллюстративный пример.

**Ключевые слова:** Линейный несамосопряженный оператор, вещественный спектр, базис, корневые векторы, полнота, теория электрических сигналов, теория плазмы, дискретный оператор, инвариантные подпространства, корневые подпространства, вполне непрерывный оператор, собственные и присоединенные векторы, внутренняя симметрия, проектор, резольвента.

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## DYNAMIC STABILITY OF WAVE PROCESSES OF A ROUND ROD

**Abstract:** This paper is devoted to the study of the stability dynamics of wave processes of flat and circular elements, and also some axisymmetric problems of oscillation of an elastic layer limited by rigid or deformable boundaries when exposed to normal or rotational shear stresses are considered. Solutions to the problems under consideration were obtained using integral transformations by coordinate or time. The work develops the dynamic stability of a round rod. The loss of stability of a round rod will be investigated on the basis of the mathematical theory and the transverse oscillations of a round rod, described in the work of I.G. Philippov.

**Key words:** oscillations, stability, wave process, axisymmetric problems, round rod, exponential transformation, shear stress.

The issues of stability in a static formulation have been developed by many authors, and the results of such studies can be found in [1,2].

However, the issues of dynamic stability of elements' construction and structures received much less attention.

In this paper, the dynamic stability of a round rod is developed.

Consider a round elastic rod of length  $l$ . We will assume that an axial compressive force of intensity is applied to the ends of the rod at any moment of time.

The loss of stability of a round rod will be investigated on the basis of the mathematical theory and the transverse oscillation of a round rod, described in the work of I.G. Philippov [3].

The fourth order equation describing the transverse oscillation of the rod has the form.

$$A_0 \frac{\partial^4 V}{\partial t^4} - A_1 \frac{\partial^4 V}{\partial z^2 \partial t^2} + A_2 \frac{\partial^4 V}{\partial z^4} + \rho \frac{\partial^2 V}{\partial t^2} = F(z, t), \quad (1)$$

where the constants  $A_j$  are equal to

$$A_0 = \frac{\mu r_0^2 (4a^2 + b^2)}{8b^4 (a^2 - b^2)}; A_1 = \frac{\mu r_0^2 (13a^2 - 8b^2)}{8b^2 (a^2 - b^2)}; A_2 = \mu r_0^2, \quad (2)$$

$r_0$  - rod radius,  $a$  and  $b$  - propagation velocity of longitudinal and transverse waves in the material of the rod,  $\mu$  - the module of shift модуль сдвига,  $\rho$  - rod material density.

Under the action of compressive force for a small period of time  $\Delta t$  when solving bending forces in cross-sectional planes of the rod  $\Delta F$ , as we know, equals to [2]

$$\Delta F = \Delta P \frac{\partial^2 V}{\partial z^2} \quad (3)$$

Summing (3) over time, we get the power  $F$  :

$$F(z, t) = \int_0^t P'(t - \xi) \frac{\partial^2 V}{\partial z^2} d\xi \quad (4)$$

where the lower limit of integration depends on the beginning of the force  $P$  and, for example if  $P$  power is applied at the  $t = 0$  then the lower integration limit is also zero.

Obviously, if the compressive force is constant and is applied at the moment  $t = 0$ , then it can be written as  $P = P_0 H(t)$ , где  $H(t)$  Heaviside function, then we get the following: (4)

$$F = -P_0 \frac{\partial^2 V}{\partial z^2} \quad (5)$$

Therefore, equation (5) can be written in the following form

$$A_0 \frac{\partial^4 V}{\partial t^4} - A_1 \frac{\partial^4 V}{dz^2 \partial t^4} + A_2 \frac{\partial^4 V}{\partial z^4} + \rho \frac{\partial^2 V}{\partial t^2} - \int_0^t P'(t - \xi) \frac{\partial^2 V}{\partial z^2} d\xi = 0 \quad (6)$$

To study the stability of the rod, it is necessary to formulate the boundary conditions at the ends of the rod. These conditions are: hinged joints  $V = \frac{\partial^2 V}{\partial z^2} = 0$ , rigid fastening  $V = \frac{\partial V}{\partial z} = 0$ , free ending

$\frac{1}{a^2} \frac{\partial^2 V}{\partial t^2} + 4 \frac{\partial^2 V}{\partial z^2} = 0; \frac{\partial^3 V}{\partial z^3} = 0$ . In order to study the stability of the rod, it is necessary to formulate the

boundary conditions at the ends of the rod. These conditions are: [hinged joints](#)  $V = \frac{\partial^2 V}{\partial z^2} = 0$ , rigid

fastening  $V = \frac{\partial V}{\partial z} = 0$ , free ending

$$\frac{1}{a^2} \frac{\partial^2 V}{\partial t^2} + 4 \frac{\partial^2 V}{\partial z^2} = 0; \frac{\partial^3 V}{\partial z^3} = 0.$$

The solution of equation (6) will be sought believing that

$$V(z, t) = V_0(z) \exp(i\omega t) \quad (7)$$

where  $\omega$  is oscillation frequency. Putting (7) into the equation (6), for для  $V_0$  we get the equation

$$A_2 \frac{d^4 V_0}{dz^4} + [A_1 \omega^4 - \int_0^\infty P'(\xi) \exp(i\omega \xi) d\xi] \frac{d^2 V_0}{dz^2} + A_0 \omega^4 V_0 - \rho \omega^2 V_0 = 0 \quad (8)$$

Denoting by  $Q(\omega)$  Fourier's exponential transform of rate of change of compressive force  $P'(t)$ , equation (8) we write in the form

$$\frac{d^4 V_0}{dz^4} + B_0 \frac{d^2 V_0}{dz^2} + B_1 V_0 = 0 \quad (9)$$

where the coefficients are  $B_0, B_1$  are equal to

$$B_0 = \frac{A_1 \omega^2 - Q(\omega)}{A_2}; \quad B_1 = \frac{A_0 \omega^4 - \rho \omega^2}{A_2} \quad (10)$$

For simplicity, we consider the case when the ends of the rod are hinged, i.e. at  $z = 0$  and  $z = l$  conditions are met

$$V = \frac{\partial^2 V}{\partial z^2} = 0 \quad (11)$$

The solution of equation (9) is sought in the form of

$$V_0 = \sum_{n=1}^{\infty} V_{0n} \sin\left(\frac{\pi n}{l} z\right) \quad (12)$$

at the same time, boundary conditions (11) are satisfied automatically.

Putting (12) into the equation (9) for the length  $Q(\omega)$  we get

$$Q(\omega) = A_1 \omega^2 - A_2 \left(\frac{l}{\pi n}\right)^2 \left[ B_1 + \left(\frac{\pi n}{l}\right)^4 \right] \quad (13)$$

Введем безразмерные величины

Let's introduce dimensionless quantities

$$Q_0(\xi) = \frac{Q(\omega)}{\mu \pi^4}; \quad \xi = \frac{\omega l}{\pi b}; \quad c = \pi \left(\frac{r_0}{l}\right); \quad \frac{b^2}{a^2} = \frac{1-2\nu}{2(1-\nu)} \quad (14)$$

And then for  $Q_0(\xi)$  we get

$$Q_0(\xi) = \frac{2}{c^2} \left[ \frac{(9-5\nu)c^2 + 4}{9-10\nu} \right] \xi^2 - \xi^4 - \frac{8}{9-10\nu} \quad (15)$$

where  $\nu$  - Poisson's ratio.

The physical meaning of the influence of the rate of change of the compressive force is that, it is important that the impulse of this force or the rate of increase of the force over time as in the impact theory.

You can also consider other types of binding. For example, you can search for solutions of equation (9) in the form

$$\begin{aligned}
 V_0 = & C_1 \left[ \frac{\cos(\alpha_0 z)}{\alpha_0^n} + \frac{\cos(\alpha_1 z)}{\alpha_1^n} \right] + C_2 \left[ \frac{\cos(\alpha_0 z)}{\alpha_0^n} - \frac{\cos(\alpha_1 z)}{\alpha_1^n} \right] + \\
 & + C_3 \left[ \frac{\sin(\alpha_0 z)}{\alpha_0^m} + \frac{\sin(\alpha_1 z)}{\alpha_1^m} \right] + C_4 \left[ \frac{\sin(\alpha_0 z)}{\alpha_0^m} - \frac{\sin(\alpha_1 z)}{\alpha_1^m} \right]
 \end{aligned} \quad (16)$$

where  $C_j$  - arbitrary constants, integers  $(n, m)$  are selected from conditions at  $Z = 0$  и  $Z = l$   
 $\alpha_j$  - the roots of the characteristic equation and are equal

$$\alpha_j = \sqrt{\frac{B_0}{2} \pm \sqrt{\frac{B_0^2}{4} - B_1}}$$

In the case of rigid fastening of both ends of the rod integers  $(n = 0, m = 1)$  and to find  $Q(\omega)$  we get the transcendental equation

$$2 - \frac{\alpha_0^2 + \alpha_1^2}{\alpha_0 \alpha_1} \sin(\alpha_0 l) \sin(\alpha_1 l) - 2 \cos(\alpha_0 l) \cos(\alpha_1 l) = 0, \quad (17)$$

Similarly, you can consider other combinations of conditions for binding the ends of the rod.

Some axisymmetric problems of oscillation of an elastic layer bounded by rigid or deformable boundaries when exposed to a normal or rotational shear stress are considered below. Solutions to the problems under consideration were obtained using integral transformations by coordinate or time.

First we consider the problem for half-space under the assumption that the half-space  $z > 0$  is an anisotropic medium with the axis of symmetry of the mechanical properties (axis) (axis  $z$ ), and the surface of which is subjected to an impulse voltage at the time  $\xi_{z0} = -f(r, t)$ .

Because of the symmetry of the mechanical properties of the medium relative to the axis  $z$  of the unique nonzero component of the displacement vector,  $U_0(r, z, t)$  only the stresses and  $\xi_{r0}$  and  $\xi_{z0}$  the ones determined by formulas

$$\begin{aligned}
 \xi_{r0} &= C_{mn} \left( \frac{\partial U_0}{\partial r} - \frac{U_0}{r} \right), \\
 \xi_{z0} &= C_{mn} \frac{\partial U_0}{\partial z}
 \end{aligned} \quad (18)$$

The equation of motion reduces to one

$$\frac{\partial \xi_{r0}}{\partial r} + \frac{\partial \xi_{r0}}{\partial z} + \frac{2\xi_{r0}}{r} = \rho \frac{\partial^2 U_0}{\partial t^2} \quad (19)$$

Substituting the expressions for  $\xi_{r0}$  and  $\xi_{z0}$  from (18) into equation (19), we bring it to the form:

$$\frac{\partial^2 U_0}{\partial r^2} + \frac{1}{r} \frac{\partial U_0}{\partial r} - \frac{U_0}{r^2} + \gamma^2 \frac{\partial^2 U_0}{\partial z^2} = \frac{1}{b^2} \frac{\partial^2 U_0}{\partial t^2} \quad (20)$$

where

$$b^2 = \frac{C_{mm}}{\rho}; \quad \gamma^2 = \frac{C_{mm}}{C_{nn}}$$

If the half-space is isotropic, then  $\gamma = 1$  and  $b = \sqrt{\frac{\mu}{\rho}}$ .

The boundary conditions for  $U_0$  has:

$$\xi_{z_0} = -f(r, t) \text{ under } z = 0, \quad t \geq 0 \quad (21)$$

$$U_0 \rightarrow 0 \text{ under } z \rightarrow \infty \quad (22)$$

The initial conditions of the problem are zero

$$U_0 = \frac{\partial U_0}{\partial t} = 0 \quad \text{under } t = 0 \quad (23)$$

The solution of equation (20) for the boundary (21) - (22) and the initial conditions (6) will be sought, by applying the Laplace transform. Suppose,

$$U(r, z, p) = \int_0^{\infty} U_0(r, z, t) e^{-pt} dt, \quad \text{Re } p > 0 \quad (24)$$

Definitely, for the function  $U(r, z, p)$  we obtain the equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \left( \frac{1}{r^2} + \frac{p^2}{b^2} \right) U + \gamma^2 \frac{\partial^2 U}{\partial z^2} = 0 \quad (25)$$

And  $U$  must satisfy the boundary conditions:

$$\frac{\partial U}{\partial z} = -\frac{f_0(r, p)}{C_{66}} \quad \text{under } z = 0, \quad t > 0 \quad (26)$$

$$U_0 \rightarrow 0 \text{ under } z \rightarrow \infty \quad (27)$$

where

$$f_0(r, p) = \int_0^{\infty} f(r, t) e^{-pt} dt.$$

The general solution of equation (25) is sought by the method of separation of variables (the Fourier method) and has the form:

$$U(r, z, p) = \int_0^{\infty} \alpha \left[ A(\alpha, p) e^{-\frac{z}{\gamma} \sqrt{\alpha^2 + \frac{p^2}{b^2}}} + B(\alpha, p) e^{\frac{z}{\gamma} \sqrt{\alpha^2 + \frac{p^2}{b^2}}} \right] J_1(\alpha r) d\alpha. \quad (28)$$

where  $A(\alpha, p)$  and  $B(\alpha, p)$  are defined from boundary conditions (26)-(27) from the condition (29) follows that

$$B(\alpha, p) = 0 \quad (29)$$

Using boundary condition (26), for defining  $A(\alpha, p)$  we'll take integral equation:

$$\int_0^\infty \alpha A(\alpha, p) \sqrt{\alpha^2 + \frac{p^2}{b^2}} J_1(\alpha r) d\alpha = \frac{\gamma}{C_{66}} f_0(r, p) \tag{30}$$

Let

$$f_0(r, p) = \int_0^\infty \alpha f_1(\alpha, p) J_1(\alpha r) d\alpha \tag{31}$$

Then

$$A(\alpha, p) = \frac{f_1(\alpha, p)}{C_{mm} \sqrt{\alpha^2 + \frac{p^2}{b^2}}} \tag{32}$$

Substituting expression (29) and (32) into formula (28), we obtain the following expression:

$$U(r, z, p) = \frac{\gamma}{C_{mm}} \int_0^\infty \frac{\alpha f_1(\alpha, p)}{\sqrt{\alpha^2 + \frac{p^2}{b^2}}} J_1(\alpha r) e^{-\frac{z}{\gamma} \sqrt{\alpha^2 + \frac{p^2}{b^2}}} d\alpha \tag{33}$$

We consider the special case when

$$f_0(r, p) = \frac{\varphi_0(p)}{r} \tag{34}$$

In case (34) function

$$f_1(\alpha, p) = \frac{\varphi_0(p)}{\alpha}$$

and (33) has

$$U(r, z, p) = \frac{\gamma}{C_{mm}} \int_0^\infty \frac{\varphi_0(p)}{\sqrt{\alpha^2 + \frac{p^2}{b^2}}} e^{-\frac{z}{\gamma} \sqrt{\alpha^2 + \frac{p^2}{b^2}}} J_1(\alpha r) d\alpha = \frac{1}{C_{mm}} \varphi_0(p) I_{\frac{1}{2}} \left[ \frac{p}{2b} \left( \sqrt{\frac{z^2}{\gamma^2} + r^2} - \frac{z}{\gamma} \right) \right] K_{\frac{1}{2}} \left[ \frac{p}{2b} \left( \sqrt{\frac{z^2}{\gamma^2} + r^2} + \frac{z}{\gamma} \right) \right]$$

where  $K_{\frac{1}{2}}, I_{\frac{1}{2}}$  Bessel functions of imaginary argument. Using the representations of the functions  $I_{\frac{1}{2}}(\zeta)$  and  $K_{\frac{1}{2}}(\zeta)$ , for  $U(r, z, p)$  we find that

$$U(r, z, p) = \frac{b\varphi_0(p)}{C_{mm}rp} \left[ e^{-\frac{pz}{\gamma a}} - e^{-\frac{p}{a} \sqrt{\frac{z^2}{\gamma^2} + r^2}} \right] \tag{35}$$

Paying attention to the expression (35) in  $p$ , for the required value  $U_0(r, z, t)$ , we'll take the expression

$$U_0(r, z, t) = \frac{b}{\gamma C_{mm} r} \int_0^t f_1(t - \xi) \left[ H\left(\xi - \frac{z}{\gamma b}\right) - H\left(\xi - \frac{1}{b} \sqrt{\frac{z^2}{\gamma^2} + r^2}\right) \right] d\xi \quad (36)$$

where

$$f(r, t) = \frac{f_1(t)}{r}.$$

The resulting expression  $U_0(r, z, t)$  consists of two terms, the first term corresponding to plane wave propagating in half-space with velocity  $\gamma b$  and parallel to the plane  $z = 0$ , and the second term to the diffracted wave, which has the form of half-ellipsoid of revolution (hemisphere at  $\gamma = 1$ ) and in contact with plane wave on the rotation axis at  $z = b\gamma t$ .

Besides, from (36) follows, that  $U_0(r, z, t)$  fades out from  $r$  as  $1/r$ .

If the acting function  $f(r, t)$  is arbitrary, then we represent it in the form of Schlemmich series:

$$f(r, t) = \frac{1}{r} \sum_{j=0}^{\infty} a_j(t) J_0(jr) = \frac{f_r(t)}{r} \quad (37)$$

where

$$a_0(t) = \frac{1}{\pi} \int_0^{\pi} \left\{ f_r(0, t) + U \int_0^1 \frac{\partial f_r(\xi U, t)}{\sqrt{1 - \xi^2}} d\xi \right\} dU; \quad (38)$$

$$a_j(t) = \frac{2}{\pi} \int H \cos(jU) \left\{ \int_0^1 \frac{\partial f_r(\xi U, t)}{\sqrt{1 - \xi^2}} d\xi \right\} dU; j = 1, 2, \dots$$

For  $f(r, t)$  type (37) function  $f_1(\alpha, p)$ , entering to the formula (33), is

$$f_1(\alpha, p) = \frac{1}{\alpha} \sum_{j=0}^{\infty} a_{j0}(p) H(\alpha - j) \quad (39)$$

where

$$a_{j0}(p) = \int_0^{\infty} a_j(p) e^{-pt} dt. \quad (40)$$

Following,

$$U(r, z, p) = \frac{\gamma}{C_{66}} \sum_{j=0}^{\infty} a_{j0}(p) \int_0^{\infty} \frac{e^{-\frac{z}{\gamma} \sqrt{\alpha^2 + \frac{p^2}{b^2}}}}{\sqrt{\alpha^2 + \frac{p^2}{b^2}}} J_1(\alpha r) d\alpha \quad (41)$$

Paying attention (41) on  $p$  and using theory about convolution, we'll take



$$U_0(r, z, p) = \frac{\gamma}{2C_{66}} \sum_{j=0}^{\infty} \int_g^t a_j(t-\xi) T_j(r, z, \xi) d\xi \quad (42)$$

where

$$T_j(r, z, \xi) = b \int_j^{\infty} J_1(\alpha r) J_0 \left[ \alpha \sqrt{b^2 t^2 - \frac{z^2}{\gamma^2}} \right] d\alpha \quad (43)$$

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### ДӨҢГЕЛЕК ӨЗЕКТІҢ ТОЛҚЫНДЫҚ ПРОЦЕСТЕРІНІҢ ДИНАМИКАЛЫҚ ТҰРАҚТЫЛЫҒЫ

**Аннотация.** Бұл жұмыс жазық және айналмалы элементтердің толқындық процестерінің орнықтылық динамикасын зерттеуге арналған, сондай-ақ мақалада қалыпты немесе айналмалы жанау кернеулерінің әсер етуі кезінде қатты немесе деформацияланатын шекаралармен шектелген серпімді қабат тербелісінің кейбір осесимметриялық есептері қарастырылады. Қарастырылатын есептердің шешімдері координат бойынша немесе уақыт бойынша интегралды түрлендірулерді пайдалана отырып алынған. Жұмыста дөңгелек өзектің динамикалық тұрақтылығы жайлы есепін терең зеріттеледі. Дөңгелек өзектің орнықтылығының ауытқуы И. Г. Филипповтың жұмысында баяндалғандай, дөңгелек өзектің көлденең тербелісі математикалық теория негізінде зерттелетін болады.

**Түйін сөз:** тербелістер, орнықтылық, толқындық процесс, осесимметриялық есептер, дөңгелек өзек, экспоненциалды түрлендіру, жанама кернеу.

**А.Ж.Сейтмуратов<sup>1</sup>, Б.К.Жүсіпбек<sup>1</sup>, Г.К.Сыдыкова<sup>1</sup>, А.Сейтжанова<sup>2</sup>, У.Ж.Айтимова<sup>3</sup>**

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### ДИНАМИЧЕСКАЯ УСТОЙЧИВОСТЬ ВОЛНОВЫХ ПРОЦЕССОВ КРУГЛОГО СТЕРЖНЯ

**Аннотация:** Данная работа посвящена изучению динамики устойчивости волновых процессов плоских и круговых элементов, а также рассматриваются некоторые осесимметричные задачи колебания упругого слоя ограниченные жесткими или деформируемыми границами при воздействии на него нормального или вращательного касательного напряжений. Решения рассматриваемых задач получены с использованием интегральных преобразований по координате или по времени. В работе развивается динамическая устойчивость круглого стержня. Потеря устойчивости круглого стержня будет исследоваться на основе математической теории и поперечного колебания круглого стержня, изложенной в работе И.Г.Филиппова.

**Ключевые слова:** колебания, устойчивость, волновой процесс, осесимметричные задачи, круглый стержень, экспоненциальное преобразование, касательная напряжения.

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**SORTING ALGORITHMS AND COMPARISON OF THEIR  
EFFECTIVENESS**

**Abstract.** The present work is dedicated to the methods of sorting data and analysis of their complexity. There are several reasons for analysis of algorithms. One of them is necessity to evaluate the boundary values for the amount of memory or time required by some algorithm for successful data processing. The sorting process can be implemented by various algorithms. The choice of algorithm depends on the structure of the data being processed. In practice two classes of sorting are used: external and internal. If the amount of input data fits within the range of available internal RAM they say about the algorithms for internal sorting. But if the input data are stored in files, i.e. external memory, they say about external sorting.

This work demonstrates the fundamental algorithms of internal sorting with quadratic time and quick algorithms with  $O(n \cdot \log n)$  complexity. Quick sorting algorithms such as merge sorting and Hoare's quicksort algorithms are given. Also simpler methods of internal sorting such as exchange sort, Shell's method, insertion and selection algorithms are discussed as well. The article describes the idea behind these methods, algorithms on which they are based, complexity of these algorithms and provides concrete examples of programs.

**Keywords:** array, data, sorting, ordering, exchange sorting, insertion, selection, merge, quicksort, algorithm complexity.

**Introduction.** Regardless of whether you are a student or a professional programmer and in which sphere of activity you work it is necessary for you to obtain a knowledge of algorithms and data structures. They are crucial building blocks for solving problems.

The concept of an algorithm is not something completely new to us as we meet them on every step in our everyday life. This can be an algorithm for computing a mathematical function, an algorithm of a technological process, an algorithm of designing a computer or some engineering construction, etc.

In all areas of its activities, in particular, in the field of information processing, a person is faced with various methods of solving problems. They determine the order of actions to obtain the desired result - we can interpret this as the initial or intuitive definition of the algorithm.

An algorithm is a finite prescription given in a language, defining a finite sequence of executable elementary operations for solving a problem, common to a class of possible input data [1].

The main algorithms of processing data structures are sorting and search algorithms.

Sorting is one of the most important procedures for processing structured data. Sorting is the process of rearrangement of a given sequence of objects in some predefined order [2]. A certain order, e.g. increasing or decreasing in alphabetic order, in the sequence of objects is necessary for convenience of using this sequence.

It is much easier to work with ordered objects than with those arranged randomly. It becomes much simpler to search for an existing element or delete or insert a new one.

The goal of sorting is to facilitate the search for the next element in a pre-sorted sequence.

The process of sorting data can be implemented by various algorithms. The choice of algorithm depends on the structure of input data. There are two classes of sorting — sorting of arrays and sorting of

files[3,PP 43]. These classes are also called internal and external respectively. If the amount of input data allows to use internal RAM then we say about algorithms of internal sorting. If the data are stored in external memory, that is files, we say about external sorting. This work puts emphasis on fundamental algorithms of internal sorting. A great diversity of sorting algorithms leads to the necessity of their analysis to achieve their maximum efficiency.

**Methods.** To solve this problem, we used method of comparative analysis, methods of theoretical and research and also methods of the analysis of data.

**Results.** Before continuing it is necessary to introduce some terms and definitions.

Let us consider the sequence of  $n$  elements:  $a_1, a_2, \dots, a_n$ . Each record  $a_i$  has a key  $k_i$ , which manages the sorting process. The goal of sorting is to rearrange these elements in such a way that all keys are in a non-decreasing order:

$$k_1 \leq k_2 \leq \dots \leq k_n$$

A sorting algorithm is called stable if during the sorting process the relative order of the elements with equals keys do not change[3, PP 45].

The most important characteristic of a sorting algorithm is speed of its work which is determined by functional relation between average time of sorting the sequences of data elements with predefined length and the length itself. The sorting time is proportional to the number of comparison and reshuffling of data elements in the process of their sorting.

To evaluate the quality of algorithm it is necessary to define the term complexity (or effectiveness) of algorithm. The more time and the memory amount it takes to implement the algorithm the greater is its complexity and effectiveness[4]. The algorithm complexity are divided into capacity and time complexities. The time capacity is determined by the time taken by the implementation of algorithm. Capacity complexity is criterium indicating the memory overload for implementation of algorithm.

Solid algorithms of internal sorting require the number of comparisons to be equal to  $n \cdot \log n$  order. The following quick algorithms can serve as examples of such solid algorithms:

- merge sorting;
- partition sorting.

We will start the analysis with direct methods which are called simple ordering methods with time complexity of  $n^2$  order. This group of algorithms is presented by the following simple algorithm of pair sorting:

- selection sorting;
- exchange sorting;
- insertion sorting.

Though these algorithms are relatively slow, nevertheless they are convenient for use in describing the characteristic features of main principles of the majority of sorting methods.

Let us start discussing the quadratic time sorting with the exchange sort algorithms. Here the sorting is based on comparison of two elements. If the order of elements does not fit the required regularity then their exchange takes place. The process repeats until all the elements are arranged. Implementation of this algorithm in C++:

```
#include <iostream>
using namespace std;
int main()
{
    int *arr;
    int size;
    cout<<" Enter the number of elements in the array n =";
    cin>> size;
    arr = new int[size];
    for (int I = 0; I < size; i++) {
        cout<<"arr["<<I<<" ] = ";
        cin>>arr[i];
```

```

    }
    int temp;
    for (int I = 0; I < size - 1; i++) {
    for (int j = 0; j < size - I - 1; j++) {
    if (arr[j] > arr[j + 1]) {
    temp = arr[j];
    arr[j] = arr[j + 1];
    arr[j + 1] = temp;
        }
    }
    }
    for (int I = 0; I < size; i++) {
    cout << arr[i] << " ";
    }
    cout << endl;
    delete [] arr;
    return 0;
}

```

Bubble sort is a specific case of exchange sort. Its idea is reflected by its name. The heaviest elements go to the top of the sequence, while the lightest are placed at the bottom. The sequence of  $n$  data elements is compared from the very beginning to the very end so that adjacent elements are swapped if the first of them is lesser (or lighter) than the second. After the comparison ends the lightest element is dragged to the bottom of the sequence [5,6].

The next algorithm with quadratic time is the selection sort. The idea of the method is that in the beginning the smallest element is selected and separated from others. Then it changes places with the very first elements. After that this operation is repeated with the remaining  $n-1$  elements. The whole process is repeated until all elements are put in their appropriate places. Its algorithm looks as follows:

```

#include <iostream>
using namespace std;
int main()
{
    int *arr;
    int size;
    int temp;
    int j;
    cout << " Enter the number of elements in the array n = ";
    cin >> size;
    arr = new int[size];
    for (int I = 0; I < size; i++) {
    cout << "arr[" << I << "] = ";
    cin >> arr[i];
    }
    for (int i = 0; i < size - 1; i++)
    {
    int min = arr[i];
    int prs = i;
    for (j = i + 1; j < size; j++)
    if (arr[j] < min)
        {
    min = arr[j];
    prs = j;
    temp = arr[i];
    arr[i] = arr[prs];

```

```
arr[prs]=temp;

}
}
for (intI = 0; I< size; i++) {
cout<<arr[i] <<" ";
}

return 0;
}
```

The insertion sort selects sequentially each element from unordered sequence of elements, compares it to a pre-ordered element and then places it in an appropriate place.

The algorithm of this method is as follows:

- at the first stage two initial elements are compared. If the next element is lesser than the first then we swap their places, i.e. the next element is moved to the place of the previous element and this previous element is shifted to the next position to the right;
- at the second stage we select an element from unordered sequence and compare it to the two previously ordered elements. If is greater than these previous elements then it retains its place. Else if it is lesser then it is shifted to the appropriate place;
- all remaining elements are analysed the similar way until the whole sequence is ordered.

The code of this method is as follows:

```
#include <iostream>
using namespace std;
int main()
{
int *arr;
int size;
int temp;
intI;
int j;
cout<<"Enter the number of elements in the array n =";
cin>> size;
arr = new int[size];
for (I = 0; I< size; i++) {
cout<<"arr["<<I<<" ] = ";
cin>>arr[i];
}
for (i=1; i<size;i++)
{
for ( j=i-1; j>=0;j--)
if (arr[j]>arr[j+1]) {
temp=arr[j];
arr[j]=arr[j+1];
arr[j+1]=temp;
}
}
for (I = 0; I< size; i++) {
cout<<arr[i] <<" ";
}
return 0;
}
```

3	2	5	6	23	4	17	7	1	9
2	3	5	6	23	4	17	7	1	9
2	3	5	6	23	4	17	7	1	9
2	3	5	6	23	4	17	7	1	9
2	3	5	6	23					
2	3	4	5	6	23	17	7	1	9
2	3	4	5	6	17	23			
2	3	4	5	6	7	17	23		
1	2	3	4	5	6	7	17	23	
1	2	3	4	5	6	7	9	17	23

The method of direct insertion was improved by D.Shell. The Shell's method does not compare neighbouring elements. Instead it compares elements located at the distance, where  $d$  — is number of steps between compared elements. If the sequence consists of  $n$  elements the initial value of  $d = \lceil n/2 \rceil$ . After each comparison the  $d$  is decreased double times. At the last comparison it is increased to  $d=1$ . In the end such method outputs an ordered sequence. Its implementation looks as follows:

```
#include <iostream>
using namespace std;
int main()
{
int *arr;
int size,d;
int temp;
int I;
int j;
cout<<"Enter the number of elements in the array n =";
cin>> size;
arr = new int[size];
for (I = 0; I< size; i++) {
cout<<"arr["<<I<<" ] = ";
cin>>arr[i];
}
{
d=size;
d=d/2;
while (d>0)
{
for (i=0; i<size-d; i++)
{
j=i;
while (j>=0 &&arr[j]>arr[j+d])
{
temp=arr[j];
arr[j]=arr[j+d];
arr[j+d]=temp;
j--;
}
}
d=d/2;
}
}
```

```
for (i=0; i<size ; i++)
cout<<arr[i]<<" ";
}
}
```

The average time of the algorithm's complexity depends on length of intervals —  $d$  which contain the sorted elements of source array of capacity  $N$  on each step of algorithm.

Now let us cover the quicksort algorithms. One version of the quicksort algorithms is merge sort. It works the following way:

- the sequence is divided to two equal parts;
- each part is sorted separately;
- separately sorted parts of the source sequence are merged.

Now let us provide the program implementation of this algorithm:

```
#include <iostream>
using namespace std;
int arr[100];
int size;
void merge(int l, int r) {
if (r == l) return;
if (r - l == 1) {
if (arr[r] < arr[l])
swap(arr[r], arr[l]);return;
}
int m = (r + l) / 2;
merge(l, m);
merge(m + 1, r);
int buf[100];
int xl = l;
int xr = m + 1;
int cur = 0;
while (r - l + 1 != cur) {
if (xl > m)
buf[cur++] = arr[xr++];
else if (xr > r)
buf[cur++] = arr[xl++];
else if (arr[xl] > arr[xr])
buf[cur++] = arr[xr++];
else buf[cur++] = arr[xl++];
}
for (int i = 0; i < cur; i++)
arr[i + l] = buf[i];
}
int main() {
cout<<"Enter the number of elements in the array n =";cin>> size;
for (int i = 0; i < size; i++)
cin>>arr[i];
merge(0, size - 1);
for (int i = 0; i < size; i++)
cout<<arr[i] <<" ";
return 0;
}
```



The next algorithm was invented by T.Hoare. In practice it is generally considered to be one of the most effective quicksort algorithms. This algorithm is known as quicksort algorithm and its complexity equals  $O(n \cdot \log n)$ . The quicksort algorithm belongs to the group of divide-and-conquer algorithms.

The essence of this algorithm is as follows. A key element is selected and fixed. With respect to this element all other elements with larger weight are shifted right and the element with lesser weight are shifted left. Also with respect to the selected key the whole sequence is divided into two parts and for each part the process is repeated. Let us provide the code of the quicksort algorithm where the role of the key element is played by the central element of the sorted sequence:

```
#include <iostream>
using namespace std;
int first, last;
// sort function
void sort(int* arr, int first, int last)
{
    int i = first, j = last;
    double tmp, x = arr[(first + last) / 2];

    do {
        while (arr[i] < x)
            i++;
        while (arr[j] > x)
            j--;

        if (i <= j)
        {
            if (i < j)
            {
                tmp=arr[i];
                arr[i]=arr[j];
                arr[j]=tmp;
            }
            i++;
            j--;
        }
    } while (i <= j);

    if (i < last)
        sort(arr, i, last);
    if (first < j)
        sort(arr, first, j);
}
//main function
int main()
{
    int size;
    cout<<"Enter the number of elements in the array n =";    cin>> size;
    int *arr=new int[size];
    for (int i=0; i<size; i++)
    {
        cout<<"arr["<<i<<" ] = ";
        cin>>arr[i];
    }
}
```

```
first=0; last=size-1;
sort (arr, first, last);
for (inti=0; i<size; i++)
cout<<arr[i]<<" ";
}
```

The merge sort algorithm based on division of the source sequence into separate parts was pretty simple, while the process of merging the sorted parts was much more complicated. On the contrary, in the partition sort algorithm the most complicated part was dividing array into parts, while the process of merging these parts was much simpler.

**Conclusions.** Finishing our review and analysis of sorting methods we attempted to compare their effectiveness. In our opinion the essence of each method is to provide effective means for rearranging given sequence in increasing or decreasing order. It is safe to say that algorithms with quadratic time are easier to understand and use while the quicksort algorithms are harder to understand but at the same time more efficient. We think that it is due to user itself to select the appropriate method for his specific case.

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#### СҰРЫПТАУ АЛГОРИТМДЕРІ ЖӘНЕ ОЛАРДЫҢ ТИІМДІЛІКТЕРІН САЛЫСТЫРУ

**Аннотация.** Бұл мақала деректерді сұрыптау әдістеріне және олардың өнімділігін талдауға арналған. Алгоритмдерді талдаудың бірқатар маңызды себептері бар. Олардың бірі – деректерді өңдеу үшін алгоритмге қажет болатын бағалау, жады көлемі үшін шекаралар немесе жұмыс уақытын алу қажеттілігі болып табылады. Деректерді сұрыптау процесі түрлі алгоритмдер арқылы жүзеге асырылуы мүмкін. Алгоритмді таңдау өңделетін деректер құрылымына тәуелді болады. Іс жүзінде екі сұрыптау класы қолданылады: ішкі және сыртқы. Егер кіріс деректерінің көлемі жедел, ішкі жадымен шектелетін болса, онда ішкі сұрыптау алгоритмдері туралы, ал егер деректер файлдарда орналастырылса, яғни, сыртқы жадыда, онда сыртқы сұрыптау туралы айтылады.

Бұл жұмыста біз ішкі сұрыптаудың негізгі: күрделілігі квадраттық уақытқа тең және  $O(n * \log n)$  күрделілікке тең жылдам сұрыптау алгоритмі деп аталатын алгоритмдерді қарастырамыз. Сұрыптаудың жылдам алгоритмдері: біріктіру арқылы сұрыптау, Хоара жылдам сұрыптауы және неғұрлым қарапайым ішкі сұрыптау әдістері: алмастыру көмегімен, тікелей кірістіру арқылы сұрыптау, Шелл әдісі, таңдау алгоритмдерінің жұмыстары келтіріледі. Мақалада бұл әдістердің негізгі идеясы мен мәнісі, жұмыс алгоритмі, алгоритмдердің күрделілігі ескеріледі, бағдарламалар мысалдары келтіріледі.

**Түйін сөздер:** массив, деректер, сұрыптау, реттілік, алмастыру, қою, біріктіру арқылы сұрыптау, жылдам сұрыптау, алгоритмнің күрделілігі.

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#### АЛГОРИТМЫ СОРТИРОВКИ И СРАВНЕНИЕ ИХ ЭФФЕКТИВНОСТИ

**Аннотация.** Данная статья посвящена методам сортировки данных и их анализа трудоемкости. Существует ряд важных причин для анализа алгоритмов. Одной из них является необходимость получения оценок или границ для объема памяти или времени работы, которое потребуется алгоритму для успешной обработки данных. Процесс сортировки данных может быть осуществлен различными алгоритмами. Выбор алгоритма зависит от структуры обрабатываемых данных. На практике применяется два класса сортировки: внутренней и внешней. Если объем входных данных позволяет обходиться оперативной, внутренней памятью, то говорят об алгоритмах внутренней сортировки, а если данные размещаются в файлы, т.е. внешней памяти, то речь идет о внешней сортировке.

В данной работе мы продемонстрируем основные алгоритмы внутренней сортировки: с квадратичным временем и алгоритмы сортировки которые называются быстрыми и имеют трудоемкость  $O(n * \log n)$ .

Приводятся быстрые алгоритмы сортировки, такие как сортировка слиянием, быстрая сортировка Хоара. Более простые методы внутренней сортировки, такие как сортировка с помощью обмена, с помощью прямого включения, метод Шелла, алгоритмы выбора. В статье рассматривается идея и суть этих методов, алгоритм работы, трудоемкость этих алгоритмов, приводятся примеры программ.

**Ключевые слова:** массив, данные, сортировка, упорядочивание, сортировка обмена, вставка, выбор, слияние, быстрая сортировка, трудоемкость алгоритма.

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