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ИЗВЕСТИЯ

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РЕСПУБЛИКИ КАЗАХСТАН

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ
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AL-FARABI KAZAKH
NATIONAL UNIVERSITY

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PHASE PORTRAITS OF THE HENON-HEILES POTENTIAL

Abstract. In this paper the Henon-Heiles potential is considered. In the second half of the 20th century, in astronomy the model of motion of stars in a cylindrically symmetric and time-independent potential was studied. Due to the symmetry of the potential, the three-dimensional problem reduces to a two-dimensional problem; nevertheless, finding the second integral of the obtained system in the analytical form turns out to be an unsolvable problem even for relatively simple polynomial potentials. In order to prove the existence of an unknown integral, the scientists Henon and Heiles carried out an analysis of research for trajectories in which the method of numerical integration of the equations of motion is used. The authors proposed the Hamiltonian of the system, which is fairly simple, which makes it easy to calculate trajectories, and is also complex enough that the resulting trajectories are far from trivial. At low energies, the Henon-Heiles system looks integrable, since independently of the initial conditions, the trajectories obtained with the help of numerical integration lie on two-dimensional surfaces, i.e. as if there existed a second independent integral. Equipotential curves, the momentum and coordinate dependences on time, and also the Poincaré section were obtained for this system. At the same time, with the increase in energy, many of these surfaces decay, which indicates the absence of the second integral. It is assumed that the obtained numerical results will serve as a basis for comparison with analytical solutions.
Keywords: Henon-Heiles model, Poincaré section, numerical solutions.

Introduction. Interest in the existence of the third integral of motion for stars moving in the potential of the galaxy revived in the late 50's and early 60's of the last century. Initially it was assumed that the potential has a symmetry and does not depend on time, therefore in cylindrical coordinates (r, θ, z) this will be only a function of r and z. There must be five integrals of motion that are constant for the six-dimensional phase space. However, the integrals can be either isolating or non-isolating. Non-isolating integrals usually fill all available phase spaces and do not restrict the orbit.

By the time Henon and Heiles wrote their pioneer article, there were only two known integrals of motion: total orbital energy and angular momentum per unit mass of the star. It is easy to show that at least two integrals are not isolated. It was also assumed that the third integral was also not isolated, because no analytical solution has been found so far. Nevertheless, observations of stars near the Sun, as well as numerical calculations of the orbits, behaved in some cases as if they obeyed the three isolating integrals of motion.

Henon and Heiles tried to find out if they could find any real proof that there must be a third isolating integral of the motion. Making numerical calculations, they did not complicate the astronomical meaning of the problem; they only demanded that the potential investigated by them be axially symmetric. The authors also suggested that the motion was tied to a plane and passed into the Cartesian phase space (x, y, $\dot{x}, \dot{y}$). After some tests they managed to find a real potential. This potential is analytically simple, so that the orbits can be calculated quite easily, but it is still quite complex, so that the types of orbits are nontrivial. This potential is now known as the potential of Henon and Heiles [1-3].

Methods and calculations. The Henon-Heiles potential is undoubtedly one of the simplest, classical and characteristic examples of open Hamiltonian systems with two degrees of freedom. The above topic was devoted to a large number of research scientists [4-25].

The potential of the Henon-Heiles system is determined by the formula:

$$U(x,y)=\frac{1}{2}(x^2+y^2+2x^2y-\frac{2}{3}y^3) \tag{1}$$

Equation (1) shows that the potential actually consists of two harmonic oscillators, which were connected by the perturbing terms $x^2y-\frac{1}{3}y^3$.

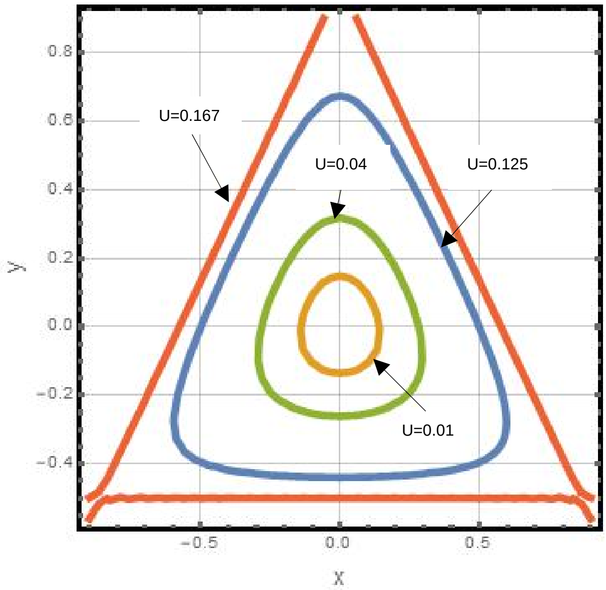

Figure 1 - Closed equipotential curves for the Henon-Heiles model for different values of U

The basic equations of motion for a test particle with a unit mass $(m=1)$ are:

$$\begin{cases} \ddot{x}=-\frac{\partial U}{\partial x}=-x-2xy \\ \ddot{y}=-\frac{\partial U}{\partial y}=-y-x^2+y^2 \end{cases} \tag{2}$$

Consequently, the Hamiltonian of system (1) has the form:

$$H=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}\left(x^2+y^2\right)+x^2y-\frac{1}{3}y^3=h, \tag{3}$$

where $\dot{x}$ and $\dot{y}$ are the momenta per unit mass, x and y are the coordinates of the system; $h>0$ the numerical value of the Hamiltonian, which is conserved. It is seen that $h>0$ the Hamiltonian is symmetric with respect to $x\to -x$, and H also exhibits a symmetry of rotation at $2\pi\ /\ 3$.

Below are the dependencies of the coordinates of the functions in time for the systems of equations (2).

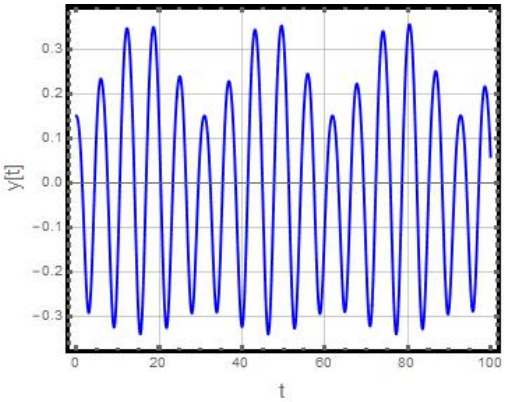

Figure 2 - Dependence of the function y on time

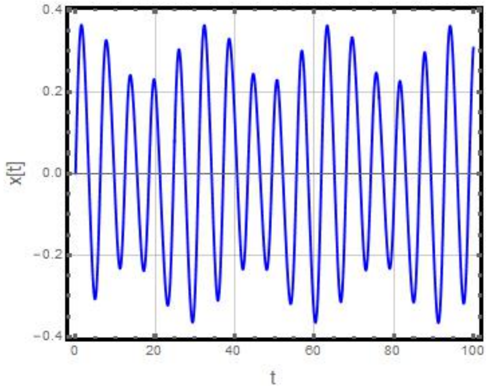

Figure 3 - Dependence of the function x on time

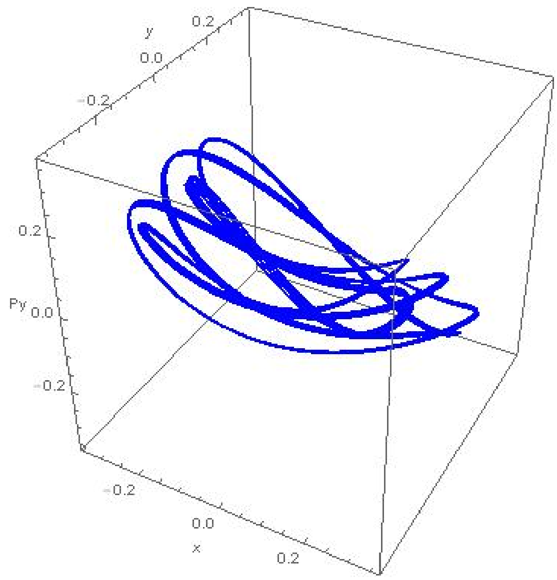

Figure 4 - Evolutionary trajectories of the functions x, y, Py

To study the Henon-Heiles system, the Poincaré section method is used. Advantages of this method are especially evident when we consider nonlinear systems for which exact solutions are unknown. In this case, the phase trajectories are calculated by numerical methods.

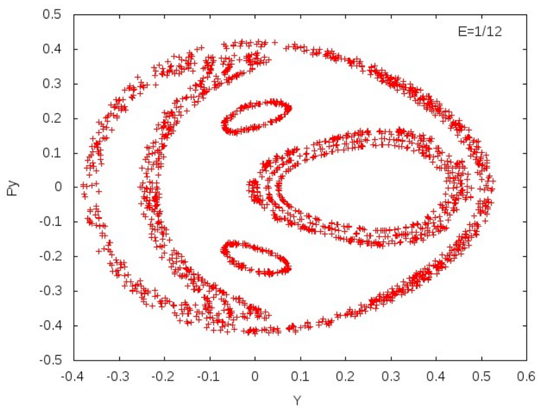

Figure 5 - Poincare section at $E = 1/12$.

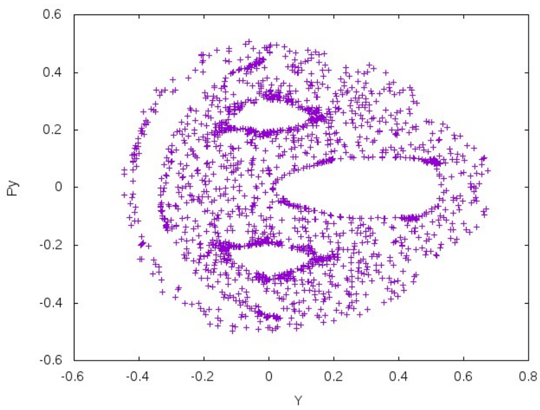

Figure 6 - Poincare section at $E = 1/8$.

To solve the systems of equations (2), boundary conditions are chosen so that they satisfy equation (3). Further, the systems of equation (2) are solved on the basis of the Runge-Kutta method. To construct the Poincaré section, those values that intersect the plane $x = 0$ are chosen. Below are the Poincaré sections for Henon-Heiles systems for different energy values: $E = 1/12$, $E = 1/8$, $E = 1/6$. With increasing energy, the structure of the cross sections is destroyed. The results obtained are in agreement with other authors [1, 2].

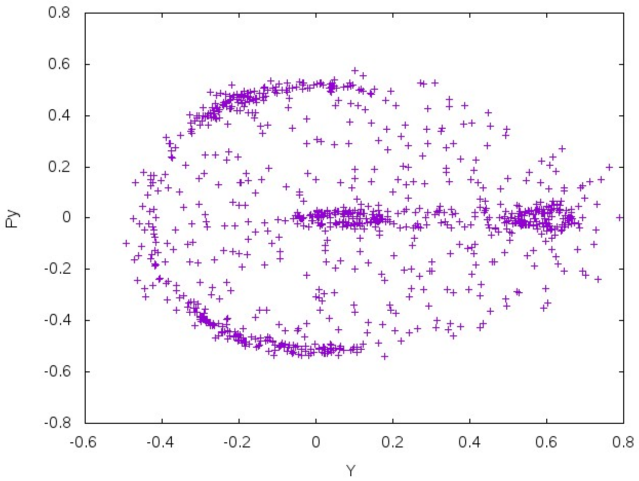

Figure 8 - Poincare section at $E = 1/6$.

Conclusion. Thus, the results obtained by the numerical method determine the oscillations for the Henon-Heiles model and serve as the basis for a comparative analysis in determining the analytical mapping.

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ФАЗОВЫЕ ПОРТРЕТЫ ПОТЕНЦИАЛА ХЕНОНА-ХЕЙЛЕСА

Аннотация. В данной работе исследуется потенциал Хенона-Хейлеса. Во второй половине XX века в астрономии изучались модели движения звезд в цилиндрически симметричном и не зависящем от времени потенциале. Из-за симметрии потенциала трехмерная задача сводится к двумерной, тем не менеенахождение второго интеграла полученной системы в аналитическом виде оказывается неразрешимой задачей даже для сравнительно простых полиномиальных потенциалов. Чтобы доказать существование неизвестного интеграла, ученые Хенон и Хейлес провели анализ исследований для траекторий, в котором используютметод численного интегрирования уравнений движения. Авторы предложили гамильтониан системы, который достаточно прост, что позволяет легко вычислять траектории, а также достаточно сложен, чтобы полученные траектории оказались далеко не тривиальными. При малых энергиях система Хенона-Хейлеса выглядит интегрируемой,так как независимо от начальных условий, траектории, полученные с помощью численного интегрирования, лежат на двумерных поверхностях, т.е. так, как если бы существовал второй независимый интеграл. Для данной системы были получены эквипотенциальные кривые, зависимости импульса и координаты от времени, также сечение Пуанкаре. В то же время с увеличением энергии многие из этих поверхностей распадаются, что указывает на отсутствие второго интеграла. Предполагается что, полученные численные результаты, послужат основой для сравнения с аналитическими решениями.

Ключевые слова: Модель Хенона-Хейлеса, сечение Пуанкаре, численные решения.

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ХЕНОН-ХЕЙЛЕС ПОТЕНЦИАЛЫНЫҢ ФАЗАЛЫҚ БЕЙНЕСІ

Аннотация.Берілген мақалада Хенон-Хейлес моделі қарастырылады. 20 ғасырдың екінші жартысында жұлдыздар қозғалысының цилиндрлік симметриялы және уақыттан тәуелсіз потенциал моделі зерттелді. Потенциалдың симметриялығы арқасында үш өлшемді есепті екі өлшемді есеп ретінде қарастыруға болады, алайда берілген жүйенің аналитикалық түрде екінші интегралын табу, тіпті салыстырмалы түрде қарапайым полиномиальды потенциалдар үшін шешілмейтін мәселе болып табылады. Белгісіз интегралдың бар екенін дәлелдеу үшін, Хенон мен Хейлес қозғалыс теңдеулерінің сандық интегралдау әдісін пайдаланып траектория бойынша зерттеулер жасады. Авторлар жүйенің гамильтонианын ұсынды, ол өте қарапайым және траекторияны есептеуді жеңілдетеді, сонымен қатар алынған траекториялар тривиалды емес күрделі болып табылады. Энергияның төмен деңгейлерінде Хенон-Хейлес жүйесі интегралданады, екінші белгісіз интергралы бар секілді бастапқы шарттардан тәуелсіз жүйенің траекториясы сандық интегралдық әдіспен шешіліп, екі өлшемді кеңістікте сипатталады. Берілген жүйе үшін эквипотенциалдық қисықтар, импульс пен координаттың уақыттан тәуелділігі, сондай-ақ Пуанкаре қимасы алынды. Сонымен қатар, энергияның өсуімен, осы беттердің көпшілігі ыдырайды, бұл екінші интегралдың болмауын көрсетеді. Алынған сандық нәтижелер аналитикалық шешімдермен салыстыру үшін негіз болады деп болжануда.

Түйін сөздер: Хенон-Хейлес моделі, Пуанкаре қимасы, сандық шешімдер.

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THE METHOD OF NUMERICAL SOLUTION OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

Abstract. When considering systems of differential equations with very general boundary conditions, exact solution methods encounter great difficulties, which become insurmountable in the study of nonlinear problems. In this case it is necessary to apply to certain numerical methods. It is important to note that the use of numerical methods often allows you to abandon the simplified interpretation of the mathematical model of the process. The problems of numerical solution of nonlinear Volterra integral equations of the first kind with a differentiable kernel, which degenerates at the initial point of the diagonal, are studied in the paper. This equation is reduced to the Volterra integral equation of the third kind and a numerical method is developed on the basis of that regularized equation. The convergence of the numerical solution to the exact solution of the Volterra integral equation of the first kind is proved, an estimate of the permissible error and a recursive formula of the computational process are obtained.

Keywords: nonlinear integral equation, system of nonlinear algebraic equations, error vectors, the Volterra equation, small parameter, numerical methods.

Introduction

The problem of solving integral equations arises as an auxiliary problem for solving boundary value problems for partial differential equations and as an independent one in the study of the operation of nuclear reactors, in solving the so-called inverse problems of geophysics, in processing the results of observations, and so on. We confine ourselves to the case of nonlinear Volterra integral equations of the first kind.

Questions about the numerical solution of linear integrated equations of Vol'terraof the first sort are explored in two case, when source K(x,t) on diagonal doesn't return zero in any points of a section and the source on diagonal is identical zero, a derivative on x on diagonal doesn't return zero in any points of the section [4-7]. In this research we considered the case non-liner integrated equations Vol'terra of the first sort with allocated source, which can return zero in some points of the section of the diagonal.

Formulation of the problem

Consider the nonlinear Volterra integral equation of the first kind:

$$\int_0^x N_0\big(x,t,u(t)\big)dt = g(x), \qquad (1)$$

where $N_0\big(x,t,u(t)\big) = K(x,t)u(t) + N\big(x,t,u(t)\big)$ and known functions $K(x,t), N\big(x,t,u(t)\big)$, $g(x)$ obey conditions:

a) $g(x) \in C^2[0,b], K(x,t) \in C^{2,1}(D), D = \{(x,t)/0 \le t \le x \le b\}$,

$g^{(i)}(0) = 0, i = 0,1, k(x) = K(x,x), k(0) = 0, 0 < k(x)\ \forall x \in (0,b]$, $k(x)$ – nondecreasing function;

b) $G(x) \ge d_1, G(x) = L(x,x) + C_1 g(x), L(x,t) = C_2 K(x,t) + K_x(x,t), 0 < d_1, C_1, C_2 = const$;

c) $\boldsymbol{N(x,t,u) \in C^{1,0,1}(D \times R^1), M_0(x,t,u) \in C^{0,0,1}(D \times R^1), M_0(x,x,u) = 0, M_0(x,t,u(t)) = C_2 N(x,t,u(t)) + N_x(x,t,u(t))}$, **for** $\boldsymbol{x > t, (x,s), (t,s) \in D, (x,s,u), (x,s,w), (t,s,w), (t,s,u) \in D \times R^1}$ **the following inequality holds true**

$\boldsymbol{|M_0(x,s,u) - M_0(x,s,w) - M_0(t,s,w) + M_0(t,s,u)| \le L_N(x-t)|u-w|}$,

$0 < L_N = const.$

The research and solution of the basic equation

We get Volterra integral equation of the third kind from equation (1) after applying $D + C_1 T + C_2 I$, where I is an identical operator, *D* is an operator of differentiation with respect to x, *T* is Volterra operator kind of $(Tv)(x) = \int_0^x u(t)v(t)dt$, [1]:

$$k(x)u(x) + \int_0^x G(t)u(t)dt = \int_0^x M(x,t,u(t))dt + C_1 \int_0^x u(t)dt \times$$
$$\times \int_t^x K(s,t)u(s)ds + C_1 \int_0^x \int_t^x N(s,t,u(t))u(s)ds\,dt + f(x), (1_0)$$

where $M(x,t,u(t)) = -M_0(x,t,u(t)) + (L(t,t) - L(x,t))u(t), \quad f(x) = C_2 g(x) + g'(x).$

Consider regularized equation with a small parameter

$$(\varepsilon + k(x))u_\varepsilon(x) + \int_0^x G(t)u_\varepsilon(t)dt = \int_0^x M(x,t,u_\varepsilon(t))dt + C_1 \int_0^x u_\varepsilon(t)dt \times$$
$$\times \int_t^x K(\tau,t)\,u_\varepsilon(\tau)d\tau + C_1 \int_0^x \int_t^x N(\tau,t,u_\varepsilon(t))u_\varepsilon(\tau)\,d\tau\,dt + \varepsilon u(0) + f(x), \qquad (2)$$

Transform equation (2) to the following form

$$u_\varepsilon(x) = -\frac{1}{\varepsilon + k(x)} \int_0^x exp\left(-\int_t^x \frac{G(\tau)}{\varepsilon + k(\tau)}d\tau\right) \frac{G(t)}{\varepsilon + k(t)} \left\{ \int_0^t M(t,\tau,u_\varepsilon(\tau))d\tau - \right.$$
$$- \int_0^x M(x,\tau,u_\varepsilon(\tau))d\tau - C_1 \left[\int_0^t u_\varepsilon(\tau)d\tau \int_s^t K(\nu,\tau)\,u_\varepsilon(\nu)d\nu + \right.$$
$$+ \int_0^x u_\varepsilon(\tau)d\tau \int_s^x K(\nu,\tau)\,u_\varepsilon(\nu)d\nu - \int_0^t \int_\tau^t N(\nu,\tau,u_\varepsilon(\tau))u_\varepsilon(\nu)d\nu\,d\tau +$$
$$\left. \left. + \int_0^x \int_\tau^x N(\nu,\tau,u_\varepsilon(\tau))u_\varepsilon(\nu)d\nu\,d\tau \right] + f(t) - f(x) \right\} dt + \frac{1}{\varepsilon + k(x)} \times$$
$$\times exp\left(-\int_0^x \frac{G(\tau)}{\varepsilon + k(\tau)}d\tau\right) \left\{ \int_0^x M(x,t,u_\varepsilon(t))\,dt + C_1 \int_0^x u_\varepsilon(t)dt \times \right.$$
$$\left. \times \int_t^x K(\tau,t)\,u_\varepsilon(\tau)d\tau + \int_0^x \int_t^x N(\tau,t,u_\varepsilon(t))u_\varepsilon(\tau)d\tau\,dt \right] + \varepsilon u_{0,h} + f(x) \Big\}. \qquad (3)$$

Introduce a uniform grid $\omega_h = \{x_i = ih, i = 0..n, b = nh\}$ on the $[0,b]$ segment, *n* – natural number and C_h – space of grid functions $u_i = u(x_i)$ with the following norm

$$\|u_i\|_{C_h} = \max_{0 \le i \le n} |u_i|.$$

Using the right Riemann sum and replacing $u(0)$ to $u_{0h} = f_1/(hG_1 + k_1)$, we obtain the next system of nonlinear algebraic equations from equation (3):

$$u_{\varepsilon,i} = -\frac{1}{\varepsilon + k_i} h \sum_{j=1}^{i-1} exp\left(-h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon + k_s}\right) \frac{G_j}{\varepsilon + k_j} \times$$

$$\times \left\{ h \sum_{s=1}^{j} [M_{j,s}(u_{\varepsilon,s}) - M_{i,s}(u_{\varepsilon,s})] - h \sum_{s=j+1}^{i-1} M_{i,s}(u_{\varepsilon,s}) - C_1 h \sum_{s=1}^{j-1} u_{\varepsilon,s} \times \right.$$

$$\times h \sum_{m=j+1}^{i} K_{m,s} u_{\varepsilon,m} - C_1 h \sum_{s=j+1}^{i-1} u_{\varepsilon,s}\, h \sum_{m=s+1}^{i} K_{m,s} u_{\varepsilon,m} - C_1 h \sum_{s=1}^{j-1} h \times$$

$$\left. \times \sum_{m=j+1}^{i} N_{m,s}(u_{\varepsilon,s})\, u_{\varepsilon,m} - C_1 h \sum_{s=j}^{i-1} h \sum_{m=s+1}^{i} N_{m,s}(u_{\varepsilon,s}) u_{\varepsilon,m} + f_j - f_i \right\} +$$

$$+ \frac{1}{\varepsilon + k_i} exp\left(-h \sum_{s=1}^{i} \frac{G_s}{\varepsilon + k_s}\right) \left\{ h \sum_{j=1}^{i-1} M_{i,j}(u_{\varepsilon,j}) + C_1 h \sum_{j=1}^{i-1} u_{\varepsilon,j}\, h \sum_{s=j+1}^{i} K_{s,j} u_{\varepsilon,s} + \right.$$

$$\left. + C_1 h \sum_{j=1}^{i-1} h \sum_{s=j+1}^{i} N_{s,j}(u_{\varepsilon,j})\, u_{\varepsilon,s} + \varepsilon u_{0,h} + f_i \right\}, i = 1..n, \quad (4)$$

where $M_{i,j}(u_{\varepsilon,j}) = M(x_i, x_j, u(x_j)), f_i = f(x_i), x_j = jh, j = 1..i, i = 1..n.$

Introduce the notations

$$q = \frac{d_2 b T_0}{d_1}(L_2 + C_2 L_1 + L_N)\left(2\frac{h}{\varepsilon} + e^{-1}\right) + \frac{2C_1 T_0 M b r d_2 h}{d_1 \varepsilon} +$$

$$+ C_1 b \left(\frac{2T_0 d_2 h}{d_1 \varepsilon} + \frac{1}{e d_1}\right)(M_N + K_N r), M = \max_D |K(x,t)|, |u_\varepsilon(x)| \le r.$$

$$L_1 = \max_D |K_x(x,t)|, L_2 = \max_D |K_{xx}(x,t)|, T_0 = \max_{x \in [0,b]} |G(x)|,$$

$$d_2 = sup\left(\sum_{j=1}^{i-1} exp\left(-h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon + k_s}\right)\left(h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon + k_s}\right)\right),$$

$$M_N = \max_{D \times R^1} |N(x,t,u)|, K_N = \max_{D \times R^1} |N_u(x,t,u)|,$$

Theorem. If the conditions *a)–c),* $q < 1$ and $\varepsilon = O(h^\alpha)$ for all $0 < \alpha < 1/2$ are satisfied, then the solution of the system of equations (4) converges uniformly to the exact solution u_i of equation (1) when $h \to 0$, thus we have the estimate

$$\|u_{\varepsilon,i} - u_i\| \le N_1 h^\alpha + N_2 h^{1-\alpha} + N_3 h, 0 < N_i = const, i = 1,2,3.$$

Proof. Adding the quantity $\varepsilon u(x)$ to both sides of equation (1_0), we reduce it to the form (3), where $u_\varepsilon(x)$ and εu_0 are respectively instead of $u(x)$ and $\varepsilon u(x)$. Putting $x = x_i, i = 1..n$ in the obtained equation, we use the formula of the right Riemann sum and consider the difference of the resulting system of algebraic equations with the remainder term and the system of equations (4). Then, using the error vector $\eta_{\varepsilon,i}^h = u_\varepsilon(x_i) - u(x_i), i = 1..n$, we obtain

$$\eta_{\varepsilon,i}^{h} = -\frac{1}{\varepsilon+k_i} h \sum_{j=1}^{i-1} exp\left(-h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon+k_s}\right) \frac{G_j}{\varepsilon+k_j} \left\{ h \sum_{s=1}^{j-1} [M_{j,s}(u_{\varepsilon,s}) - \right.$$

$$-M_{j,s}(u_s) - M_{i,s}(u_{\varepsilon,s}) + M_{i,s}(u_s)] - h \sum_{s=j+1}^{i-1} [M_{i,s}(u_{\varepsilon,s}) - M_{i,s}(u_s)] -$$

$$-C_1 h \sum_{s=1}^{j-1} u_s\, h \sum_{m=j+1}^{i} K_{m,s} \eta_{\varepsilon,m}^{h} - C_1 h \sum_{s=j+1}^{i-1} u_s\, h \sum_{m=s+1}^{i} K_{m,s}\, \eta_{\varepsilon,m}^{h} -$$

$$- - C_1 h \sum_{s=1}^{j-1} \eta_{\varepsilon,s}^{h}\, h \sum_{m=j+1}^{i} K_{m,s}\, u_m - C_1 h \sum_{s=j+1}^{i-1} \eta_{\varepsilon,s}^{h}\, h \sum_{m=s+1}^{i} K_{m,s} u_m -$$

$$-C_1 h \sum_{s=1}^{j-1} h \sum_{m=j+1}^{i} N_{m,s}(u_s)\, \eta_{\varepsilon,m}^{h} - C_1 h \sum_{s=j+1}^{i-1} h \sum_{m=s+1}^{i} N_{m,s}(u_s)\, \eta_{\varepsilon,m}^{h} -$$

$$-C_1 h \sum_{s=1}^{j-1} h \sum_{m=j+1}^{i} [N_{m,s}(u_{\varepsilon,s}) - N_{m,s}(u_s)] u_m -$$

$$\left. -C_1 h \sum_{s=j+1}^{i-1} h \sum_{m=s+1}^{i} [N_{m,s}(u_{\varepsilon,s}) - N_{m,s}(u_s)] u_m + \varepsilon(u_j - u_i) \right\} +$$

$$+\frac{1}{\varepsilon+k_i} exp\left(-h \sum_{s=1}^{i} \frac{G_s}{\varepsilon+k_s}\right) \left\{ h \sum_{j=1}^{i-1} [M_{i,j}(u_{\varepsilon,j}) - M_{i,j}(u_j)] + \right.$$

$$+C_1 h \sum_{j=1}^{i-1} \eta_{\varepsilon,j}^{h} \sum_{s=j+1}^{i} K_{s,j} u_s + C_1 h \sum_{j=1}^{i-1} u_j \times h \sum_{s=j+1}^{i} K_{s,j} \eta_{\varepsilon,s}^{h} +$$

$$+C_1 h \sum_{j=1}^{i-1} h \sum_{s=j+1}^{i} [N_{s,j}(u_{\varepsilon,j}) - N_{s,j}(u_j)]\, u_s +$$

$$\left. +C_1 h \sum_{j=1}^{i-1} h \sum_{s=j+1}^{i} N_{s,j}(u_j) \eta_{\varepsilon,s}^{h} + \varepsilon u_{0,h} - \varepsilon u_i \right\} - R_i, i = 1..n, \qquad (5)$$

where R_i is a remainder term. We have the following estimates from (5):

1) $$\left| -\frac{1}{\varepsilon+k_i} h \sum_{j=1}^{i-1} exp\left(-h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon+k_s}\right) \frac{G_j}{\varepsilon+k_j} \left\{ h \sum_{s=1}^{j-1} [M_{j,s}(u_{\varepsilon,s}) - \right.\right.$$

$$\left. -M_{j,s}(u_s) - M_{i,s}(u_{\varepsilon,s}) + M_{i,s}(u_s)] - h \sum_{s=j+1}^{i-1} [M_{i,s}(u_{\varepsilon,s}) - M_{i,s}(u_s)] \right| \le$$

$$\le \frac{2 d_2 b T_0 h}{d_1 \varepsilon} (L_2 + C_2 L_1 + L_N) \left\| \eta_{\varepsilon,i}^{h} \right\|_{C_h},$$

where $L_1 = max_D |K_x(x,t)|, L_2 = max_D |K_{xx}(x,t)|\, , T_0 = max_{x \in [0,b]} |G(x)|$,

$$d_2 = sup\left(\sum_{j=1}^{i-1} exp\left(-h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon+k_s}\right) \left(h \sum_{s=j+1}^{i} \frac{G_s}{\varepsilon+k_s} \right) \right);$$

$$2)\ \left|\frac{C_1}{\varepsilon+k_i}h\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\left\{h\sum_{s=1}^{j-1}h\sum_{m=j+1}^{i}N_{m,s}(u_s)\,\eta_{\varepsilon,m}^h+\right.\right.$$

$$\left.+h\sum_{s=j+1}^{i-1}h\sum_{m=s+1}^{i}N_{m,s}(u_s)\,\eta_{\varepsilon,m}^h\right|\le\frac{2C_1T_0M_Nd_2bh}{d_1\varepsilon}\left\|\eta_{\varepsilon,i}^h\right\|_{C_h},$$

$where\ M_N=\max_{D\times R^1}|N(x,t,u)|;$

$$3)\ \left|\frac{C_1}{\varepsilon+k_i}h\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\left\{h\sum_{s=1}^{j-1}h\sum_{m=j+1}^{i}[N_{m,s}(u_{\varepsilon,s})-\right.\right.$$

$$\left.\left.-N_{m,s}(u_s)]u_m+h\sum_{s=j+1}^{i-1}h\sum_{m=s+1}^{i}[N_{m,s}(u_{\varepsilon,s})-N_{m,s}(u_s)]u_m\right\}\right|\le$$

$$\le\frac{2C_1T_0K_Nrbd_2h}{d_1\varepsilon}\left\|\eta_{\varepsilon,i}^h\right\|_{C_h},K_N=\max_{D\times R^1}|N_u(x,t,u)|;$$

$$4)\ \left|\frac{C_1h}{\varepsilon+k_i}\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\left\{h\sum_{s=1}^{j-1}u_s\,h\sum_{m=j+1}^{i}K_{m,s}\eta_{\varepsilon,m}^h+\right.\right.$$

$$\left.\left.+h\sum_{s=j+1}^{i-1}u_s\,h\sum_{m=s+1}^{i}K_{m,s}\eta_{\varepsilon,m}^h\right\}\right|\le\frac{C_1T_0Mh}{\varepsilon+p_i}\sum_{j=1}^{i-1}exp\left(-h\sum_{k=j+1}^{i}\frac{G_k}{\varepsilon+k_s}\right)\frac{1}{\varepsilon+k_j}\times$$

$$\times\left\{h\sum_{s=1}^{j}|u_s|\,h\sum_{m=j+1}^{i}\left|\eta_{\varepsilon,m}^h\right|+h\sum_{s=j+1}^{i-1}|u_s|\,h\sum_{m=j+1}^{i}\left|\eta_{\varepsilon,m}^h\right|\right\}\le$$

$$\le\frac{C_1T_0Mbrd_2h}{d_1\varepsilon}\left\|\eta_{\varepsilon,i}^h\right\|_{C_h};$$

where $M=\max_{D}|K(x,t)|,|u(x)|\le r;$

$$5)\ \left|\frac{C_1h}{\varepsilon+\ k_i}\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\left\{h\sum_{s=1}^{j-1}\eta_{\varepsilon,s}^h\,h\sum_{m=j+1}^{i}K_{m,s}\,u_{\varepsilon,m}+\right.\right.$$

$$\left.\left.+h\sum_{s=j+1}^{i-1}\eta_{\varepsilon,s}^h\,h\sum_{m=s+1}^{i}K_{m,s}u_{\varepsilon,m}\right\}\right|\le\frac{C_1T_0Mh}{\varepsilon+k_i}\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{1}{\varepsilon+k_j}\times$$

$$\times\left\{h\sum_{s=1}^{j-1}\left|\eta_{\varepsilon,s}^h\right|h\sum_{m=j+1}^{i}\left|u_{\varepsilon,m}\right|+h\sum_{s=j+1}^{i-1}\left|\eta_{\varepsilon,s}^h\right|h\sum_{m=j+1}^{i}\left|u_{\varepsilon,m}\right|\right\}\le$$

$$\le\frac{C_1T_0Mrd_2h}{d_1\varepsilon}h\sum_{j=1}^{i-1}\left|\eta_{\varepsilon,j}^h\right|\le\frac{C_1T_0Mbrd_2h}{d_1\varepsilon}\left\|\eta_{\varepsilon,i}^h\right\|_{C_h};$$

$$6)\ \left|\frac{1}{\varepsilon+k_i}exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)h\sum_{j=1}^{i-1}[M_{i,j}\left(u_{\varepsilon,j}\right)-M_{i,j}(u_j)]\right|\le$$

$$\le\frac{d_2bT_0}{d_1e}(L_2+C_2L_1+L_N)\left\|\eta_{\varepsilon,i}^h\right\|_{C_h}$$

7) $\left|\frac{C_1}{\varepsilon+k_i} exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)h\sum_{j=1}^{i-1}h\sum_{s=j+1}^{i}N_{s,j}\left(u_j\right)\eta_{\varepsilon,s}^{h}\right| \le \frac{C_1 M_N b}{ed_1}\left\|\eta_{\varepsilon,i}^{h}\right\|_{C_h};$

8) $\left|\frac{C_1}{\varepsilon+k_i} exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)h\sum_{j=1}^{i-1}h\sum_{s=j+1}^{i}\left[N_{s,j}\left(u_{\varepsilon,j}\right)-N_{s,j}\left(u_j\right)\right]u_j\right| \le$
$\le \frac{C_1 K_N br}{ed_1}\left\|\eta_{\varepsilon,i}^{h}\right\|_{C_h}.$

On the basis of estimates 1) -8) for the error vector $\eta_{\varepsilon,i}^{h}$ from (5) we obtain

$$\left|\eta_{\varepsilon,i}^{h}\right| \le q\left\|\eta_{\varepsilon,i}^{h}\right\|_{C_h} + |\varepsilon H_\varepsilon u_i| + |R_i|, (6)$$

Missing the cumbersome calculations, note that the next estimate for R_i holds true as in [2]:

$$\|R_i\|_{C_h} \le N_2 h/\varepsilon + N_3 h, 0 < N_2,\ N_3 = const.$$

Since according to [3]:

$$\left\|\varepsilon H_\varepsilon^h[u_i]\right\|_{C_h} \le N_1\varepsilon, 0 < N_1 = const,$$

then we get estimate by the grid norm from (6)

$$\left\|\eta_{\varepsilon,i}^{h}\right\|_{C_h} \le (1-q)^{-1}(N_1\varepsilon + N_2 h/\varepsilon + N_3 h).$$

Taking into account that $\varepsilon = O(h^\alpha)$, we arrive at the estimate of the theorem, which was to be proved.

Equation (4) is a system of nonlinear, therefore we obtain the following system of equations with respect to $u_{\varepsilon,i}$

$$u_{\varepsilon,i} = \frac{C_1}{\varepsilon+k_i}\left\{h\sum_{j=1}^{i-1} exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j} + exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\right\}\times$$

$$\times h\left\{\left(h\sum_{s=1}^{i-1}K_{i,s}u_{\varepsilon,s}\right)+\left(h\sum_{s=1}^{i-1}N_{i,s}\left(u_{\varepsilon,s}\right)\right)\right\}u_{\varepsilon,i} +$$

$$-\frac{1}{\varepsilon+k_i}h\sum_{j=1}^{i-1} exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\times$$

$$\times\left\{h\sum_{s=1}^{j}\left[M_{j,s}\left(u_{\varepsilon,s}\right)-M_{i,s}\left(u_{\varepsilon,s}\right)\right] - h\sum_{s=j+1}^{i-1}M_{i,s}\left(u_{\varepsilon,s}\right) - C_1 h\sum_{s=1}^{j-1}u_{\varepsilon,s}\times\right.$$
$$\times h\sum_{m=j+1}^{i-1}K_{m,s}u_{\varepsilon,m} - C_1 h\sum_{s=j+1}^{i-1}u_{\varepsilon,s}\,h\sum_{m=s+1}^{i-1}K_{m,s}u_{\varepsilon,m} - C_1 h\sum_{s=1}^{j-1}h\times$$
$$\left.\times\sum_{m=j+1}^{i-1}N_{m,s}\left(u_{\varepsilon,s}\right)u_{\varepsilon,m} - C_1 h\sum_{s=j}^{i-1}h\sum_{m=s+1}^{i-1}N_{m,s}\left(u_{\varepsilon,s}\right)u_{\varepsilon,m} + f_j - f_i\right\}+$$

$$+\frac{1}{\varepsilon+k_i}exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\left\{h\sum_{j=1}^{i-1}M_{i,j}\left(u_{\varepsilon,j}\right)+C_1h\sum_{j=1}^{i-1}u_{\varepsilon,j}\,h\sum_{s=j+1}^{i-1}K_{s,j}u_{\varepsilon,s}+\right.$$
$$\left.+C_1h\sum_{j=1}^{i-1}h\sum_{s=j+1}^{i-1}N_{s,j}\left(u_{\varepsilon,j}\right)u_{\varepsilon,s}+\varepsilon u_{0,h}+f_i\right\},i=1..n,\qquad (7)$$

Estimate the expression

$$U_{i-1}(u_1,\dots,u_{i-1})=\left\{exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)+h\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\times\right.$$
$$\left.\times\frac{G_j}{\varepsilon+k_j}\right\}\frac{C_1h}{\varepsilon+k_i}\left\{\left(h\sum_{s=1}^{i-1}K_{i,s}u_{\varepsilon,s}\right)+\left(h\sum_{s=1}^{i-1}N_{i,s}\left(u_{\varepsilon,s}\right)\right)\right\}$$

putting $C_1=C_0h^2, 0<C_0=const.$

Then

$$|U_{i-1}(u_1,\dots,u_{i-1})|\le\frac{(Mr+M_N)C_1h}{ed_1}+T_0\bar{d}_2b\frac{(Mr+M_N)C_0h}{d_1},$$

$$\bar{d}_2=sup\left|h\sum_{j=1}^{i-1}\left(h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\right|.$$

If

$$h<\frac{ed_1}{T_0\bar{d}_2bMrC_0},\quad (8)$$

then $|U_{i-1}(u_1,\dots,u_{i-1})|<1$. If condition (8) is satisfied, the system (7) can be rewritten in the form

$$u_{\varepsilon,i}=\left(1-U_{i-1}(u_1,\dots,u_{i-1})\right)^{-1}\left[-\frac{h}{\varepsilon+k_i}\sum_{j=1}^{i-1}exp\left(-h\sum_{s=j+1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\frac{G_j}{\varepsilon+k_j}\times\right.$$
$$\times\left\{h\sum_{s=1}^{j}[M_{j,s}\left(u_{\varepsilon,s}\right)-M_{i,s}\left(u_{\varepsilon,s}\right)]-h\sum_{s=j+1}^{i-1}M_{i,s}\left(u_{\varepsilon,s}\right)-C_1h\sum_{s=1}^{j-1}u_{\varepsilon,s}\times\right.$$
$$\times h\sum_{m=j+1}^{i-1}K_{m,s}u_{\varepsilon,m}-C_1h\sum_{s=j+1}^{i-1}u_{\varepsilon,s}\,h\sum_{m=s+1}^{i-1}K_{m,s}u_{\varepsilon,m}-C_1h\sum_{s=1}^{j-1}h\times$$
$$\left.\times\sum_{m=j+1}^{i-1}N_{m,s}\left(u_{\varepsilon,s}\right)u_{\varepsilon,m}-C_1h\sum_{s=j}^{i-1}h\sum_{m=s+1}^{i-1}N_{m,s}\left(u_{\varepsilon,s}\right)u_{\varepsilon,m}+f_j-f_i\right\}+$$
$$+\frac{1}{\varepsilon+k_i}exp\left(-h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+k_s}\right)\left\{h\sum_{j=1}^{i-1}M_{i,j}\left(u_{\varepsilon,j}\right)+C_1h\sum_{j=1}^{i-1}u_{\varepsilon,j}\,h\sum_{s=j+1}^{i-1}K_{s,j}u_{\varepsilon,s}+\right.$$
$$\left.+C_1h\sum_{j=1}^{i-1}h\sum_{s=j+1}^{i-1}N_{s,j}\left(u_{\varepsilon,j}\right)u_{\varepsilon,s}+\varepsilon u_{0,h}+f_i\right\},i=1..n,\qquad (9)$$

It is not difficult to see that (9) is a recursive formula.

Results

This equation is reduced to the Volterra integral equation of the third kind and a numerical method is developed on the basis of that regularized equation. The convergence of the numerical solution to the exact solution of the Volterra integral equation of the first kind is proved, an estimate of the permissible error and a recursive formula of the computational process are obtained.

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МЕТОД ЧИСЛЕННОГО РЕШЕНИЯ НЕЛИНЕЙНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ВОЛЬТЕРРА ПЕРВОГО РОДА

Аннотация. При рассмотрении систем дифференциальных уравнений с весьма общими краевыми условиями, точные методы решения наталкиваются на большие трудности, которые становятся непреодолимыми при рассмотреиии нелинейных задач. В этих случаях приходится обращаться к тем или иным численным методам решения. Важно отметить, что использование численных методов зачастую позволяет отказаться от упрощенной трактовки математической модели процесса. В работе изучаются вопросы численного решения нелинейных интегральных уравнений Вольтерра первого рода с дифференцируемым ядром, которое вырождается в начальной точке диагонали. Рассматриваемое уравнение сводится к интегральному уравнению Вольтерра третьего рода и на основе регуляризованного уравнения разработан численный метод. Доказана сходимость численного решения к точному решению интегрального уравнения Вольтерра первого рода, получены оценка допускаемой погрешности и рекурсивная формула вычислительного процесса.

Ключевые слова: нелинейное интегральное уравнение, систему нелинейных алгебраических уравнений, вектора погрешности, уравнение Вольтерра, малый параметр, численный метод.

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БІРІНШІ ТҮРДЕГІ СЫЗЫҚТЫ ЕМЕС ИНТЕГРАЛДЫ ВОЛЬТЕРРА ТЕҢДЕУЛЕРІН САНДЫҚ ШЕШУ ӘДІСІ

Аннотация. Дифференциалдық теңдеулер жүйесін өте жалпы шекаралық шарттармен қарастырған кезде, сызықты емес проблемаларды қарастыру кезінде шешілмейтін қиындықтарға айналдырудың дәл әдістері. Мұндай жағдайларда белгілі бір сандық әдістерге жүгіну керек. Сандық әдістерді қолдану, процестің математикалық моделін оңайлатылған түсіндіруден бас тартуға мүмкіндік береді. Алғашқы түрдегі сызықты емес Вольтерра интегралдық теңдеулерін диагональды бастапқы нүктесінде нөлге келтіретін дифференциалды ядро сандық шешудің сандық мәселелері қарастырылады. Қарастырылып отырған теңдеу Вольтерра интегралдық теңдеуін үшінші түрге дейін азайтады және реттелген теңдеудің негізінде сандық әдіс әзірленеді. Сандық шешімнің бірінші түрдегі Вольтерра интегралдық теңдеуінің дәл шешіміне дәлелденді, рұқсат етілген қателікті бағалау және есептеу үдерісінің рекурсивті формуласы алынды.

Түйінді сөздер: сызықты емес интегралдық теңдеу, сызықтық алгебралық теңдеулер жүйесі, қателік векторы, Вольтерра теңдеуі, кіші параметр, сандық әдіс.

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STEFAN PROBLEM IN ELLIPSOIDAL COORDINATES

Abstract. This paper presents the quasi-stationary Stefan problem in symmetric electrical contacts.The method of the solution can be obtained from the suggestion that the identity of equipotential and isothermal surfaces in contacts, which is correct for stationary fields in linear case, keeps safe for non-linear case as well. The idea is,transform the system of problem which is given in cylindrical coordinates into ellipsoidal coordinates.The analytical solution of stationary Stefan problem is found. Based on that decision was constructed the temperature profile to the approximate solution of heat problem with Joule heating in ellipsoidal coordinates.
Keywords: quasi-stationary model, Stefan problem, integral method.

Introduction

Stationary temperature and electromagnetic fields in symmetric electrical contacts have been described in [1]. Working with the scale of a mile second range, we think that every time the stationary state manages to instantly achieve stationary. And therefore this solution is suitable for constructing a temperature profile of the quasi-stationary problem.

Quasi-stationary nonlinear mathematical model of melting in ellipsoidal coordinates

The system of equations for the temperature $T_i(r,z)$ and electrical potential $\Phi_i(r,z)$ can be written in the form

$$div(\lambda_1 \,\mathrm{gradT_1}) + \frac{1}{\rho_1} grad^2 \Phi_1 = 0$$

$$div(\frac{1}{\rho_1} grad\Phi_1) = 0$$

$$div(\lambda_2 \,\mathrm{gradT_2}) + \frac{1}{\rho_2} grad^2 \Phi_2 = 0$$

$$div(\frac{1}{\rho_2} grad\Phi_2) = 0$$

where Φ_i, λ_i, ρ_i are electrical potential, heat conductance and electrical resistivity respectively.

In cylindrical coordinates these equations can be written as

$$\rho_i \frac{d\lambda_i}{dT_i}\left[\left(\frac{\partial T_i}{\partial r}\right)^2 + \left(\frac{\partial T_i}{\partial z}\right)^2\right] + \rho_i \lambda_i \Delta T_i + \left(\frac{\partial \Phi_i}{\partial r}\right)^2 + \left(\frac{\partial \Phi_i}{\partial z}\right)^2 = 0 \quad (1)$$

$$\frac{1}{\rho_i}\Delta\Phi_i - \frac{d\rho_i}{dT}\frac{1}{\rho_i^2}\left(\frac{\partial T_i}{\partial r}\frac{\partial \Phi_i}{\partial r} + \frac{\partial T_i}{\partial z}\frac{\partial \Phi_i}{\partial z}\right) = 0 \quad (2)$$

The index $i=1$ relates to the melted zone occupying the domain $D_1(0<z<\infty,\ \ r_0<r<r_m(t)$, and $i=2$ corresponds to the solid zone in the domain $D_2(0<z<\infty,\ \ r_m(t)<r<\infty)$.

It has to be mentioned that this problem is essentially non-linear due to temperature dependence of thermal conductivity $\lambda_i=\lambda_i(T_i)$ and electrical conductivity $\rho_i=\rho_i(T_i)$. The method of the solution can be obtained from the suggestion that the identity of equipotential and isothermal surfaces in contacts, which is correct for stationary fields in linear case, keeps safe for non-linear case as well. In linear case these surfaces are ellipsoids of revolution.

Equations (1) and (2) can be transformed into ellipsoidal coordinates and using well known relations among cylindrical and elliptical coordinates, if we suggest similarly like above that

$$\varPhi_i=\varPhi_i(\xi)\ ,\quad T_i=T_i(\xi)\ , \tag{3}$$

where

$$\xi=\sqrt{s+\sqrt{s^2+4r_0^{\ 2}z^2}}\ ,\qquad s=r^2+z^2-r_0^{\ 2}$$

then the equations (1) and (2) should be replaced by the equation

$$\rho_i\frac{d\lambda_i}{d\mathrm{T}_i}\cdot\left(\frac{d\mathrm{T}_i}{d\xi}\right)^2+\rho_i\lambda_i\frac{d^2\mathrm{T}_i}{d\xi^2}+\rho_i\lambda_i\frac{d\mathrm{T}_i}{d\xi}\cdot\frac{2\xi}{r_0^{\ 2}+\xi^2}+\left(\frac{d\Phi_i}{d\xi}\right)^2=0 \tag{4}$$

$$\frac{d^2\varPhi_i}{d\xi^2}+\frac{2\xi}{r_0^{\ 2}+\xi^2}\cdot\frac{d\varPhi_i}{d\xi}-\frac{1}{\rho_i}\frac{d\rho_i}{d\mathrm{T}_i}\cdot\frac{d\mathrm{T}_i}{d\xi}\frac{d\varPhi_i}{d\xi}=0 \tag{5}$$

$$D:0<r<\infty,\ 0<z<\infty\ ,\ z=0,\cup\ 0\le r<r_0\ ,\ 0<\xi<\infty\ ,\ 0\le\eta<r_0 \tag{6}$$

The boundary conditions are

$z=0\ (\xi=0)$ $\qquad \dfrac{dT_1}{d\xi}=0 \quad (7) \qquad \varPhi_1\big|_{0\le r\le r_0}=0 \quad (8) \qquad \left.\dfrac{\partial\varPhi_1}{\partial z}\right|_{r<r_m(\mathrm{t})}=0 \quad (9)$

$z=\sigma(\mathrm{r,t})\ (\xi=\xi_m(\mathrm{t}))$ $\qquad T_1=T_2=T_m \quad (10) \qquad \varPhi_1=\varPhi \quad (11)$

$\qquad \lambda_1\dfrac{dT_1}{d\xi}=\lambda_2\dfrac{dT_2}{d\xi} \quad (12) \qquad \dfrac{1}{\rho_1}\dfrac{d\varPhi_1}{d\xi}=\dfrac{1}{\rho_2}\dfrac{d\varPhi_2}{d\xi} \quad (13)$

$z=\infty\ or\ r=\infty\ (\xi=\infty)$ $\qquad T_2=0 \quad (14) \qquad \varPhi_2=\dfrac{U_c}{2} \quad (15)$

while the solution for electric potentials

$$\varPhi_1'(\xi)=\frac{I^2\rho_1(\mathrm{T}_1)}{2\pi(\mathrm{r}_0^2+\xi^2)},\qquad \varPhi_2'(\xi)=\frac{I^2\rho_2(\mathrm{T}_2)}{2\pi(\mathrm{r}_0^2+\xi^2)} \tag{16}$$

Putting (16) into (4) we get

$$\frac{1}{\lambda_i}\frac{d\lambda_i}{d\mathrm{T}_i}\cdot\left(\frac{d\mathrm{T}_i}{d\xi}\right)^2+\frac{d^2\mathrm{T}_i}{d\xi^2}+\frac{d\mathrm{T}_i}{d\xi}\cdot\frac{2\xi}{r_0^{\ 2}+\xi^2}+\frac{I^2\rho_i}{4\pi^2(\mathrm{r}_0^2+\xi^2)}=0 \tag{17}$$

Let us introduce the new independent variable ζ using formula and consider the case when thermal conductivity doesn't depend on temperature, $\frac{d\lambda}{dT} = 0$

$$\zeta = \arctan\frac{\xi}{r_0} \tag{18}$$

Taking into account that $\rho_1 = \rho_{10}(1+\alpha_{10}(T_1 - T_m)),\ \rho_2 = \rho_{20}(1+\alpha_{20}T_2)$

And using[2] $\omega_i^2 = \frac{I^2\rho_{i0}\alpha_{i0}}{4\pi^2 r_0^2\lambda_i}$

then the equation (17) for melted zone can be reduced to the form

$$\frac{d^2T_1}{d\zeta^2} + \frac{\omega_1^2}{\alpha_{10}}[1+\alpha_{10}(\mathrm{T}_1 - \mathrm{T}_m)] = 0 \tag{19}$$

The general solution of this equation is

$$T_1 = \frac{A_1}{\alpha_{10}}\cos\omega_1\zeta + \frac{B_1}{\alpha_{10}}\sin\omega_1\zeta + T_m - \frac{1}{\alpha_{10}} \tag{20}$$

and A_1, B_1 are arbitrary constants, which can be found from the boundary conditions (7) and (8)
From (7) and (10)
$B_1 = 0$

$$A_1 = \frac{1}{\cos\omega_1\frac{\pi}{2}}$$

Finally,

$$T_1 = \frac{1}{\alpha_{10}}\left(\frac{\cos\omega_1\zeta}{\cos\omega_1\zeta_m} + \alpha_{10}T_m - 1\right)$$

The equation (17) for solid zone can be reduced to the form

$$\frac{d^2T_2}{d\zeta^2} + \frac{\omega_2^2}{\alpha_{20}}[1+\alpha_{20}\,\mathrm{T}_2] = 0 \tag{21}$$

the general solution can be represented

$$T_2 = \frac{1}{\alpha_{20}}\left(A_2\frac{\cos\omega_2\zeta}{\cos\omega_2\frac{\pi}{2}} + B_2\frac{\sin\omega_2\zeta}{\sin\omega_2\frac{\pi}{2}} - 1\right) \tag{22}$$

From (14) and (10) can be found A_2, B_2 and temperature T_2 will be in the form

$$T_2 = \frac{1}{\alpha_{20}\sin\omega_2(\frac{\pi}{2} - \zeta_m)}\left\{(1+\alpha_{20}T_m)\sin\omega_2(\frac{\pi}{2} - \zeta) - \sin\omega_2(\zeta_m - \zeta) - \sin\omega_2(\frac{\pi}{2} - \zeta_m)\right\}$$

Noting first that

$$\left.\frac{dT_1}{d\xi}\right|_{\xi=\xi_m(t)} = \frac{dT_1}{d\zeta}\cdot\left.\frac{d\zeta}{d\xi}\right|_{\xi=\xi_m(t)} = -\frac{\omega_1 \sin\omega_1\zeta_m}{\alpha_{10}\cos\omega_1\zeta_m}\cdot\frac{r_0}{r_0^2+\xi_m^2(t)}$$

$$\left.\frac{dT_2}{d\xi}\right|_{\xi=\xi_m(t)} = \frac{dT_2}{d\zeta}\cdot\left.\frac{d\zeta}{d\xi}\right|_{\xi=\xi_m(t)} = -\frac{\omega_2\left[\left(1+\alpha_{20}T_m\right)\cos\left(\frac{\pi}{2}-\zeta_m\right)-1\right]}{\alpha_{20}\sin\omega_2\left(\frac{\pi}{2}-\zeta_m\right)}\cdot\frac{r_0}{r_0^2+\xi_m^2(t)}$$

from (12),

$$\frac{\lambda_1\omega_1\sin\omega_1\zeta_m}{\alpha_{10}\cos\omega_1\zeta_m} = \frac{\lambda_2\omega_2\left[\left(1+\alpha_{20}T_m\right)\cos\omega_2\left(\frac{\pi}{2}-\zeta_m\right)-1\right]}{\alpha_{20}\sin\omega_2\left(\frac{\pi}{2}-\zeta_m\right)}$$

finally we get

$$\zeta_m = \frac{1}{\omega_1}\arctan\frac{\lambda_2\omega_2\alpha_{10}}{\lambda_1\omega_1\alpha_{20}}\left[\left(1+\alpha_{20}T_m\right)\cot\omega_2\left(\frac{\pi}{2}-\zeta_m\right)-\cos ec\omega_2\left(\frac{\pi}{2}-\zeta_m\right)\right]$$

Approximate solution of heat problem in ellipsoidal coordinates

Considering the problem from the class of Stefan type problem, in first stage of heating electrical contact, where contact material is solid and temperature attains softening point. In this case we consider the heat equation

$$\frac{\partial\theta_1}{\partial t} = \frac{a_1^2}{r_0^2}\cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2}+\omega_1^2\left(\theta_1+\frac{1}{\alpha_1}\right)\right] \quad 0<\zeta<\pi/2 \tag{23}$$

subjected to boundary conditions

$\zeta = 0$:

$$\frac{\partial\theta_1(0,t)}{\partial\zeta} = 0 \tag{24}$$

$\zeta = \pi/2$:

$$\theta_1(\pi/2\,,t) = 0 \tag{25}$$

and initial condition

$t = 0$:

$$\theta_1(\zeta\,,0) = 0 \tag{26}$$

where

$$\omega_1 = \frac{I}{2\pi r_0^2}\sqrt{\frac{\rho_{10}\alpha_1}{c_1\gamma_1}}$$

For the temperature distribution $\theta_1(\zeta\,,t)$, let us assume that the temperature profile as given in the form

$$\theta_1(\zeta,t) = A_1(t)\cos(\omega_1\zeta) + B_1(t)\sin(\omega_1\zeta) + C_1(t) \text{ in } 0 \le \zeta \le \frac{\pi}{2} \quad (27)$$

where the coefficients are in general functions of time.

Using conditions (24) and (25) we get

$$\begin{cases} B_1(t) = 0 \\ A_1(t)\cos(\omega_1 \frac{\pi}{2}) + C_1(t) = 0 \end{cases} \quad (28)$$

Integration equation (23) with respect to the space variable form $\zeta = 0$ to $\zeta = \pi/2$, noting first that

$$\int_0^{\pi/2} \cos^4(\zeta)\left[\frac{\partial^2\theta_1}{\partial\zeta^2} + \omega_1^2\left(\theta_1 + \frac{1}{\alpha_1}\right)\right]d\zeta = \cos^4(\zeta)\frac{\partial\theta_1}{\partial\zeta}\Big|_0^{\pi/2} + 4\theta_1\cos^3(\zeta)\sin(\zeta)\Big|_0^{\pi/2} +$$

$$+\frac{3\pi\omega_1^2}{16\alpha_1} + \int_0^{\pi/2}\left[12\cos^2(\zeta) + \left(\omega_1^2 - 16\right)\cos^4(\zeta)\right]\theta_1 d\zeta$$

then we have

$$\frac{r_0^2}{a_1^2}\int_0^{\pi/2}\frac{\partial\theta_1}{\partial t}d\zeta = \frac{3\pi\omega_1^2}{16\alpha_1} + \int_0^{\pi/2}\left[12\cos^2(\zeta) + \left(\omega_1^2 - 16\right)\cos^4(\zeta)\right]\theta_1 d\zeta$$

When the integral on the left-hand side is performed using Leibniz's integral formula, we obtain

$$\frac{r_0^2}{a_1^2}\frac{d}{dt}\left[\int_0^{\pi/2}\theta_1 d\zeta\right] = \frac{3\pi\omega_1^2}{16\alpha_1} + \int_0^{\pi/2}\left[12\cos^2(\zeta) + \left(\omega_1^2 - 16\right)\cos^4(\zeta)\right]\theta_1 d\zeta \quad (29)$$

(29) is called the energy integral equation for the problem considered here.

Substituting (27) and (28) the above into the energy integral equation (29) we obtain the following ordinary for $C_1(t)$

$$\frac{r_0^2}{a_1^2}\left(\frac{\pi}{2} - \frac{\tan(\omega_1\pi/2)}{\omega_1}\right)\frac{dC_1(\mathrm{t})}{dt} = \frac{3\pi\omega_1^2}{16\alpha_1}\left(\frac{1}{\alpha_1} + C_1(\mathrm{t})\right)$$

$$\begin{cases} C_1(0) = 0 \\ A_1(t) = -C_1(t)\sec\left(\omega_1\frac{\pi}{2}\right) \end{cases}$$

The solution of equation

$$C_1(\mathrm{t}) = \exp\left(\left(\frac{\pi}{2} - \frac{\tan(\omega_1\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2 a_1^2}{16\alpha_1 r_0^2}t - \ln(\alpha_1)\right) - \frac{1}{\alpha_1}$$

and

$$A_1(\mathrm{t}) = \left[\frac{1}{\alpha_1} - \exp\left(\left(\frac{\pi}{2} - \frac{\tan(\omega_1\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2 a_1^2}{16\alpha_1 r_0^2}t - \ln(\alpha_1)\right)\right]\sec\left(\omega_1\frac{\pi}{2}\right)$$

Finally temperature profile

$$\theta_1(\zeta,t)=\left[\frac{1}{\alpha_1}-\exp\left(\left(\frac{\pi}{2}-\frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2a_1^2}{16\alpha_1r_0^2}t-\ln(\alpha_1)\right)\right]\sec\left(\omega_1\frac{\pi}{2}\right)\cos(\omega_1\zeta)+$$

$$+\exp\left(\left(\frac{\pi}{2}-\frac{\tan(\omega_1\,\pi/2)}{\omega_1}\right)^{-1}\frac{3\pi\omega_1^2a_1^2}{16\alpha_1r_0^2}t-\ln(\alpha_1)\right)-\frac{1}{\alpha_1}$$

Conclusion

The problem (23)-(26) is solved by integral method. All coefficients of temperature profile is found. This method is useful to apply to solving the phase-change problem with moving boundary.

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ЭЛЛИПСОИДТТІК КООРДИНАТТАРДАҒЫ СТЕФАН ЕСЕБІ

Аннотация. Бұл мақалада симметриялық электрлік байланыста квазистационарлық Стефан мәселесі берілген. Ерітінді әдісі сызықтық жағдайда стационарлық өрістерге дұрыс болатын контактілерде тең потенциалды және изотермиялық беттердің идентификациясы, сондай-ақсызықты емес жағдайда да қауіпсіз болуын ұсыныспен алуға болады. Бұл идея цилиндрлік координаттарда эллипсоидтік координаттарға берілген проблема жүйесін өзгерту болып табылады. Стефанның стационарлық мәселесінің аналитикалық шешімі табылды. Осы шешім негізінде эллипсоидтік координаттарда Джоул жылумен жылу проблемасын жуықтап шешу үшін температуралық профиль құрылды.

Түйіндісөздер: квази-стационарлық үлгі, Стефан проблемасы, интеграл әдісі

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ЗАДАЧА СТЕФАНА В ЭЛЛИПСОИДАЛЬНЫХ КООРДИНАТАХ

Абстрактные. В настоящей работе представлена квазистационарная задача Стефана в симметричных электрических контактах. Метод решения может быть получен из предположения, что идентичность эквипотенциальных и изотермических поверхностей в контактах, которая правильна для стационарного поля в линейном случае также и для нелинейного случая. Идея состоит в том, чтобы преобразовать систему задач, заданную в цилиндрических координатах, в эллипсоидальные координаты. Получено аналитическое решение стационарной задачи Стефана. На основании этого решения был построен профиль температуры приближенному решению тепловой задачи с Джоулевым нагревом в эллипсоидальных координатах.

Ключевые слова: квазистационарная модель, проблема Стефана, интегральный метод.

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AN ALGORITHM FOR SOLVING A CONTROL PROBLEM FOR A DIFFERENTIAL EQUATION WITH A PARAMETER

Abstract. On a finite interval, a control problem for a linear ordinary differential equations with a parameter is considered. By partitioning the interval and introducing additional parameters, considered problem is reduced to the equivalent multipoint boundary value problem with parameters. To find the parameters introduced, the continuity conditions of the solution at the interior points of partition and boundary condition are used. For the fixed values of the parameters, the Cauchy problems for ordinary differential equations are solved. By substituting the Cauchy problem's solutions into the boundary condition and the continuity conditions of the solution, a system of linear algebraic equations with respect to parameters is constructed. The solvability of this system ensures the existence of a solution to the original control problem. The system of linear algebraic equations is composed by the solutions of the matrix and vector Cauchy problems for ordinary differential equations on the subintervals. A numerical method for solving the origin control problem is offered based on the Runge-Kutta method of the 4-th order for solving the Cauchy problem for ordinary differential equations.

Key words: boundary value problem with parameter, differential equation, solvability, algorithm.

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In the present paper we consider the control problem for the linear ordinary differential equations with parameter

$$\frac{dx}{dt} = A(t)x + B(t)\mu + f(t), \quad x, \mu \in R^n, \quad t \in (0,T), \tag{1}$$

$$x(0) = x^0, \quad x^0 \in R^n, \tag{2}$$

$$x(T) = x^1, \quad x^1 \in R^n, \tag{3}$$

where the $(n \times n)$-matrices $A(t)$, $B(t)$ and n-vector-function $f(t)$ are continuous on $[0,T]$.

Let $C([0,T],R^n)$ denote the space of continuous functions $x:[0,T] \to R^n$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|.$

Solution to problem (1), (2) is a pair $(\mu^*, x^*(t))$, where the function $x^*(t) \in C([0,T],R^n)$ is continuously differentiable on $(0,T)$ and satisfies Eq. (1) with $\mu = \mu^*$ and additional conditions (2), (3).

Qualitative properties of problem (1), (2) and methods for solving boundary value problems with parameters studied by many authors (see [1-16, 19,20] and references cited therein).

In the present paper problem (1), (2) is solved by parametrization's method [17, 18].

Given the points: $t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N = T$, and let Δ_N be the partition of interval $[0,T)$ into N subintervals: $[0,T) = \bigcup_{r=1}^{N} [t_{r-1}, t_r)$.

By $C([0,T],\Delta_N, R^{nN})$ we denote the space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$, where $x_r : [t_{r-1}, t_r) \to R^n$ are continuous and have finite left-hand limits $\lim_{t \to t_r - 0} x_r(t)$ for all $r = \overline{1,N}$, with the norm $\|x\|_2 = \max_{r=1,N} \sup_{t \in [t_{r-1}, t_r)} \|x_r(t)\|$.

Denote by $x_r(t)$ the restriction of function $x(t)$ to the r-th interval $[t_{r-1}, t_r)$ and reduce problem (1)-(3) to the equivalent multipoint boundary-value problem

$$\frac{dx_r}{dt} = A(t)x_r + B(t)\mu + f(t), \qquad t \in [t_{r-1}, t_r), \ r = \overline{1,N}, \tag{4}$$

$$x_1(0) = x^0, \tag{5}$$

$$\lim_{t \to T-0} x_N(t) = x^1, \tag{6}$$

$$\lim_{t \to t_s - 0} x_s(t) = x_{s+1}(t_s), \quad s = \overline{1, N-1}, \tag{7}$$

where (7) are the continuity conditions of the solution at the interior points of the partition.

A pair $(\mu^*, x^*[t])$ with $\mu^* \in R^n$ and $x^*[t] = (x_1^*(t), x_2^*(t), \ldots, x_N^*(t)) \in C([0,T],\Delta_N, R^{nN})$ is called a solution to problem (4)-(7), if it satisfies the system of differential equations (4) and conditions (5)-(7).

Introducing the additional parameters $\lambda_1 = \mu$, $\lambda_p = x_p(t_{p-1})$, $p = \overline{2,N}$, and performing the substitutions $u_1(t) = x_1(t) - x^0$, $u_p(t) = x_p(t) - \lambda_p$, $t \in [t_{p-1}, t_p)$, $p = \overline{2,N}$, we obtain the boundary value problem with parameters

$$\frac{du_1}{dt} = A(t)(u_1 + x^0) + B(t)\lambda_1 + f(t), \qquad t \in [t_0, t_1), \tag{8}$$

$$u_1(t_0) = 0, \tag{9}$$

$$\frac{du_p}{dt} = A(t)(u_p + \lambda_p) + B(t)\lambda_1 + f(t), \quad t \in [t_{p-1}, t_p), \quad p = \overline{2,N}, \tag{10}$$

$$u_p(t_{p-1}) = 0, \ p = \overline{2,N}, \tag{11}$$

$$\lambda_N + \lim_{t \to T-0} u_N(t) = x^1, \tag{12}$$

$$x^0 + \lim_{t \to t_1 - 0} u_1(t) = \lambda_2, \tag{13}$$

$$\lambda_s + \lim_{t \to t_s - 0} u_s(t) = \lambda_{s+1}, \quad s = \overline{2, N-1}. \tag{14}$$

Solution to problem (8)-(14) is a pair $(\lambda, u[t])$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in R^{nN}$, $u[t] = (u_1(t), u_2(t), \ldots, u_N(t)) \in C([0,T],\Delta_N, R^{nN})$. If $(\lambda, u[t])$ is a solution to problem (8)-(14), then the pair $(\mu, x(t))$ with parameter μ and the function $x(t)$, defined by the equalities: $\mu = \lambda_1$, $x(t) = x^0 + u_1(t)$, $t \in [t_0, t_1)$, $x(t) = \lambda_p + u_p(t)$, $t \in [t_{p-1}, t_p)$, $p = \overline{2,N}$, $x(T) = \lambda_N + \lim_{t \to T-0} u_N(t)$, is a solution to problem (1)- (3). Conversely, if $(\tilde{\mu}, \tilde{x}(t))$ is a solution to problem (1)-(3), then the pair $(\widetilde{\lambda}, \widetilde{u}[t])$, with $\widetilde{\lambda} = (\widetilde{\mu}, \widetilde{x}(t_1), \dots, \widetilde{x}(t_{N-1}))$, $\widetilde{u}[t] = (\widetilde{x}(t) - \widetilde{x}^0, \ \widetilde{x}(t) - \widetilde{x}(t_1), \ \dots, \ \widetilde{x}(t) - \widetilde{x}(t_{N-1}))$, is a solution to problem (8)-(14).

Let $\Phi(t)$ be a fundamental matrix to the ordinary differential equation

$$\frac{dx}{dt} = A(t)x, \quad t \in [0,T].$$

Then a unique solution of the Cauchy problem for the system of ordinary differential equations (8)-(11) at the fixed values $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ has the following form

$$u_1(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)B(\tau)d\tau \cdot \lambda_1 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(\tau)\left[A(\tau)x^0 + f(\tau)\right]d\tau, \ t \in [t_0, t_1), \quad (15)$$

$$u_p(t) = \Phi(t)\int_{t_{p-1}}^{t} \Phi^{-1}(\tau)A(\tau)d\tau\lambda_p + \Phi(t)\int_{t_{p-1}}^{t} \Phi^{-1}(\tau)B(\tau)d\tau \cdot \lambda_1 +$$
$$+ \Phi(t)\int_{t_{p-1}}^{t} \Phi^{-1}(\tau)f(\tau)d\tau, \quad t \in [t_{p-1}, t_p), \quad p = \overline{2,N}. \quad (16)$$

Substituting (15) and (16) into (12)-(14) yields the system of algebraic equations for finding the unknown parameters $\lambda_1, \lambda_2, \dots, \lambda_N$:

$$\lambda_N + \Phi(T)\int_{t_{N-1}}^{T} \Phi^{-1}(\tau)A(\tau)d\tau\lambda_N + \Phi(T)\int_{t_{N-1}}^{T} \Phi^{-1}(\tau)B(\tau)d\tau \cdot \lambda_1 + \Phi(T)\int_{t_{N-1}}^{T} \Phi^{-1}(\tau)f(\tau)d\tau = x^1, \quad (17)$$

$$x^0 + \Phi(t_1)\int_{t_0}^{t_1} \Phi^{-1}(\tau)A(\tau)x^0 d\tau + \Phi(t_1)\int_{t_0}^{t_1} \Phi^{-1}(\tau)B(\tau)\lambda_1 d\tau + \Phi(t_1)\int_{t_0}^{t_1} \Phi^{-1}(\tau)f(\tau)d\tau = \lambda_2, \quad (18)$$

$$\lambda_s + \Phi(t_s)\int_{t_{s-1}}^{t_s} \Phi^{-1}(\tau)\{A(\tau)\lambda_s + B(\tau)\lambda_1\}d\tau + \Phi(t_s)\int_{t_{s-1}}^{t_s} \Phi^{-1}(\tau)f(\tau)d\tau = \lambda_{s+1}, \ s = \overline{2, N-1}. \quad (19)$$

Let $Q_*(\Delta_N)$ denote the matrix corresponding to the left-hand side of system (17)- (19). Then the system can be written as

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \ \lambda \in R^{nN}, \quad (20)$$

where

$$F_*(\Delta_N) = \Bigg(-x^1 + \Phi(T)\int_{t_{N-1}}^{T} \Phi^{-1}(\tau)f(\tau)d\tau, x^0 + \Phi(t_1)\int_{t_0}^{t_1} X^{-1}(\tau)A(\tau)x^0 d\tau + \Phi(t_1)\int_{t_0}^{t_1} X^{-1}(\tau)f(\tau)d\tau,$$
$$\Phi(t_2)\int_{t_1}^{t_2} X^{-1}(\tau)f(\tau)d\tau, \dots, \Phi(t_{N-1})\int_{t_{N-2}}^{t_{N-1}} \Phi^{-1}(\tau)f(\tau)d\tau \Bigg).$$

Lemma 1. The following assertions hold:

(a) The vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{nN}$, consisting of $\lambda_1^* = \mu^*$ and the values of the solution $x^*(t)$ to problem (1)-(3) at the partition points $\lambda_p^* = x^*(t_{p-1})$, $r = \overline{2,N}$, satisfies system (20).

(b) The pair $(\widetilde{\mu}, \widetilde{x}(t))$ defined by the equalities $\widetilde{\mu} = \widetilde{\lambda}_1$, $\widetilde{x}(t) = \widetilde{u}_1(t) + x^0$, $t \in [t_0, t_1)$, $\widetilde{x}(t) = \widetilde{u}_p(t) + \widetilde{\lambda}_p$, $t \in [t_{p-1}, t_p)$, $p = \overline{2,N}$, $\widetilde{x}(T) = \widetilde{\lambda}_N + \lim_{t \to T-0} \widetilde{u}_N(t)$ is a solution to problem (1)-(3), where $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_N) \in R^{nN}$ is a solution to system (20) and the system of functions $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \dots, \widetilde{u}_N(t)) \in C([0,T], \Delta_N, R^{nN})$ is solution to the Cauchy problem (8)- (11) for $\widetilde{\lambda}_1 = \mu$, $\lambda_p = \widetilde{\lambda}_p$, $p = \overline{2,N}$.

Definition. Problem (1)-(3) is called uniquely solvable if it has a unique solution for any $x^0, x^1 \in R^n$ and $f(t) \in C([0,T], R^n)$.

Theorem 1. Problem (1)-(3) is uniquely solvable if and only if the matrix $Q_*(\Delta_N): R^{nN} \to R^{nN}$ has an inverse one.

Proof. Necessity. Assume the opposite, i.e., that $Q_*(\Delta_N)$ is not invertible. Then the homogeneous system of equations

$$Q_*(\Delta_N)\lambda = 0 \tag{22}$$

has a nontrivial solution $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_N) \in R^{nN}$. In the case of a homogeneous boundary value problem for the ordinary differential equation, i.e., for problem (1)-(3) with $x^0 = 0,\ x^1 = 0,\ f(t) = 0,$ system (20) becomes (22). Therefore, by Lemma 1, the pair $(\widetilde{\mu}, \widetilde{x}(t))$ defined by the equalities $\widetilde{\mu} = \widetilde{\lambda}_1$, $\widetilde{x}(t) = \widetilde{u}_1(t) + x^0$, $t \in [t_0, t_1)$, $\widetilde{x}(t) = \widetilde{u}_p(t) + \widetilde{\lambda}_p$, $t \in [t_{p-1}, t_p)$, $p = \overline{2, N}$, $\widetilde{x}(T) = \widetilde{\lambda}_N + \lim\limits_{t \to T-0} \widetilde{u}_N(t)$, where the system of functions $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t))$ is solution to following problem

$$\frac{du_1}{dt} = A(t)u_1 + B(t)\lambda_1, \qquad t \in [t_0, t_1),\ u_1(t_0) = 0,$$

$$\frac{du_p}{dt} = A(t)(u_p + \lambda_p) + B(t)\lambda_1, \quad t \in [t_{p-1}, t_p), \quad u_p(t_{p-1}) = 0,\ p = \overline{2, N},$$

is a nontrivial solution of the homogeneous boundary value problem (1)-(3) ($x^0 = 0,\ x^1 = 0,\ f(t) = 0$). Since, problem (1)-(3) has the trivial solution $\widetilde{x}(t) = 0,\ \widetilde{\mu} = 0$. This contradicts to a unique solvability of problem (1)-(3).

Sufficiency. Due to the invertibility of the matrix $Q_*(\Delta_N)$, we can find unique solution of system equation (20) $\lambda^* = -[Q_*(\Delta_N)]^{-1} F_*(\Delta_N)$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in R^{nN}$. Solving Cauchy problem (8)-(11) for $\lambda_r = \lambda_r^*$, $r = \overline{1, N}$, we obtain the system of functions $\widetilde{u}[t] = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_N(t))$. According to Lemma 1, the pair $(\mu^*, x^*(t))$ defined by equalities $\mu^* = \lambda_1^*$, $x^*(t) = u_1^*(t) + x^0$, $t \in [t_0, t_1)$, $x^*(t) = \lambda_p^* + u_p^*(t)$, $t \in [t_{p-1}, t_p)$, $p = \overline{2, N}$, $x^*(T) = \lambda_N^* + \lim\limits_{t \to T-0} u_N^*(t)$, is a solution to problem (1)-(3).

Establish the uniqueness of the solution. Suppose that the problem (1)-(3) has another solution $(\widetilde{\mu}, \widetilde{x}(t))$ except $(\mu^*, x^*(t))$. Then the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ composed by the parameter $\widetilde{\mu}$ and $\widetilde{x}(t)$ is also a solution of the boundary value problem with a parameter (8)-(14). By Lemma 1, λ^* and $\widetilde{\lambda}$ satisfy the system of equations (20):

$$Q_*(\Delta_N)\lambda^* = -F_*(\Delta_N),\ Q_*(\Delta_N)\widetilde{\lambda} = -F_*(\Delta_N).$$

Invertibility of the matrix $Q_*(\Delta_N)$, yields: $\lambda^* = \widetilde{\lambda}$. Unique solvability of the Cauchy problem (8)-(11) provides fulfillment of relationships $u_r^*(t) = \widetilde{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, $\lim\limits_{t \to T-0} u_N^*(t) = \lim\limits_{t \to T-0} \widetilde{u}_N(t)$. Therefore, $\mu^* = \widetilde{\mu}$ and $x^*(t) = \widetilde{x}(t)$ for all $t \in [0, T]$. The proof is complete.

The Cauchy problems for ordinary differential equations on the subintervals

$$\frac{dz}{dt} = A(t)z + P(t), \quad z(t_{r-1}) = 0, \quad r = \overline{1, N} \tag{23}$$

are a significant part of the proposed algorithm. Here $P(t)$ is either $(n \times n)$ matrix or n vector, continuous on $[t_{r-1}, t_r)$, $r = \overline{1, N}$. Consequently, solution to problem (23) is a square matrix or a vector of dimension n. Denote by $a_r(P, t)$ the solution to the Cauchy problem (23). Obviously,

$$a_r(P,t) = \Phi(t) \int_{t_{r-1}}^{t} \Phi^{-1}(\tau) P(\tau) d\tau, \quad [t_{r-1}, t_r), \quad r = \overline{1, N}, \tag{24}$$

where $\Phi(t)$ is a fundamental matrix of differential equation (23) on the r-th interval.

We offer the following numerical implementation of algorithm for finding solution of problem (1)-(3) based on solving Cauchy problems by the Runge-Kutta method of 4-th order.

I. Suppose we have a partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$. Divide each $r-$th interval $[t_{r-1}, t_r)$, $r = \overline{1, N}$, into N_r parts with the step $h_r = (t_r - t_{r-1}) / N_r$. Assume that on each interval $[t_{r-1}, t_r)$, the variable $\hat{t}$ takes its discrete values: $\hat{t} = t_{r-1}$, $\hat{t} = t_{r-1} + h_r$, …, $\hat{t} = t_{r-1} + (N_r - 1) h_r$, $\hat{t} = t_r$, and denote by $\{t_{r-1}, t_r\}$ the set of such points.

II. Solving the Cauchy problem for ordinary differential equations

$$\frac{dz}{dt} = A(t)z + A(t), \ z(t_{r-1}) = 0, \ r = \overline{1, N},$$

$$\frac{dz}{dt} = A(t)z + B(t), \ z(t_{r-1}) = 0, \ r = \overline{1, N},$$

$$\frac{dz}{dt} = A(t)z + f(t), \ z(t_{r-1}) = 0, \ r = \overline{1, N},$$

by using Runge-Kutta method of the 4-th order, we find the values of the $(n \times n)$-matrix $a_r(A, \hat{t})$, $a_r(B, \hat{t})$, and n-vector $a_r(f, \hat{t})$ on $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$.

III. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde{h}}(\Delta_N)\lambda = -F_*^{\tilde{h}}(\Delta_N), \ \lambda \in R^{nN}, \ \tilde{h} = (h_1, h_2, \ldots, h_N). \tag{25}$$

Solving the system of algebraic equations (25) we find $\lambda^{\tilde{h}} \in R^{nN}$.

IV. To define the values of the approximate solution at the remaining points of the set $\{t_{r-1}, t_r\}$, we solve the following Cauchy problems by applying the Runge-Kutta method of the 4-th order

$$\frac{dx}{dt} = A(t)x + B(t)\lambda_1^{\tilde{h}} + f(t), \ \ t \in [t_0, t_1), \quad x(t_0) = x^0, \tag{26}$$

$$\frac{dx}{dt} = A(t)x + B(t)\lambda_1^{\tilde{h}} + f(t), \ \ t \in [t_{r-1}, t_r), \quad x(t_{r-1}) = \lambda_r^{\tilde{h}}, \quad r = \overline{2, N}. \tag{27}$$

To illustrate the proposed method for control problem (1)-(3) we consider the following example.

Example. Consider the control problem for differential equations with parameter:

$$\frac{dx}{dt} = A(t)x + B(t)\mu + f(t), \ t \in [0,1], \ x \in R^2, \tag{28}$$

$$x(0) = x^0, \tag{29}$$

$$x(1) = x^1, \tag{30}$$

where $A(t) = \begin{pmatrix} t & t+4 \\ 2t^2 & 7 \end{pmatrix}$, $B(t) = \begin{pmatrix} t+1 & 1 \\ t & 2t^2 \end{pmatrix}$, $f(t) = \begin{pmatrix} -t^4 - 4t^3 + t^2 + 2t - 18 \\ -9t^3 - 9t^2 + 12t - 23 \end{pmatrix}$, $x^0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $x^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

In this example, the matrix of the differential part is variable and construction of a fundamental matrix breaks down. We use the numerical implementation of the algorithm proposed. We provide the results of the numerical implementation of the algorithm by partitioning the subintervals [0,0.5], [0.5,1] with step $h_1 = h_2 = 0.05$.

Solving the system of equations (20) we obtain the numerical values of the parameters

$$\lambda_1^{\tilde{h}} = \begin{pmatrix} 1.9997698 \\ 5.000474 \end{pmatrix}, \ \lambda_2^{\tilde{h}} = \begin{pmatrix} 1.500072 \\ 2.1249704 \end{pmatrix}.$$

We find the numerical solutions at the other points of the subintervals applying the Runge-Kutta method of the 4th order to the following Cauchy problems:

$$\frac{dx_1}{dt} = A(t)x_1 + B(t)\lambda_1^{\tilde{h}} + f(t), \ t \in \left[0, \frac{1}{2}\right), \quad x_1(0) = x^0,$$

$$\frac{dx_2}{dt} = A(t)x_2 + B(t)\lambda_1^{\tilde{h}} + f(t), \ t \in \left[\frac{1}{2}, 1\right), \quad x_2\left(\frac{1}{2}\right) = \lambda_2^{\tilde{h}}.$$

The exact solution of the problem (28)-(30) is a pair $(\mu^*, x^*(t))$, where $\mu^* = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$,

$$x^*(t) = \begin{pmatrix} t+1 \\ t^3 - 2t + 3 \end{pmatrix}, \quad t \in [0,1].$$

The results of calculations of numerical and exact solutions at the partition points are presented in the following table:

t	$\tilde{x}_1(t)$ (numerical solution)	$x_1^*(t)$	$\tilde{x}_2(t)$ (numerical solution)	$x_2^*(t)$
0.05	1.050012	1.05	2.900125	2.900125
0.1	1.1000233	1.1	2.8009992	2.801
0.15	1.1500338	1.15	2.7033727	2.703375
0.2	1.2000432	1.2	2.6079955	2.608
0.25	1.2500516	1.25	2.5156176	2.515625
0.3	1.3000587	1.3	2.426989	2.427
0.35	1.3500644	1.35	2.3428599	2.342875
0.4	1.4000686	1.4	2.2639803	2.264
0.45	1.4500712	1.45	2.1911004	2.191125
0.5	1.500072	1.5	2.1249704	2.125
0.55	1.5500709	1.55	2.0663403	2.066375
0.6	1.6000679	1.6	2.0159606	2.016
0.65	1.650063	1.65	1.9745814	1.974625
0.7	1.7000563	1.7	1.9429532	1.943
0.75	1.7500478	1.75	1.9218265	1.921875
0.8	1.800038	1.8	1.9119519	1.912
0.85	1.8500272	1.85	1.9140803	1.914125
0.9	1.9000163	1.9	1.9289631	1.929
0.95	1.9500066	1.95	1.957352	1.957375
1	2	2	2	2

$\tilde{\mu}_1 = \lambda_{11}^{\tilde{h}}$ (numerical solution)	μ_1^*	$\tilde{\mu}_2 = \lambda_{12}^{\tilde{h}}$ (numerical solution)	μ_2^*
1.9997698	2	5. 000474	5

For the difference of the corresponding values of the exact and constructed solutions of the problem the following estimate is true:

$$\max \|\mu^* - \tilde{\mu}\| < 0.0002,$$

$$\max_{j=\overline{0,20}} \left\| x^*(t_j) - \widetilde{x}(t_j) \right\| < \varepsilon, \ \ \varepsilon = 0.000072.$$

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ПАРАМЕТРІ БАР ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН БАСҚАРУ ЕСЕБІН ШЕШУДІҢ БІР АЛГОРИТМІ ТУРАЛЫ

Аннотация. Шектелген аралықта параметрі бар сызықты жәй дифференциалдық теңдеу үшін басқару есебі қарастырылады. Аралықты бөлу мен қосымша параметрлер енгізу арқылы қарастырылып отырған сызықты басқару есебі параметрлері бар эквивалентті көпнүктелі шеттік есепке келтіріледі. Енгізілген параметрлерді анықтау үшін шешімнің ішкі бөліктеу нүктелерінің үзіліссіздік шарттарымен шеттік шарт пайдаланылады. Параметрлердің бекітілген мәнінде жәй дифференциалдық теңдеулер үшін Коши есептері шешіледі. Коши есептерінің шешімдерін шеттік шартқа және шешімнің үзіліссіздік шарттарына қою арқылы енгізілген параметрлерге қатысты сызықтық алгебралық теңдеулер жүйесі құрылады. Осы жүйенің шешілімділігі бастапқы басқару есебінің шешімінің бар болуын қамтамасыз етеді. Сызықтық алгебралық теңдеулер жүйесінің шешімі жәй дифференциалдық теңдеулер үшін ішкі интервалдағы матрицалық және векторлық Коши есептерінің шешімдері көмегімен жүзеге асырылады. Бастапқы басқару есебінің шешімін табудың жәй дифференциалдық теңдеулер үшін Коши есептерін шешуге арналған төртінші ретті Рунге-Куттаның әдісіне негізделген сандық әдісі ұсынылады.

Түйін сөздер: параметрі бар шеттік есеп, дифференциалдық теңдеу, шешілімділік, алгоритм.

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ОБ ОДНОМ АЛГОРИТМЕ РЕШЕНИЯ ЗАДАЧИ УПРАВЛЕНИЯ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПАРАМЕТРОМ

Аннотация. На ограниченном отрезке рассматривается задача управления для линейного обыкновенного дифференциального уравнения, содержащего параметр. Разбиением интервала и введением дополнительных параметров рассматриваемая линейная задача управления сводится к эквивалентной многоточечной краевой задаче с параметрами. Для определения введенных параметров используются условия непрерывности решения во внутренних точках разбиения и краевое условие. При фиксированных значениях параметров решаются задачи Коши для обыкновенных дифференциальных уравнений. Подставляя решения задач Коши в краевое условие и условия непрерывности решения составляется система линейных алгебраических уравнений относительно введенных параметров. Разрешимость этой системы обеспечивает существование решения исходной задачи управления. Нахождение системы линейных алгебраических уравнений осуществляется с помощью решений матричных и векторных задач Коши для обыкновенных дифференциальных уравнений на подинтервалах. Предлагается численный метод решения исходной задачи управления, основанный на методе Рунге-Кутта четвертого порядка для решения задач Коши обыкновенных дифференциальных уравнений.

Ключевые слова: краевая задача с параметром, дифференциальное уравнение, разрешимость, алгоритм.

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WELL-POSEDNESS OF A NONLOCAL PROBLEM WITH INTEGRAL CONDITIONS FOR THIRD ORDER SYSTEM OF THE PARTIAL DIFFERENTIAL EQUATIONS

Abstract. The nonlocal problem with integral conditions for the system of partial differential equations third-order is considered. The existence and uniqueness of classical solution to nonlocal problem with integral conditions for third-order system of partial differential equations are studied and the method for constructing their approximate solutions is proposed. Conditions of an unique solvability to nonlocal problem with integral conditions for third order system of partial differential equations are established. By introduction of new unknown functions, we have reduced the considered problem to an equivalent problem consisting of a nonlocal problem with integral conditions and parameters for a system of hyperbolic equations of second order and a integral relations. We have offered the algorithm for finding approximate solution to investigated problem and have proved its convergence. Sufficient conditions for the existence of unique solution to the equivalent problem with parameters are obtained. Well-posedness of the nonlocal problem with integral conditions for third order system of partial differential equations are established in the terms of well-posedness to nonlocal problem with integral conditions for system of hyperbolic equations second order.
Key Words: third order partial differential equations, nonlocal problem, integral condition, system of hyperbolic equations second order, solvability, algorithm.

1. Introduction. In recent years, there has been a great interest to a nonlocal problems for third order partial differential equations and systems. Such problems are appeared in the mathematical modeling of various natural science processes [1-9]. A lot of many works devoted to the investigate of various problems for third order partial differential equations with two independent variables, bibliography and analysis can be see in [1, 2, 5, 7-9]. The third order system of partial differential equations began to be studied relatively recently [5, 7-9].

In the present paper we consider the nonlocal problem with integral conditions for third order system of partial differential equations at a rectangular domain. We investigate the questions of existence and uniqueness of the classical solution to nonlocal problem with integral conditions for third order system of partial differential equations and its applications. Some types of initial-boundary value problems for third order system of partial differential equations are considered in [7-9].

Methods. For solve to considered problem we use a method of introduction additional functional parameters [10-32]. The original problem is reduced to an equivalent problem consisting from nonlocal problem with integral conditions for system of hyperbolic equations second order, containing functional parameters and integral relations. Sufficient conditions of the unique solvability to investigated problem are established in the terms of unique solvability to nonlocal problem with integral conditions for system of hyperbolic equations. Algorithms for finding of approximate solution to the equivalent problem are constructed. Well-posedness to nonlocal problem with integral conditions for third order system of partial differential equations are established in the terms of well-posedness to nonlocal problem with integral conditions for system of hyperbolic equations second order.

2. Statement of problem. At the domain $\Omega = [0,T]\times[0,\omega]$ we consider the following initial-boundary value problem for the special system of partial differential equations

$$\frac{\partial^3 u}{\partial t\partial x^2} = A(t,x)\frac{\partial^2 u}{\partial x^2} + B(t,x)\frac{\partial^2 u}{\partial t\partial x} + C(t,x)\frac{\partial u}{\partial x} + D(t,x)\frac{\partial u}{\partial t} + E(t,x)u + f(t,x),\ (t,x)\in\Omega, \quad (1)$$

$$\int_0^T L(\tau,x)\frac{\partial u(\tau,x)}{\partial x}d\tau = \varphi(x),\ x\in[0,\omega], \quad (2)$$

$$u(t,0) = \psi_0(t),\ t\in[0,T], \quad (3)$$

$$\int_0^\omega P(t,\xi)\frac{\partial u(t,\xi)}{\partial \xi}d\xi = \psi_1(t),\ t\in[0,T], \quad (4)$$

where $u(t,x) = col(u_1(t,x),u_2(t,x),...,u_n(t,x))$ is unknown function, the $n\times n$-matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$, $E(t,x)$ and n-vector function $f(t,x)$ are continuous on Ω, the $n\times n$-matrix $L(t,x)$ is continuous and continuously differentiable by x on Ω, the n-vector-function $\varphi(x)$ is continuously differentiable on $[0,\omega]$, the $n\times n$-matrix $P(t,x)$ is continuous and continuously differentiable by t on Ω, the n-vector-functions $\psi_0(t)$ and $\psi_1(t)$ are continuously differentiable on $[0,T]$.

A function $u(t,x)\in C(\Omega,R^n)$ having partial derivatives $\frac{\partial u(t,x)}{\partial x}\in C(\Omega,R^n)$, $\frac{\partial u(t,x)}{\partial t}\in C(\Omega,R^n)$, $\frac{\partial^2 u(t,x)}{\partial x^2}\in C(\Omega,R^n)$, $\frac{\partial^2 u(t,x)}{\partial t\partial x}\in C(\Omega,R^n)$, $\frac{\partial^3 u(t,x)}{\partial t\partial x^2}\in C(\Omega,R^n)$ is called a classical solution to problem (1)--(4) if it satisfies system (1) for all $(t,x)\in\Omega$, and integral and initial conditions (2), (3) and (4).

We investigate the questions of existence and uniqueness of the classical solutions to nonlocal problem with integral conditions for system of partial differential equations of third order (1)--(4) and the approaches of constructing its approximate solutions. For this goals, we applied the method of introduction additional functional parameters proposed in [10-32] for the solve of nonlocal boundary value problems for systems of hyperbolic equations with mixed derivatives. Considered problem is provided to nonlocal problem with integral conditions for system of hyperbolic equations of second order, including additional functions and integral relations. The algorithm of finding the approximate solution of the investigated problem is proposed and its convergence proved. Sufficient conditions of the existence unique classical solution to problem (1)--(4) are obtained in the terms of unique solvability to nonlocal problem with integral conditions for system of hyperbolic equations second order.

3. Scheme of the method and reduction to equivalent problem.

We introduce a new unknown function $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and re-write nonlocal problem with integral conditions (1)--(4) in the following form

$$\frac{\partial^2 v}{\partial t\partial x} = A(t,x)\frac{\partial v}{\partial x} + B(t,x)\frac{\partial v}{\partial t} + C(t,x)v + D(t,x)\frac{\partial u}{\partial t} + E(t,x)u + f(t,x),\ (t,x)\in\Omega, \quad (5)$$

$$\int_0^T L(\tau,x)v(\tau,x)d\tau = \varphi(x),\ x\in[0,\omega], \quad (6)$$

$$\int_0^{\omega} P(t,\xi)v(t,\xi)d\xi = \psi_1(t), \ t \in [0,T], \tag{7}$$

$$u(t,x) = \psi_0(t) + \int_0^x v(t,\xi)d\xi, \ \frac{\partial u(t,x)}{\partial t} = \dot{\psi}_0(t) + \int_0^x \frac{\partial v(t,\xi)}{\partial t} d\xi, \ (t,x) \in \Omega. \tag{8}$$

Here the condition (3) is taken account in (8).

A pair functions $(v(t,x),u(t,x))$, where the function $v(t,x) \in C(\Omega,R^n)$ has partial derivatives $\frac{\partial v(t,x)}{\partial x} \in C(\Omega,R^n)$, $\frac{\partial v(t,x)}{\partial t} \in C(\Omega,R^n)$, $\frac{\partial^2 v(t,x)}{\partial t \partial x} \in C(\Omega,R^n)$, the function $u(t,x) \in C(\Omega,R^n)$ has partial derivatives $\frac{\partial u(t,x)}{\partial x} \in C(\Omega,R^n)$, $\frac{\partial u(t,x)}{\partial t} \in C(\Omega,R^n)$, $\frac{\partial^2 u(t,x)}{\partial x^2} \in C(\Omega,R^n)$, $\frac{\partial^2 u(t,x)}{\partial t \partial x} \in C(\Omega,R^n)$, $\frac{\partial^3 u(t,x)}{\partial t \partial x^2} \in C(\Omega,R^n)$, is called a solution to problem (5)--(8) if it satisfies of the system of hyperbolic equations (5) for all $(t,x) \in \Omega$, the boundary conditions (6), (7), and the integral relation (8).

At fixed $u(t,x)$ the problem (5)--(7) is the nonlocal problem with integral conditions for the system of hyperbolic equations with respect to $v(t,x)$ on Ω. The integral relations (8) allow us to determine the unknown functions $u(t,x)$ and its partial derivative $\frac{\partial u(t,x)}{\partial t}$ for all $(t,x) \in \Omega$.

4. Algorithm for finding of the approximate solution to problem (5—(8).

The unknown function $v(t,x)$ will be determined from nonlocal problem with integral conditions for system of hyperbolic equations (5)--(7). The unknown function $u(t,x)$ and its partial derivative $\frac{\partial u(t,x)}{\partial t}$ will be found from integral relations (8). If we known the function $u(t,x)$ and its partial derivative $\frac{\partial u(t,x)}{\partial t}$, then from nonlocal problem with integral conditions (5)--(7) we find the function $v(t,x)$. Conversely, if we known the function $v(t,x)$ and its partial derivative $\frac{\partial v(t,x)}{\partial t}$, then from integral relations (8) we find the function $u(t,x)$ and its partial derivative $\frac{\partial u(t,x)}{\partial t}$. Since the functions $u(t,x)$ and $v(t,x)$ are unknowns together for finding of the solution to problem (5)--(8) we use an iterative method.

The solution to problem (5)--(8) is the pair functions $(v^*(t,x),u^*(t,x))$ we defined as a limit of sequence of pairs $(v^{(k)}(t,x),u^{(k)}(t,x))$, $k = 0,1,2,...$, according to the following algorithm:

Step 0. 1) Suppose in the right-hand part of system (5) $u(t,x) = \psi_0(t)$, $\frac{\partial u(t,x)}{\partial t} = \dot{\psi}_0(t)$, from nonlocal problem with integral conditions (5)--(7) we find the initial approximation $v^{(0)}(t,x)$ and its partial derivatives for all $(t,x) \in \Omega$;

2) From integral relations (8) under $v(t,x) = v^{(0)}(t,x)$, $\frac{\partial v(t,x)}{\partial t} = \frac{\partial v^{(0)}(t,x)}{\partial t}$, we find the functions $u^{(0)}(t,x)$ and $\frac{\partial u^{(0)}(t,x)}{\partial t}$ for all $(t,x) \in \Omega$.

Step 1. 1) Suppose in the right-hand part of system (5) $u(t,x)=u^{(0)}(t,x)$, $\dfrac{\partial u(t,x)}{\partial t}=\dfrac{\partial u^{(0)}(t,x)}{\partial t}$, from nonlocal problem with integral conditions (5)--(7) we find the first approximation $v^{(1)}(t,x)$ and its partial derivatives for all $(t,x)\in\Omega$.

2) From integral relations (8) under $v(t,x)=v^{(1)}(t,x)$, $\dfrac{\partial v(t,x)}{\partial t}=\dfrac{\partial v^{(1)}(t,x)}{\partial t}$, we find the functions $u^{(1)}(t,x)$ and $\dfrac{\partial u^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

And so on.

Step k. 1) Suppose in the right-hand part of system (5) $u(t,x)=u^{(k-1)}(t,x)$, $\dfrac{\partial u(t,x)}{\partial t}=\dfrac{\partial u^{(k-1)}(t,x)}{\partial t}$, from nonlocal problem with integral conditions (5)--(7) we find the k-th approximation $v^{(k)}(t,x)$ and its partial derivatives for all $(t,x)\in\Omega$:

$$\frac{\partial^2 v^{(k)}}{\partial t\partial x}=A(t,x)\frac{\partial v^{(k)}}{\partial x}+B(t,x)\frac{\partial v^{(k)}}{\partial t}+C(t,x)v^{(k)}+D(t,x)\frac{\partial u^{(k-1)}}{\partial t}+E(t,x)u^{(k-1)}+f(t,x),\ (t,x)\in\Omega, \quad (9)$$

$$\int_0^T L(\tau,x)v^{(k)}(\tau,x)d\tau=\varphi(x),\ x\in[0,\omega], \quad (10)$$

$$\int_0^\omega P(t,\xi)v^{(k)}(t,\xi)d\xi=\psi_1(t),\ t\in[0,T]. \quad (11)$$

2) From integral relations (8) under $v(t,x)=v^{(k)}(t,x)$, $\dfrac{\partial v(t,x)}{\partial t}=\dfrac{\partial v^{(k)}(t,x)}{\partial t}$, we find the function $u^{(k)}(t,x)$ and $\dfrac{\partial u^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$:

$$u^{(k)}(t,x)=\psi_0(t)+\int_0^x v^{(k)}(t,\xi)d\xi,\ \frac{\partial u^{(k)}(t,x)}{\partial t}=\dot{\psi}_0(t)+\int_0^x\frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi,\ (t,x)\in\Omega. \quad (12)$$

$k=1,2,3,\dots$.

5. The main result.

Consider nonlocal problem with integral conditions (5)-(7) at fixed $u(t,x)$. Then nonlocal problem with integral conditions for system of hyperbolic equations can be have the following form

$$\frac{\partial^2 v}{\partial t\partial x}=A(t,x)\frac{\partial v}{\partial x}+B(t,x)\frac{\partial v}{\partial t}+C(t,x)v+F(t,x),\ (t,x)\in\Omega, \quad (13)$$

$$\int_0^T L(\tau,x)v(\tau,x)d\tau=\varphi(x),\ x\in[0,\omega], \quad (14)$$

$$\int_0^\omega P(t,\xi)v(t,\xi)d\xi=\psi_1(t),\ t\in[0,T], \quad (15)$$

where n-vector function $F(t,x)$ is continuous on Ω.

The following theorem gives conditions of feasibility and convergence of the constructed algorithm and the conditions of the existence unique solution to problem (5)--(8).

Theorem 1. *Suppose that*

i) the $n\times n$*-matrices* $A(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$, $E(t,x)$, *and* n*-vector function* $f(t,x)$

are continuous on Ω*;*

ii) the $n\times n$ *-matrix* $L(t,x)$ *is continuous and continuously differentiable by* x *on* Ω*; and the* n*-vector-function* $\varphi(x)$ *continuously differentiable on* $[0,\omega]$*;*

iii) the $n\times n$ *-matrix* $P(t,x)$ *is continuous and continuously differentiable by* t *on* Ω*; the* n*-vector-functions* $\psi_0(t)$ *and* $\psi_1(t)$ *are continuously differentiable on* $[0,T]$*;*

iv) Nonlocal problem with integral conditions for system of hyperbolic equations (13)--(15) has a unique classical solution.

Then equivalent nonlocal problem for system of hyperbolic equations with integral conditions and parameters (5)—(8) has a unique solution.

Proof. Let the conditions i) - iv) of Theorem 1 are fulfilled. By the algorithm, proposing above, on 0^{th} step we have

$$\frac{\partial^2 v^{(0)}}{\partial t\partial x}=A(t,x)\frac{\partial v^{(0)}}{\partial x}+B(t,x)\frac{\partial v^{(0)}}{\partial t}+C(t,x)v^{(0)}+D(t,x)\dot\psi_0(t)+E(t,x)\psi_0(t)+f(t,x),\ (t,x)\in\Omega, \quad (16)$$

$$\int_0^T L(\tau,x)v^{(0)}(\tau,x)d\tau=\varphi(x),\ x\in[0,\omega], \quad (17)$$

$$\int_0^\omega P(t,\xi)v^{(0)}(t,\xi)d\xi=\psi_1(t),\ t\in[0,T]. \quad (18)$$

Since condition iv) is valid, problem (16)—(18) has a unique classical solution $v^{(0)}(t,x)$ and the following inequality holds

$$\max\left(\max_{(t,x)\in\Omega}\left\|v^{(0)}(t,x)\right\|,\max_{(t,x)\in\Omega}\left\|\frac{\partial v^{(0)}(t,x)}{\partial x}\right\|,\max_{(t,x)\in\Omega}\left\|\frac{\partial v^{(0)}(t,x)}{\partial t}\right\|\right)\le$$

$$\le K(x)\max\left(\max_{(t,x)\in\Omega}\left\|f(t,x)\right\|,\max_{t\in[0,T]}\left\|\psi_1(t)\right\|,\max_{x\in[0,\omega]}\left\|\varphi(x)\right\|,\max_{(t,x)\in\Omega}\left\|F^{(0)}(t,x)\right\|\right),$$

where the function $K(x)$ is continuous on $[0,\omega]$, positive and independent on functions $f(t,x)$, $\psi_1(t)$, $\varphi(x)$, $F^{(0)}(t,x)=D(t,x)\dot\psi_0(t)+E(t,x)\psi_0(t)$.

Initial approximations $u^{(0)}(t,x)$ and $\frac{\partial u^{(0)}(t,x)}{\partial t}$ are determined by the following form

$$u^{(0)}(t,x)=\psi_0(t)+\int_0^x v^{(0)}(t,\xi)d\xi,\ \frac{\partial u^{(0)}(t,x)}{\partial t}=\dot\psi_0(t)+\int_0^x\frac{\partial v^{(0)}(t,\xi)}{\partial t}d\xi,\ (t,x)\in\Omega. \quad (19)$$

Then functions $u^{(0)}(t,x)$ and $\frac{\partial u^{(0)}(t,x)}{\partial t}$ satisfy the estimate

$$\max\left(\max_{t\in[0,T]}\left\|u^{(0)}(t,x)\right\|,\max_{t\in[0,T]}\left\|\frac{\partial u^{(0)}(t,x)}{\partial t}\right\|\right)\le\max\left(\max_{t\in[0,T]}\left\|\psi_0(t)\right\|,\max_{t\in[0,T]}\left\|\dot\psi_0(t)\right\|\right)+$$

$$+\int_0^x\max\left(\max_{t\in[0,T]}\left\|v^{(0)}(t,\xi)\right\|,\max_{t\in[0,T]}\left\|\frac{\partial v^{(0)}(t,\xi)}{\partial t}\right\|\right)d\xi.$$

For kth approximations determined by relations (9)—(12), we have

$$\max\left(\max_{(t,x)\in\Omega}\left\|v^{(k)}(t,x)\right\|,\max_{(t,x)\in\Omega}\left\|\frac{\partial v^{(k)}(t,x)}{\partial x}\right\|,\max_{(t,x)\in\Omega}\left\|\frac{\partial v^{(k)}(t,x)}{\partial t}\right\|\right)\le$$

$$\leq K(x)\max\left(\max_{(t,x)\in\Omega}\|f(t,x)\|,\max_{t\in[0,T]}\|\psi_1(t)\|,\max_{x\in[0,\omega]}\|\varphi(x)\|,\max_{(t,x)\in\Omega}\|F^{(k-1)}(t,x)\|\right), \quad (20)$$

$$\max\left(\max_{t\in[0,T]}\|u^{(k)}(t,x)\|,\max_{t\in[0,T]}\left\|\frac{\partial u^{(k)}(t,x)}{\partial t}\right\|\right)\leq \max\left(\max_{t\in[0,T]}\|\psi_0(t)\|,\max_{t\in[0,T]}\|\dot{\psi}_0(t)\|\right)+$$

$$+\int_0^x \max\left(\max_{t\in[0,T]}\|v^{(k)}(t,\xi)\|,\max_{t\in[0,T]}\left\|\frac{\partial v^{(k)}(t,\xi)}{\partial t}\right\|\right)d\xi, \quad (21)$$

where $F^{(k-1)}(t,x)=D(t,x)\dfrac{\partial u^{(k-1)}(t,x)}{\partial t}+E(t,x)u^{(k-1)}(t,x)$.

Let $\Delta v^{(k)}(t,x)=v^{(k+1)}(t,x)-v^{(k)}(t,x)$, $\Delta_x v^{(k)}(t,x)=\dfrac{\partial v^{(k+1)}(t,x)}{\partial x}-\dfrac{\partial v^{(k)}(t,x)}{\partial x}$,

$\Delta_t v^{(k)}(t,x)=\dfrac{\partial v^{(k+1)}(t,x)}{\partial t}-\dfrac{\partial v^{(k)}(t,x)}{\partial t}$, $\Delta u^{(k)}(t,x)=u^{(k+1)}(t,x)-u^{(k)}(t,x)$,

$\Delta_t u^{(k)}(t,x)=\dfrac{\partial u^{(k+1)}(t,x)}{\partial t}-\dfrac{\partial u^{(k)}(t,x)}{\partial t}$, $k=1,2,3,\ldots$.

Then, for differences $\Delta v^{(k)}(t,x)$, $\Delta_x v^{(k)}(t,x)$, $\Delta_t v^{(k)}(t,x)$, $\Delta u^{(k)}(t,x)$, $\Delta_t u^{(k)}(t,x)$ are valid the following inequalities

$$\max\left(\max_{(t,x)\in\Omega}\|\Delta v^{(k)}(t,x)\|,\max_{(t,x)\in\Omega}\|\Delta_x v^{(k)}(t,x)\|,\max_{(t,x)\in\Omega]}\|\Delta_t v^{(k)}(t,x)\|\right)\leq$$

$$\leq K(x)\max\left(\max_{(t,x)\in\Omega}\|\Delta u^{(k-1)}(t,x)\|,\max_{(t,x)\in\Omega]}\|\Delta_t u^{(k-1)}(t,x)\|\right), \quad (22)$$

$$\max\left(\max_{t\in[0,T]}\|\Delta u^{(k)}(t,x)\|,\max_{t\in[0,T]}\|\Delta_t u^{(k)}(t,x)\|\right)\leq\int_0^x \max\left(\max_{t\in[0,T]}\|\Delta v^{(k)}(t,\xi)\|,\max_{t\in[0,T]}\|\Delta_t v^{(k)}(t,\xi)\|\right)d\xi. \quad (23)$$

From estimates (20)—(23) is follows

$$\max\left(\max_{t\in[0,T]}\|\Delta u^{(k)}(t,x)\|,\max_{t\in[0,T]}\|\Delta_t u^{(k)}(t,x)\|\right)\leq\frac{[\hat{K}x]^k}{k!}\cdot\max\left(\max_{(t,x)\in\Omega}\|\Delta v^{(0)}(t,x)\|,\max_{(t,x)\in\Omega}\|\Delta_t v^{(0)}(t,x)\|\right),$$

where $\hat{K}=\max_{x\in[0,\omega]}K(x)$.

Hence, we obtain the uniform convergence of the sequences $\{u^{(k)}(t,x)\}$ and $\left\{\dfrac{\partial u^{(k)}(t,x)}{\partial t}\right\}$ to functions $u^*(t,x)$ and $\dfrac{\partial u^*(t,x)}{\partial t}$ on Ω, respectively, as $k\to\infty$.

Then the sequences $\{v^{(k)}(t,x)\}$, $\left\{\dfrac{\partial v^{(k)}(t,x)}{\partial x}\right\}$, and $\left\{\dfrac{\partial v^{(k)}(t,x)}{\partial t}\right\}$ also will be convergent to functions $v^*(t,x)$, $\dfrac{\partial v^*(t,x)}{\partial x}$, and $\dfrac{\partial v^*(t,x)}{\partial t}$ on Ω, respectively, as $k\to\infty$.

The uniqueness of solution to problem (5)—(8) is proved by contradiction.

Theorem 1 is proved.

From equivalence of problem (1)—(4) and (5)—(8) follows the following assertion.

Theorem 2. *Suppose that the conditions i) - iv) of Theorem 1 are fulfilled.*

Then nonlocal problem with integral conditions for third order system of partial differential *equations (1)--(4) has a unique classical solution.*

Conditions of Theorem 1 are sufficient of the well-posedness to nonlocal problem with integral solutions for third order system of partial differential equations (1)—(4).

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МРНТИ 27.31.15

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КОРРЕКТНАЯ РАЗРЕШИМОСТЬ НЕЛОКАЛЬНОЙ ЗАДАЧИ С ИНТЕГРАЛЬНЫМИ ДЛЯ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ ТРЕТЬЕГО ПОРЯДКА

Аннотация. Рассматривается нелокальная задача с интегральными условиями для системы дифференциальных уравнений в частных производных третьего порядка. Исследуются вопросы существования и едиственности классического решения нелокальной задачи для системы дифференциальных уравнений в частных производных третьего порядка и предлагаются методы построения их приближенных решений. Установлены условия однозначной разрешимости нелокальной задачи для системы дифференциальных уравнений в частных производных третьего порядка. Путем введения новой неизвестной функции исследуемая задача сведена к эквивалентной задаче, состоящей из нелокальной задачи для системы гиперболических уравнений второго порядка с интегральными условиями и функциональными параметрами и интегрального соотношения. Предложены алгоритмы нахождения приближенного решения исследуемой задачи и доказана их сходимость. Получены достаточные условия существования единственного решения эквивалентной задачи с параметрами. Корректная разрешимость нелокальной задачи с интегральными условиями для системы дифференциальных уравнений в частных производных третьего порядка получены в терминах корректной разрешимости нелокальной задачи с интегральными условиями для системы гиперболических уравнений второго порядка.

Ключевые слова: дифференциальное уравнение в частных производных третьего порядка, нелокальная задача, интегральное условие, система гиперболических уравнений второго порядка, разрешимость,

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ҮШІНШІ РЕТТІ ДЕРБЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІ ҮШІН ИНТЕГРАЛДЫҚ ШАРТТАРЫ БАР БЕЙЛОКАЛ ЕСЕПТІҢ КОРРЕКТІЛІ ШЕШІЛІМДІЛІГІ

Аннотация. Үшінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есеп қарастырылады. Үшінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін бейлокал есептің классикалық шешімінің бар болуы мен жалғыздығы мәселелері зерттеледі және олардың жуық шешімдерін тұрғызу әдістері ұсынылады. Үшінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің бірмәнді шешілімдігінің шарттары тағайындалған. Жаңа белгісіз функция енгізу арқылы зерттеліп отырған есеп гиперболалық теңдеулер жүйесі үшін интегралдық шарттары және параметрлері бар бейлокал есептен және интегралдық қатынастан тұратын пара-пар есепке келтірілген. Зерттеліп отырған есептің жуық шешімін табу алгоритмдері ұсынылған және олардың жинақтылығы дәлелденген. Параметрлері бар пара-пар есептің жалғыз шешімінің бар болуының жеткілікті шарттары алынған. Үшінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің коррректілі шешілімділігінің шарттары екінші ретті гиперболалық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің коррректілі шешілімділігі терминінде алынған.

Түйін сөздер: үшінші ретті дербес туындылы дифференциалдық теңдеу, бейлокал есеп, интегралдық шарт, екінші ретті гиперболалық теңдеулер жүйесі, шешілімділік, алгоритм.

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N E W S
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THE PROBLEM OF THE OSCILLATION OF THE ELASTIC LAYER BOUNDED BY RIGID BOUHDARIES

Abstract: In the case of harmonic oscillations of a cylindrical shell, the phase velocity is expressed in terms of the frequency of natural oscillations freely supported along the edges of the shell, and therefore, the study of waves in plane and circular elements has the most direct relation to the problem of determining its own forms and oscillation frequencies shells finite length. Below let us consider some problems of oscillation of an elastic layer bounded by rigid boundaries under the influence of a normal or rotational shear stress. The solutions of the problems under consideration are obtained by using integral transformations by the coordinate.

Key words: harmonic oscillations, cylindrical shells, phase velocity, frequency, eigenvibrations, Bessel function, wave, anisotropic, layer.

First we consider the problem for a half-space under the assumption that the half-space $z > 0$ is an anisotropic medium with the axis of symmetry of the mechanical properties (axis z), and the surface of which at the moment $t = 0$ impulse voltage applied $\xi_{z0} = -f(r,t)$.

Because of the symmetry of the mechanical properties of the medium relative to the axis z of the unique nonzero component of the displacement vector $U_0(r,z,t)$, only the voltage ξ_{r0} and ξ_{z0} the ones determined by formulas

$$\begin{aligned}\xi_{r0} &= C_{44}\left(\frac{\partial U_0}{\partial r} - \frac{U_0}{r}\right),\\ \xi_{z0} &= C_{66}\frac{\partial U_0}{\partial z}\end{aligned} \tag{1}$$

The equation of motion reduces to one

$$\frac{\partial \xi_{r0}}{\partial r} + \frac{\partial \xi_{r0}}{\partial z} + \frac{2\xi_{r0}}{r} = \rho\frac{\partial^2 U_0}{\partial t^2} \tag{2}$$

Substituting the expressions for ξ_{r0} and ξ_{z0} from relations (1) into equation (2), we bring it to the form:

$$\frac{\partial^2 U_0}{\partial r^2} + \frac{1}{r}\frac{\partial U_0}{\partial r} - \frac{U_0}{r^2} + \gamma^2\frac{\partial^2 U_0}{\partial z^2} = \frac{1}{b^2}\frac{\partial^2 U_0}{\partial t^2} \tag{3}$$

where

$$b^2 = \frac{C_{44}}{\rho}; \ \gamma^2 = \frac{C_{66}}{C_{44}}$$

If the half-space is isotropic, then $\gamma = 1$ and $b = \sqrt{\frac{\mu}{\rho}}$.

The boundary conditions for U_0 have the form:

$$\xi_{z0} = -f(r,t) \text{ at } z = 0, \ t \geq 0 \tag{4}$$

$$U_0 \to 0 \text{ at } z \to \infty \tag{5}$$

The initial conditions of the problem are zero, i.e.

$$U_0 = \frac{\partial U_0}{\partial t} = 0 \ \text{ at } \ t = 0 \tag{6}$$

The solution of equation (3) for the boundary (4) - (5) and the initial conditions (6) will be sought, by applying the Laplace transform t. Assuming that

$$U(r,z,p) = \int_0^\infty U_0(r,z,t)e^{-pt}dt, \ \operatorname{Re} p > 0 \tag{7}$$

Obviously, for the function $U(r,z,p)$ we obtain equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \left(\frac{1}{r^2} + \frac{p^2}{b^2}\right)U + \gamma^2 \frac{\partial^2 U}{\partial z^2} = 0 \tag{8}$$

Moreover, U must satisfy the boundary conditions:

$$\frac{\partial U}{\partial z} = -\frac{f_0(r,p)}{C_{66}} \quad \text{at } z = 0, \quad t > 0 \tag{9}$$

$$U_0 \to 0 \text{ at } z \to \infty \tag{10}$$

Where

$$f_0(r,p) = \int_0^\infty (r,t)e^{-pt}dt.$$

The general solution of equation (8) is sought by the method of separation of variables (the Fourier method) and has the form:

$$U(r,z,p) = \int_0^\infty \alpha \left[A(\alpha,p)e^{-\frac{z}{\gamma}\sqrt{\alpha^2 + \frac{p^2}{b^2}}} + B(\alpha,p)e^{\frac{z}{\gamma}\sqrt{\alpha^2 + \frac{p^2}{b^2}}} \right] J_1(\alpha r)d\alpha. \tag{11}$$

where $A(\alpha,p)$ and $B(\alpha,p)$ are determined from the boundary conditions (9) - (10) and from (10) it follows that

$$B(\alpha, p) = 0 \tag{12}$$

Using the boundary condition (9), to determine $A(\alpha, p)$ we obtain the integral equation:

$$\int_0^\infty \alpha A(\alpha, p)\sqrt{\alpha^2 + \frac{p^2}{b^2}} J_1(\alpha r) d\alpha = \frac{\gamma}{C_{66}} f_0(r, p) \tag{13}$$

Suppose

$$f_0(r, p) = \int_0^\infty \alpha f_1(\alpha, p) J_1(\alpha r) d\alpha \tag{14}$$

Then

$$A(\alpha, p) = \frac{\gamma f_1(\alpha, p)}{C_{66}\sqrt{\alpha^2 + \frac{p^2}{b^2}}} \tag{15}$$

Substituting expression (12) and (15) into formula (11), for $U(r, z, p)$, we obtain the following expression:

$$U(r, z, p) = \frac{\gamma}{C_{66}} \int_0^\infty \frac{\alpha f_1(\alpha, p)}{\sqrt{\alpha^2 + \frac{p^2}{b^2}}} J_1(\alpha r) e^{-\frac{z}{\gamma}\sqrt{\alpha^2 + \frac{p^2}{b^2}}} d\alpha \tag{16}$$

Let us consider a special case, when

$$f_0(r, p) = \frac{\varphi_0(p)}{r} \tag{17}$$

In the case (17), the function

$$f_1(\alpha, p) = \frac{\varphi_0(p)}{\alpha}$$

and (16)

$$U(r,z,p) = \frac{\gamma}{C_{66}} \int_0^\infty \frac{\varphi_0(p)}{\sqrt{\alpha^2 + \frac{p^2}{b^2}}} e^{-\frac{z}{\gamma}\sqrt{\alpha^2 + \frac{p^2}{b^2}}} J_1(\alpha r) d\alpha = \frac{1}{C_{66}} \varphi_0(p) I_{\frac{1}{2}}\left[\frac{p}{2b}\left(\sqrt{\frac{z^2}{\gamma^2} + r^2} - \frac{z}{\gamma}\right)\right] K_{\frac{1}{2}}\left[\frac{p}{2b}\left(\sqrt{\frac{z^2}{\gamma^2} + r^2} + \frac{z}{\gamma}\right)\right]$$

where $K_{\frac{1}{2}}, I_{\frac{1}{2}}$ is the Bessel function of the imaginary argument. Using the representations of the functions $I_{\frac{1}{2}}(\zeta)$ and $K_{\frac{1}{2}}(\zeta)$, for $U(r, z, p)$, we find that

$$U(r, z, p) = \frac{b\varphi_0(p)}{C_{66} r p}\left[e^{-\frac{pz}{\gamma a}} - e^{-\frac{p}{a}\sqrt{\frac{z^2}{\gamma^2} + r^2}}\right] \tag{18}$$

Turning expression (18) to p , for the sought quantity $U_0(r,z,t)$, we obtain expression

$$U_0(r,z,t)=\frac{b}{\gamma C_{66}r}\int_0^t f_1(t-\xi)\left[H\left(\xi-\frac{z}{\gamma b}\right)-H\left(\xi-\frac{1}{b}\sqrt{\frac{z^2}{\gamma^2}+r^2}\right)\right]d\xi \tag{19}$$

Where

$$f(r,t)=\frac{f_1(t)}{r}.$$

The resulting expression for $U_0(r,z,t)$ consists of two terms, the first term corresponding to a plane wave propagating in a half-space with a velocity γb and parallel to the plane $z=0$, and the second term to a diffracted wave having the form of a semi-ellipsoid of revolution (hemispheres at $\gamma=1$) and in contact with a plane wave on the axis of rotation at $z=b\gamma t$.

In addition, it follows from (19) that $U_0(r,z,t)$ decays in r as $1/r$.

If the acting function $f(r,t)$ is arbitrary, then we represent it in the form of a Schlemmich series:

$$f(r,t)=\frac{1}{r}\sum_{j=0}^{\infty}a_j(t)J_0(jr)=\frac{f_r(t)}{r} \tag{20}$$

Where

$$a_0(t)=\frac{1}{\pi}\int_0^{\pi}\left\{f_r(0,t)+U\int_0^1\frac{\frac{\partial f_r(\xi U,t)}{\partial r}}{\sqrt{1-\xi^2}}d\xi\right\}dU;$$

$$a_j(t)=\frac{2}{\pi}\int H\cos(jU)\left\{\int_0^1\frac{\frac{\partial f_r(\xi U,t)}{\partial r}}{\sqrt{1-\xi^2}}d\xi\right\}dU;\ j=1,2,... \tag{21}$$

For $f(r,t)$ the form (20), the function $f_1(\alpha,p)$ in formula (16) is equal to

$$f_1(\alpha,p)=\frac{1}{\alpha}\sum_{j=0}^{\infty}a_{j0}(p)H(\alpha-j) \tag{22}$$

Where

$$a_{j0}(p)=\int_0^{\infty}a_j(p)e^{-pt}dt. \tag{23}$$

Therefore,

$$U(r,z,p)=\frac{\gamma}{C_{66}}\sum_{j=0}^{\infty}a_{j0}(p)\int_0^{\infty}\frac{e^{-\frac{z}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}}{\sqrt{\alpha^2+\frac{p^2}{b^2}}}J_1(\alpha r)d\alpha \tag{24}$$

Turning (24) to p and applying the convolution theorem, we obtain

$$U_0(r,z,p)=\frac{\gamma}{2C_{66}}\sum_{j=0}^{\infty}\int_g^t a_j(t-\xi)T_j(r,z,\xi)d\xi \tag{25}$$

where

$$T_j(r,z,\xi)=b\int_j^{\infty} J_1(\alpha r)J_0\left[\alpha\sqrt{b^2t^2-\frac{z^2}{\gamma^2}}\right]d\alpha \tag{26}$$

We generalize the problem for an anisotropic layer of thickness h. For $z=h$ there can be two types of boundary conditions:

1) $\tau_{z0}=-F(r,t)$ at $z=h$

2) $U_0=0$ at $z=h$

If $F(r,t)=0$, then condition (27) means that surface $z=h$ is voltages -free.

First we consider the problem when the boundary condition (27) is given for $z=h$.

The general solution of the problem still has the form (11), and

$$A(\alpha,p)=B(\alpha,p)+\frac{\gamma}{C_{66}}f_1(\alpha,p)$$

$$B(\alpha,p)=\frac{\gamma}{2C_{66}}\frac{f_1(\alpha,p)e^{-\frac{h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}-f_2(\alpha,p)}{sh\left[\frac{h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}\right]} \tag{29}$$

where $f_1(\alpha,p)$ is determined from equation (13), and $f_2(\alpha,p)$ is determined from equation

$$F_0(r,p)=\int_0^{\infty}\alpha f_2(\alpha,p)J_1(\alpha r)d\alpha \tag{30}$$

Suppose

$F_0(r,p)=0$,

i.e. $f_2(\alpha,p)=0$ and $f(r,t)=f_1(r,t)$.

Then

$$U(r,z,p)=\frac{1}{\gamma C_{66}}\int_0^{\infty}\frac{\varphi_0(p)}{\sqrt{\alpha^2+\frac{p^2}{b^2}}}\times\left[\frac{e^{-\frac{z-h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}+e^{\frac{z-h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}}{e^{\frac{h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}-e^{-\frac{h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}}\right]J_1(\alpha r)d\alpha$$

or

$$U(r,z,p)=\frac{1}{\gamma C_{66}}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\varphi_0(p)}{\sqrt{\alpha^2+\frac{p^2}{b^2}}}\times\left\{e^{-\frac{z+2nh}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}+e^{\frac{z-2(n+1)h}{\gamma}\sqrt{\alpha^2+\frac{p^2}{b^2}}}\right\}J_1(\alpha r)d\alpha \tag{31}$$

Calculating the quadratures by α in (31) and then reversing by p, for $U_0(r,z,t)$ we obtain the expression:

$$U_0(r,z,t)=\frac{b}{r\gamma C_{66}}\sum_{n=0}^{n_1}\left\{\int_0^t f_1(t-\xi)H\left[\xi-\frac{z+2nh}{\gamma b}\right]d\xi-\right.$$

$$-\int_0^t f_1(t-\xi)H\left[\xi-\frac{1}{b}\sqrt{r^2+\left(\frac{z+2nh}{\gamma}\right)^2}\right]d\xi\Bigg\}+$$

$$+\frac{b}{r\gamma C_{66}}\sum_{n=0}^{n_2}\left\{\int_0^t f_1(t-\xi)H\left[\xi+\frac{z-2(n+1)h}{\gamma b}\right]d\xi-\right.$$

$$-\int_0^t f_1(t-\xi)H\left[\xi-\frac{1}{b}\sqrt{r^2+\left(\frac{z-2(n+1)h}{\gamma}\right)^2}\right]d\xi\Bigg\} \qquad (32)$$

where

$$n_1=\left[\frac{z+2nh}{\gamma bt}\right],\quad n_2=\left[\frac{-z+2(n+1)h}{\gamma bt}\right]$$

$[\xi]$ is the integer part of the number ξ.

Formula (32) contains all flat and diffracted incident and reflected waves.

Similarly, the problem for the layer under boundary condition (28) is solved and we have

$$U_0(r,z,t)=\frac{b}{r\gamma C_{66}}\sum_{n=0}^{n_1}(-1)^n\left\{\int_0^t f_1(t-\xi)H\left[\xi\ \frac{z+2nh}{\gamma b}\right]d\xi-\right.$$

$$-\int_0^t f_1(t-\xi)H\left[\xi-\frac{1}{b}\sqrt{r^2+\left(\frac{z+2nh}{\gamma}\right)^2}\right]d\xi\Bigg\}-$$

$$(33)$$

$$-\frac{b}{r\gamma C_{66}}\sum_{n=0}^{\infty}(-1)^n\left\{\int_0^t f_1(t-\xi)H\left[\xi+\frac{z-2(n+1)h}{\gamma b}\right]d\xi-\right.$$

$$-\int_0^t f_1(t-\xi)H\left[\xi-\frac{1}{b}\sqrt{r^2+\left(\frac{z-2(n+1)h}{\gamma}\right)^2}\right]d\xi\Bigg\}$$

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[14] Seitmuratov,A.Zh B.M.Nurlanova, N.K.Medeubaev. BULLETIN of the Karaganda University 109-117 №3(81)/2017 Mathematics Series ISSN 2518-2729/ Equations of vibration of a two-dimensionally layered plate strictly based on the decision of various boundary-value

А.Ж. Сейтмұратов[1], С.Ш. Тілеубай[1], С.К. Тоқсанова[1],
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ҚАТАҢ ШЕКАРАЛАРМЕН ШЕКТЕЛГЕН СЕРПІМДІ ҚАБАТ ТЕРБЕЛІСІ ЖАЙЛЫ ЕСЕП

Аннотация. Цилиндірлік қабықшалардың гармоникалық тербелісі жағдайында фазалық жылдамдық сол қабықшалардың шетіне еркін бекітілген өзіндік жиілік теңдеуі арқылы өрнектеледі, сондықтан жалпақ және айналмалы элементтердің тербелісін зерттеу түпкілікті ұзындықтағы өзіндік пішіндері мен тербеліс жиіліге тікелей қатысты. Берілген төмендегі есепте қалыпты немесе айналмалы керілу кернеуі жағдайында, қатаң шегарада шектелген серпімді қабат тербеліс теңдеулері қарастырылады. Қарастырылатын есепті шешу мәселелердің шешімдері координат бойынша интегралдық түрлендіру әдістерін қолдану арқылы алынды.
Түйін: гармоникалық тербеліс, цилиндрлік қабықшалар, фазалық жылдамдық, жиілік, өзіндік тербеліс, Бессель функциясы, толқын, анизотропты, қатпар

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ЗАДАЧА О КОЛЕБАНИИ УПРУГОГО СЛОЯ ОГРАНИЧЕННЫЕ ЖЕСТКИМИ ГРАНИЦАМИ

Аннотация В случае гармонических колебаний цилиндрической оболочки фазовая скорость выражается через частоту собственных колебаний свободно опертой по краям оболочки, и поэтому, исследование волн в плоских и круговых элементах имеет самое прямое отношение к проблеме определения собственных форм и частот колебаний оболочек конечной длины. Ниже рассматриваются некоторые задачи колебания упругого слоя ограниченые жесткими границами при воздействии на него нормального или вращательного касательного напряжения. Решения рассматриваемых задач получены с использованием интегральных преобразований по координате.
Ключевые слова: гармоническая колебания, цилиндрические оболочки, фазовая скорость , частота, собственная колебания, функция Бесселя, волна, анизотропный, слой.

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VERIFICATION OF RELIABILITY TECHNICAL DEVICES THROUGH RESOLVING PROBABILITY OF FAILURE AND FAILURE

Abstract. This article describes the basic concepts and definitions of reliability theory and its applications to problems of probability theory. Probability theory makes it possible to take into account the random nature of events and processes occurring in the system, to form the mathematical foundations of the theory of reliability. The problems on the probability of failure-free operation of the element are considered.One of the main tasks solved in the course of operation and maintenance of technical devices is to ensure their reliable operation. The importance of this problem is due to the complexity of modern technical devices and high values of operating loads (temperature, pressure, humidity, etc.). Reliability refers to the ability of technical devices to perform specified functions, maintaining their performance within the specified limits for the required period of time or the required operating time in certain operating conditions.Reliability as a qualitative characteristic has always been taken into account when solving various issues of operation and maintenance.

Keywords: failure, probability, technical devices, information system, reliability.

The functional quality of technical devices, including information systems, largely depends on their reliability.

Information system is a complex software and hardware system, which includes ergatic (man-machine) links, technical or hardware and software. Speaking about the reliability of the information system, it is necessary to take into account its two components: the reliability of hardware and software reliability. If the methods of research and ensuring the reliability of the technical (hardware) component of information systems are similar to the corresponding activities of other technical devices, the software differs from such a methodology. Thus, the study of these structures refers to the reliability of the information, its correctness, correctness of its interpretation. We will note that further, speaking about technical devices, we will mean, including, and hardware components information systems (computers, the peripheral equipment, the switching equipment, the cable equipment, etc.).

These categories do not exclude, but complement each other, because in a complex system such as information systems it is possible to provide the necessary level of reliability only taking into account the peculiarities of its components. The most advanced initial technical characteristics of technical devices are necessary, but insufficient conditions of high operational qualities of these devices. Initial characteristics of technical devices show its potential technical capabilities. Important is the ability of technical devices to maintain these characteristics throughout their life cycle or during operation [1].

The ability to maintain its original technical quality during operation is called reliability. This ability depends both on the properties that were incorporated in the technical devices during the design and manufacture, and on the intensity of operation, correctness and timeliness of maintenance.

Therefore, the physical meaning of reliability is the ability to maintain these properties, to resist aggressive operational factors.

Reliability can act as an independent operational characteristic, and serve as a component of other operational characteristics.

One of the main tasks solved in the course of operation and maintenance of technical devices is to ensure their reliable operation. The importance of this problem is due to the complexity of modern technical devices and high values of operating loads (temperature, pressure, humidity, etc.).

Reliability refers to the ability of technical devices to perform specified functions, maintaining their performance within the specified limits for the required period of time or the required operating time in certain operating conditions.

Reliability as a qualitative characteristic has always been taken into account when solving various issues of operation and maintenance. Quantitative determination of reliability appeared with the emergence of the theory of reliability. Mathematical platform of reliability theory is probability theory and mathematical statistics.

Indeed, failures in technical devices occur at random at unexpected times. This is typical even for many similar devices manufactured at the same plant and put into operation at the same time. Despite the single project, the same production technology-each of them has an individual ability to maintain its original quality. Initially it seems that there is no regularity in the appearance of cracks. However, such a pattern exists. It manifests itself when not one but many technical devices in operation are monitored.

As the main quantitative measure of reliability of technical devices, characterizing the regularity of occurrence of failures in time, adopted the probability of failure-free operation.

The probability of failure – free operation is the probability that during a certain time of operation of technical devices and in specified operating conditions failure does not occur. Since the occurrence of a failure is a random event, the time of its occurrence t0 - is also a random event. Therefore, the probability of failure-free operation:

$$p(t) = p(t_0 \geq t)$$

where t is the specified operating time.

The probability of failure is the probability of the opposite event:

$$q(t) = p(t_0 < t)$$

But the event of failure and the reliability of event – opposite the essence of the event. Therefore, according to the probability property of opposite events, it is possible to record[2]

$$p(t) + q(t) = 1$$

In practice, estimates of these probabilities are determined. Let N - be the total number of the same type of technical devices operated during time t . During this time N (t) the technical device worked smoothly, and n(t) – refused. Thus:

$$N = N(t) + n(t)$$

that is, after a time t the total number of both serviceable and failed technical devices is equal to the original. The statistical probability of failure-free operation is determined by the expression

$$p^*(t) = \frac{N(t)}{N}$$

a failure rate

$$q^*(t) = \frac{n(t)}{N}$$

Find the sum of these frequencies:

$$p^*(t) + q^*(t) = \frac{N(t)}{N} + \frac{n(t)}{N} = \frac{N(t) + n(t)}{N} = \frac{N}{N} = 1$$

that corresponds to the theoretical conclusions. For the transition from p*(t) and q*(t) to p(t) and q(t) need to take the limit relations of frequencies:

$$p(t) = \lim_{N\to\infty} \frac{N(t)}{N}$$
$$q(t) = \lim_{N\to\infty} \frac{n(t)}{N}$$

As N→∞ cannot be achieved under this Declaration into practice can mean the whole Park is set on the operation of the same technical devices.

It is obvious that over time the total number of failures in technical devices increases. Consequently, q(t) increases and, hence, p(t) decreases. The curves that determine the nature of these changes are as follows:

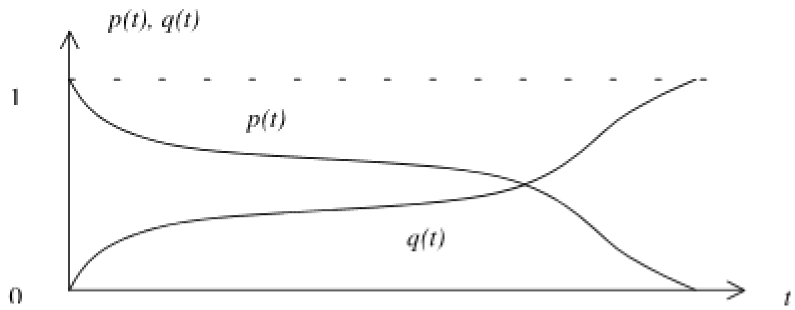


Рис. 1. Характер изменения кривых $p(t)$ и $q(t)$

In practice, it is often necessary to determine the reliability of technical devices for some time interval from ta to tb (for example, during the period of operation of this device), provided that it has already been in operation for some time tb. The TDB of technical devices during the time (tb-ta), provided that it has worked smoothly for ta hours, is determined by the conditional probability[3]

$$p\left(\frac{tb - ta}{ta}\right) = p(ta \geq tb)$$

This conditional probability is numerically equal to the probability $p\left(\frac{tb}{ta}\right)$. Indeed, the probability that an object has not failed during the time (tb-ta), provided that it has run smoothly for ta hours, consists of the probability of failure during ta hours and the probability of failure during the hours from ta to tb. According to the concept of conditional probability,

$$p\left(\frac{tb - ta}{ta}\right) = p\left(\frac{tb}{ta}\right) = \frac{p(tb)}{p(ta)}$$

But p(tb, ta) is numerically equal to the probability that the technical device will run smoothly for tb hours:

$$p(tb, ta) = p(tb)$$

Then

$$p\left(\frac{tb}{ta}\right) = \frac{p(tb)}{p(ta)}$$

In private, this formula will take the form of

$$p^{*}\left(\frac{tb}{ta}\right)=\frac{N(tb)}{N(ta)}$$

Since

$$p^{*}(ta)=\frac{N(ta)}{N}p^{*}(tb)=\frac{N(tb)}{N}$$

Using the probability of failure p(t), it is possible to estimate the average number of elements or devices of information systems (for example, computer networks or its periphery) n(t), which may fail during the time interval Δt at the known operating time t[4]:

$$n(t)=Np(t)-Np(t+\Delta t)$$

where N – number of serviceable elements of information systems at the beginning of its operation.

Example. Reliability function

Reliability function R (t) is a function that determines the probability of failure of the element during the duration of t:

$$P(T<t)=e^{-\lambda t}$$

λ - is the failure rate (the average number of failures per unit time.

An example of solving the problem of the probability of failure of the element:

The duration of the uptime of an element has exponential distribution

$$F(t)=1-e^{-0,01t}\ (t>0).$$

Find the probability that during the duration of t=50 h:

a) the item will be denied;

b) element will not refuse,

c)find the reliability function

<u>Task solution</u>

a) Since the distribution function

F (t) = 1-e-0.01 t determines the probability of failure of the element during the duration of t, then substituting t = 50 in the distribution function, we obtain the probability of failure:

$$F(50)=1-e^{-0,01*50}=0,394;$$

b) events "element will refuse" and "element will not refuse" are opposite, so the probability that the element will fail P = 1-0, 394 = 0.606.

The same result can be obtained directly using the reliability function, which determines the probability of failure of the element during the duration of t:

$$R(50)=e^{-0,01*50}=0,606.$$

c) reliability function

$$R(t)=e^{-0,01*t}$$

Example 1.1.On a test set of 1000 of the same type of electron tubes. For 3000 hours refused 80 lamps. It is required to determine the probability of failure-free operation and the probability of failure of electronic lamps within 3000 hours.

Decision. By formulas (1.1) and (1.2) we define

$$\widetilde{P}(3000) = \frac{N_O - n(t)}{N_O} = \frac{1000-80}{1000} = 0{,}92$$

$$\widetilde{Q}(3000) = \frac{n(t)}{N_O} = \frac{80}{1000} = 0{,}08$$

or

$$Q(3000) = 1 - P(t) = 1 - 0{,}92 = 0{,}08$$

Example 1.2. The test delivered 1000 of the same type of lamps. For the first 3000 hours of work refused 80 lamps, and for an interval of 3000 h – 4000 h refused 50 more lamps. Determine the frequency and failure rate of electronic lamps in the interval 3000 h-4000 h[5].

Decision. By formulas (1.3) determine the failure rate

$$\widetilde{\alpha}(3500) = \frac{n(\Delta t)}{N_O \Delta t} = \frac{50}{1000 \cdot 1000} = 5 \cdot 10^{-5} \quad (1 / \text{h})$$

Determine the average number of working items in the interval Δt.

$$N_{cp} = \frac{\mathrm{N}_i + N_{i+1}}{2} = \frac{920 + 870}{2} = 895 \text{ (PCs)}$$

By the formula (1.5) we find the failure rate

$$\widetilde{\lambda}(3500) = \frac{n(\Delta t)}{N_{cp} \Delta \mathrm{t}} = \frac{50}{895 \cdot 1000} = 5{,}6 \cdot 10^{-5} \quad (1 / \text{h})$$

Example 1.3. The test was 1000 samples of the equipment beyond repair. The number of failures was recorded every 100 hours of operation (h). Data on failures are given in the table. 1.2. It is required to calculate quantitative characteristics and to construct dependence of characteristics on time.

Table 1.2 - The data on failures for example 1.3

Δt_i, ч	$n(\Delta t_i)$	Δt_i, ч	$n(\Delta t_i)$	Δt_i, ч	$n(\Delta t_i)$
0 – 100	50	1000 - 1100	15	2000 – 2100	12
100 – 200	40	1100 - 1200	14	2100 – 2200	13
200 – 300	32	1200 – 1300	14	2200 – 2300	12
300 – 400	25	1300 – 1400	13	2300 – 2400	13
400 – 500	20	1400 – 1500	14	2400 – 2500	14
500 -600	17	1500 – 1600	13	2500 – 2600	16
600 – 700	16	1600 – 1700	13	2600 – 2700	20
700 – 800	16	1700 – 1800	13	2700 – 2800	25
800 – 900	15	1800 – 1900	14	2800 – 2900	30
900 - 1000	14	1900 - 2000	12	2900 - 3000	40

Decision. The equipment belongs to the class of non-renewable products. Therefore, reliability indicators are $P(t)$, $\alpha(t)$, $\lambda(t)$, $\overline{\overline{T}}_O$.

Calculate $P(t)$. Based on the formula (1.1) we have

$$\widetilde{P}(100)=\frac{N_O-n(100)}{N_O}=\frac{1000-50}{1000}=0{,}95,$$

$$\widetilde{P}(200)=\frac{N_O-n(200)}{N_O}=\frac{1000-90}{1000}=0{,}91,$$

…………………………………………

$$\widetilde{P}(3000)=\frac{N_O-n(3000)}{N_O}=\frac{1000-575}{1000}=0{,}425.$$

To calculate the characteristics $\alpha(t)$ and $\lambda(t)$ apply the formulas (1.3) and (1.5), then

$$\widetilde{\alpha}(50)=\frac{n(\Delta t)}{N_O\Delta t}=\frac{50}{1000\cdot 100}=5\cdot 10^{-4}\ \text{(1/h)},$$

$$\widetilde{\alpha}(150)=\frac{n(\Delta t)}{N_O\Delta t}=\frac{40}{1000\cdot 100}=4\cdot 10^{-4}\ \text{(1/h)},$$

…………………………………………

$$\widetilde{\alpha}(2950)=\frac{n(\Delta t)}{N_O\Delta t}=\frac{40}{1000\cdot 100}=4\cdot 10^{-4}\ \text{(1/h)},$$

$$\widetilde{\lambda}(50)=\frac{n(\Delta t)}{N_{cp}\Delta \text{t}}=\frac{50}{100\cdot\left(\frac{1000+950}{2}\right)}=5{,}13\cdot 10^{-4}\ \text{(1/h)},$$

$$\widetilde{\lambda}(150)=\frac{n(\Delta t)}{N_{cp}\Delta \text{t}}=\frac{40}{100\cdot\left(\frac{950+910}{2}\right)}=4{,}3\cdot 10^{-4}\ \text{(1/h)},$$

…………………………………………

$$\widetilde{\lambda}(2950)=\frac{n(\Delta t)}{N_{cp}\Delta \text{t}}=\frac{40}{100\cdot\left(\frac{465+425}{2}\right)}=9\cdot 10^{-4}\ \text{(1/h)}$$

The values $\widetilde{P}(t)$, $\widetilde{\alpha}(t)$, $\widetilde{\lambda}(t)$, calculated for all Δt_i are shown in the table. 1.3, and their dependence on time on rice. 1.3 and 1.4.

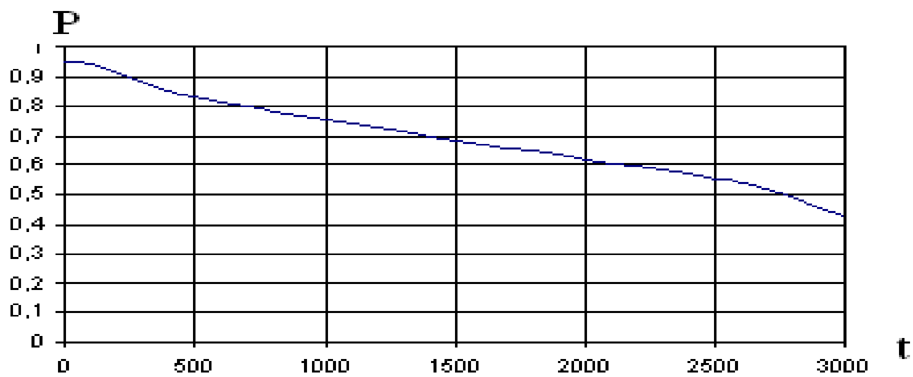
Calculate the average uptime, assuming that the test were only those samples that failed.

The calculation is carried out using the formula (1.11), given that $m=\frac{t_n}{\Delta t}=\frac{3000}{100}=30$ and $N_O=575$, have $\widetilde{\widetilde{T}}_O=\frac{\sum_{i=1}^{m}n_i t_{cpi}}{N_O}=\frac{50\cdot 50+40\cdot 150+32\cdot 250+...+30\cdot 2850+40\cdot 2950}{575}=1400$ (h).

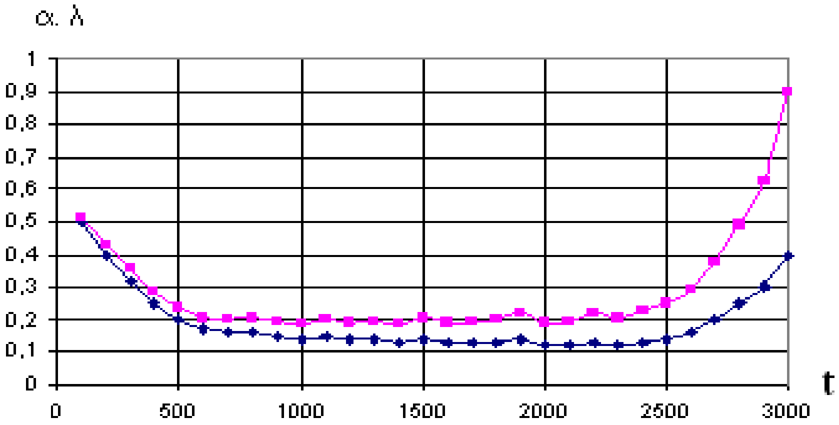
The mean value obtained before the first failure is underestimated, because the experiment was terminated after the failure of 575 samples out of 1000, put to the test.

Table 1.3 - Calculated value $\widetilde{P}(t)$, $\widetilde{\alpha}(t)$, $\widetilde{\lambda}(t)$ for example 1.3

Δt_i, h	$\widetilde{P}(t)$	$\widetilde{\alpha}(t)\cdot 10^{-4}$, (1/h)	$\widetilde{\lambda}(t)\cdot 10^{-4}$, (1/h)
0 – 100	0,950	5	5,14
100 – 200	0,910	4	4,3
200 – 300	0,878	3,2	3,58
300 – 400	0,853	2,5	2,89
400 – 500	0,833	2	2,38
500 -600	0,816	1,7	2,06
600 – 700	0,800	1,6	1,98
700 – 800	0,784	1,6	2,02
800 – 900	0,769	1,5	1,93
900 – 1000	0,755	1,4	1,84
1000 - 1100	0,740	1,5	2
1100 - 1200	0,726	1,4	1,91
1200 – 1300	0,712	1,4	1,95
1300 – 1400	0,699	!.3	1,84
1400 – 1500	0,685	1,4	2,02
1500 – 1600	0,672	1,3	1,92
1600 – 1700	0,659	1,3	1,95
1700 – 1800	0,646	1,3	2
1800 – 1900	0,632	1,4	2,2
1900 – 2000	0,620	1,2	1,92
2000 – 2100	0,608	1,2	1,95
2100 – 2200	0,595	1,3	2,17
2200 – 2300	0,583	1,2	2,04
2300 – 2400	0,570	1,3	2,25
2400 – 2500	0,556	1,4	2,48
2500 – 2600	0,540	1,6	2,9
2600 – 2700	0,520	2	3,76
2700 – 2800	0,495	2,5	4,9
2800 – 2900	0,465	3	6,24
2900 - 3000	0,425	4	9



Rice. 2 - The dependence of P from t to example 1.3.



Rice. 3 - Dependence α and λ from t to example 1.3

Example 1.4**.** For some time, the operation of one copy of the radar station was monitored. During the whole observation period, 15 failures were recorded. Before the beginning of the observation, the station worked for 258 hours, and by the end of the observation, the operating time of the station was 1233 hours. You want to determine the mean time between failures $\tilde{\tilde{T}}$.

Decision. Operating time of the radar station for the observed period is equal to $t = t_2 - t_1 = 1233 - 258 = 975$ (h)

By accepting $\sum_{i=1}^{n} t_i = 975$ h, according to the formula (1.13) we find the mean time between failures: $\tilde{\tilde{T}} = \frac{\sum_{i=1}^{n} t_i}{n} = \frac{975}{15} = 65$ (h).

Example 1.5. The work of three copies of the same type of equipment was monitored. During the observation period, the first copy was recorded 6 failures, the second and third – 11 and 8 failures, respectively[6]. Operating time of the first copy made 181 h, the second – 329 h and the third 245 h. it is Required to define operating time of the equipment on failure $\tilde{\tilde{T}}$.

Decision. Let's define the total operating time of three samples of the equipment:

$$t_{\Sigma} = \sum_{j=1}^{N} \sum_{i=1}^{nj} t_{ij} = 181 + 329 + 245 = 755 \text{ (h)}$$

Determine the total number of failures:

$$n_{\Sigma} = \sum_{j=1}^{N} n_j = 6 + 11 + 8 = 25 \text{ (refusal).}$$

Find the mean time to failure by the formula (1.14)

$$\tilde{\tilde{T}} = \frac{\sum_{j=1}^{N} \sum_{i=1}^{nj} t_{ij}}{\sum_{j=1}^{N} n_j} = \frac{t_{\Sigma}}{n_{\Sigma}} = \frac{755}{25} = 30{,}2 \text{ (h).}$$

Example 1.6. The system consists of 5 devices, and failure of any one of them leads to failure of the system. It is known that the first device refused 34 times during 952 hours of work, the second -24 times during 960 hours of work, and other devices during 210 hours of work refused 4, 6 and 5 times respectively. It is required to determine the time to failure of the system as a whole, if the exponential law of reliability for each of the five devices[7].

Decision. To solve this problem, we use the following relations:

$$\lambda_c = \sum_{i=1}^{n} \lambda_i \ \overline{T}_O = \frac{1}{\lambda_c}.$$

Determine the failure rate for each device:

$$\tilde{\lambda}_1 = \frac{34}{952} = 0{,}0357 \text{(1/h)},\ \tilde{\lambda}_2 = \frac{24}{960} = 0{,}025 \text{(1/h)},$$

$$\tilde{\lambda}_{3,4,5} = \frac{4+6+5}{210} = 0{,}0714 \text{(1/h)}.$$

The failure rate of the system is equal to

$$\lambda_c = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \lambda_{3,4,5} = 0{,}0357 + 0{,}025 + 0{,}0714 = 0{,}1321 \text{(1/h)}$$

Mean time to system failure

$$\overline{T}_O = \frac{1}{\lambda_c} = \frac{1}{0{,}1321} = 7{,}57 \text{ (h)}.$$

Example 1.7. During the observed period of operation 8 failures were recorded in the equipment. Recovery time was t1 = 12 min, t2 = 23 min, T3 = 15 min, t4 = 9 min, T5 = 17 min, T6 = 28 min, T7 = 25 min, T8 = 31 min. Determine the average recovery time of the equipment $\overline{\tilde{T}}_B$.

Decision.

$$\overline{\tilde{T}}_B = \frac{\sum_{i=1}^{n_i} t_i}{n_i} = \frac{\sum_{i=1}^{8} t_i}{8} = \frac{600}{8} = 75 \text{(min)}.$$

Example 1.8. The instrument had an average MTBF of 65 hours and the average recovery time of 1.25 h. it is Required to determine availability[8].

Decision. By formula (1.17) we have

$$K_Г = \frac{\overline{T}}{\overline{T} + \overline{T}_B} = \frac{65}{65 + 1{,}25} = 0{,}98$$

Example 1.9. Let the time of the element to failure is subject to the exponential distribution law with the parameter $\lambda = 2{,}5 \cdot 10^{-5}$ 1/h.

You want to calculate the quantitative characteristics of the reliability of the element $P(t)$, $\alpha(t)$, $\overline{T}_O$, if t = 500, 1000, 2000 h.

Decision. Use the formulas for $P(t)$, $\alpha(t)$ and $\overline{T}_O$, shown in table.1.1.

Calculate the probability of failure-free operation:

$P(t) = e^{-\lambda \cdot t} = e^{-2,5 \cdot 10^{-5} \cdot t}$.

$P(500) = e^{-2,5 \cdot 10^{-5} \cdot 500} = 0,9875$;

$P(1000) = e^{-2,5 \cdot 10^{-5} \cdot 1000} = 0,9753$;

$P(2000) = e^{-2,5 \cdot 10^{-5} \cdot 2000} = 0,9512$.

Calculate the failure rate:

$\alpha(t) = \lambda(t) \cdot P(t) = 2,5 \cdot 10^{-5} \cdot e^{-2,5 \cdot 10^{-5} \cdot t}$.

$\alpha(500) = 2,5 \cdot 10^{-5} \cdot e^{-2,5 \cdot 10^{-5} \cdot 500} = 2,5 \cdot 10^{-5} \cdot 0,9875 = 2,469 \cdot 10^{-5}$ (1/h);

$\alpha(500) = 2,5 \cdot 10^{-5} \cdot e^{-2,5 \cdot 10^{-5} \cdot 1000} = 2,5 \cdot 10^{-5} \cdot 0,9753 = 2,439 \cdot 10^{-5}$ (1/h);

$\alpha(500) = 2,5 \cdot 10^{-5} \cdot e^{-2,5 \cdot 10^{-5} \cdot 2000} = 2,5 \cdot 10^{-5} \cdot 0,9512 = 2,378 \cdot 10^{-5}$ (1/h).

Calculate the mean time to the first failure:

$$\overline{T}_O = \frac{1}{\lambda} = \frac{1}{2,5 \cdot 10^{-5}} = 40000 \text{ (h)}.$$

Example 1.10. In the operation of the system was n = 40 failures. Distribution of failures by groups of elements and time spent on recovery are given in table 1.4. It is necessary to find the value of the average recovery time of the system[30].

Decision. We determine the average recovery time of the equipment by groups of elements.

For semiconductor devices:

$$\tilde{\tilde{T}}_B = \frac{\sum_{i=1}^{n_i} t_i}{n_i} = \frac{\sum_{i=1}^{8} t_i}{8} = \frac{600}{8} = 75 \text{ (min)}$$

Likewise:

- for resistors and capacitors 76 min;
- for relays, transformers, chokes 113 min;
- EVP for 50 min.;
- for other elements 120 min.

Calculate the average recovery time of the system by the formula:

$$\tilde{\tilde{T}}_{BC} = \sum_{i=1}^{m} t_{вi} \cdot m_i ,$$

Where $t_{вi}$ average time of restoration of elements of the i-th group; m_i the weight of failures on groups of elements.

Table 1.4 - The number of registered failures in groups for example 1.10

Groupofelements	Number of failures per group n_i	The weight of failure in the group $m_i = \frac{n_i}{n}$	Recoverytime t_i, мин
SPT	8	0,2	80 59 110 91 45 43 99 73
Resistorsandcapacitors	10	0,25	61 73 91 58 44 112 82 54 91 94
Relays, transformers, chokes	4	0,1	102 98 124 128
EVP	14	0,35	60 64 56 36 65 44 42 33 32 23 86 75 61 23
Otherelements	4	0,1	125 133 115 107

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ПРОВЕРКА ТЕХНИЧЕСКИХ УСТРОЙСТВ НАДЕЖНОСТИ ЧЕРЕЗ РЕШЕНИЕ ВЕРОЯТНОСТИ НЕИСПРАВНОСТИ И НЕИСПРАВНОСТИ

Аннотация.В этой статье описываются основные понятия и определения теории надежности и ее приложения к задачам теории вероятностей. Теория вероятностей позволяет учесть случайный характер происходящих в системе событий и процессов, сформировать математические основы теории надежности. Рассмотрены проблемы с вероятностью безотказной работы элемента. Одной из основных задач, решаемых в процессе эксплуатации и технического обслуживания технических устройств, является обеспечение их надежной работы. Важность этой проблемы связана с сложностью современных технических устройств и высокими значениями рабочих нагрузок (температура, давление, влажность и т. Д.). Надежность - это способность технических устройств выполнять определенные функции, поддерживая их работу в указанных пределах на требуемый период времени или требуемое время работы в определенных условиях эксплуатации. Надежность как качественная характеристика всегда учитывалась при решении различных вопросов эксплуатации и обслуживания.

Ключевые слова: отказ, вероятность, технические устройства, информационная система, надежность.

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ТЕХНИКАЛЫҚ ҚҰРЫЛҒЫЛАРДЫҢ ЖАУАПКЕРШІЛІКТІ ЖӘНЕ АТАЛҒАН МҮМКІНДІКТЕРДІ ТИІМДІЛІКТІ АРҚЫЛЫ

Аннотация. Бұл мақалада сенімділік теориясының негізгі ұғымдары мен анықтамалары және ықтималдықтар теориясының мәселелеріне қолданылуы сипатталған. Ықтималдық теориясы жүйедегі оқиғалар мен процестердің кездейсоқ сипатын ескеріп, сенімділік теориясының математикалық негіздерін қалыптастыруға мүмкіндік береді. Элементтің сәтсіз жұмыс істеу ықтималдығы мәселесі қарастырылады. Техникалық құрылғыларды пайдалану мен күту процесінде шешілетін басты міндеттердің бірі олардың сенімді жұмысын қамтамасыз ету болып табылады. Бұл проблеманың маңыздылығы заманауи техникалық құрылғылардың және жұмыс жүктемелерінің жоғары мәндерінің (температура, қысым, ылғалдылық және т.б.) күрделілігіне байланысты.Сенімділік - техникалық құрылғылардың белгілі бір функцияларды орындауы, белгілі бір уақыт ішінде белгілі бір лимиттерде жұмыс істеуін немесе белгілі бір пайдалану жағдайларында талап етілетін жұмыс уақытын сақтау қабілеттілігі. Сапалы сипат ретінде сенімділік әртүрлі мәселелерді шешу кезінде әрқашан ескеріледі.

Түйін сөздер: бас тарту, ықтималдық, техникалық құрылғылар, ақпараттық жүйе, сенімділік.

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MATHEMATICAL MODELING OF THE PROBLEM OF OPTIMAL CONTROL OF ELECTRIC POWER SYSTEMS

Abstract: The problems of optimal control of nonlinear ordinary differential equations systems are considered in this paper. The considered mathematical model describes the transient processes in the electric power system. And the problem of optimal control of electric power systems is considered in more detail. Numerical experiments have shown that the control found is optimal for the given problem.

Keywords: mathematical model, electric power system, optimal control.

1 INTRODUCTION

The proper functioning of electric power systems (EPS) as an important component of large energy systems forms one of the successful development foundations of the country's economy as a whole. The presence of not only technical but also economic aspects of reliability predetermines the complexity of studying the above objects and their interaction with other components of the economy and the social sphere in order to determine the best control actions to achieve economic benefits and to maintain a constant readiness of the energy systems to overcome the threats to their normal functioning arising in periods of economic, political crises, in case of catastrophes, disasters, etc.

The importance of the problem of optimal control of processes in various fields of science and technology is well known[1-6]. It also has great importance for electric power systems. Without a reliable solution to this problem, reliable and high-quality supply of electricity to consumers in virtually all sectors of the national economy is impossible.

2 Mathematical model of unsteady processes in the electrical system

One of the mathematical models that describes unsteady processes in an electrical system is the following system of differential equations[1-3]:

$$\frac{d\delta_i}{dt} = S_i,$$

$$H_i \frac{dS_i}{dt} = -D_i S_i - E_i^2 Y_{ii} \sin a_{ii} - - P_i \sin(\delta_i - a_i) - \sum_{j=1, j\neq i}^{l} P_{ij} \sin(\delta_{ij} - a_{ij}) + u_i \tag{1}$$

$$\delta_{ij} = \delta_i - \delta_j,\ P_i = E_i U Y_{i,n+1},\ P_{ij} = E_i E_j Y_{i,j},$$

where – angle of rotor rotation of the i-th generator concerning some synchronous rotation axis; - slide of the i-th generator; - the inertia constant of the i-th machine; mechanical power, which are brought to the generator; - EMF of the i-th synchronous machine; - mutual conductance of the i-th and j-th branches of the system; U=const - bus voltage of constant voltage; - characterizes the connection (conductivity) of the i-th generator with the buses of constant voltage; - mechanical damping; − constant values that consider the effect of active resistance in the stator generator circuits. .

Let the state variables and control in the steady-state post-emergency condition have the following values:

$$S_i = 0, \quad \delta_i = \delta_i^f, \quad u_i = u_i^f, \quad i = 1, l \tag{2}$$

In order to obtain a perturbed motion system we proceed to equations in deviations, assuming

$$u_i = u_i^F + \Delta u_i, \delta_i = \delta_i^F + \Delta\delta_i, S_i = \Delta S_i, i = \overline{1, l} \tag{3}$$

Further, for convenience, we again denote by and use the formula

$$\sin(\delta_{ij} - \alpha_{ij}) = \cos\alpha_{ij}\sin\delta_{ij} - \sin\alpha_{ij}\cos\delta_{ij}$$

from the system (1) we obtain

$$\frac{d\delta_i}{dt} = S_i$$
$$\frac{dS_i}{dt} = \frac{1}{H_i}\left(-D_i S_i - f_i(\delta_i) - N_i(\delta) + M_i(\delta) + u_i\right) \tag{4}$$
$$i = 1, l, t \in [0, T]$$

Where

$$f_i(\delta_i) = P_i\left[\sin(\delta_i + \delta_i^F - \alpha_i) - \sin(\delta_i^F - \alpha_i)\right],$$
$$N_i(\delta) = \sum_{j=1, i \neq j}^{l} \Gamma_{ij}^1\left[\sin(\delta_{ij} + \delta_{ij}^F) - \sin(\delta_{ij}^F)\right],$$
$$M_i(\delta) = \sum_{j=1, i \neq j}^{l} \Gamma_{ij}^2\left[\cos(\delta_{ij} + \delta_{ij}^F) - \cos(\delta_{ij}^F)\right],$$
$$\Gamma_{ij}^1 = P_{ij}\cos\alpha_i, \Gamma_{ij}^2 = P_{ij}\sin\alpha_i,$$

Note thatsince $P_{ij} = P_{ji}$, then

$$\Gamma_{ij}^1 = \Gamma_{ji}^1, \Gamma_{ij}^2 = \Gamma_{ji}^2.$$

Controls $u_i, i = 1, l$ are chosen so as to compensate for the non-conservative term - $M_i(\delta), i = 1, l$ i.e.

$$u_i = v_i - M_i(\delta), i = 1, l \tag{5}$$

2.1 The problem of optimal control of electric power systems

We consider the following problem of minimizing the functional:

$$J(v) = J(v_1, ..., v_l) = 0.5\sum_{i=1}^{l}\int_0^T (w_{s_i} S_i^2 + w_{v_i} v_i^2) ** \exp\{\gamma_i t\} dt + \Lambda(\delta(T), S(T)), \tag{6}$$

under conditions

$$\frac{d\delta_i}{dt} = S_i,$$
$$H_i\frac{dS_i}{dt} = -D_i S_i - f_i(\delta_i) - N_i(\delta_i) + v_i, \tag{7}$$
$$\delta = (\delta_1, ..., \delta_l), S = (S_1, S_2, ..., S_l)$$

where

$f_i(\delta_i) = P_i\left[\sin(\delta_i + \delta_i^F - \alpha_i) - \sin(\delta_i^F - \alpha_i)\right]$, $N_i(\delta) = \sum_{j=1, i \neq j}^{l} P_{ij}\cos\alpha_i\left[\sin(\delta_{ij} + \delta_{ij}^F) - \sin(\delta_{ij}^F)\right]$, and w_{S_i}, w_{v_i} - positive constant weight coefficients, $F_i(\delta_i) - 2\pi$ periodic continuously differentiable functions, $N_i(\delta) - 2\pi$ Periodic continuously differentiable functions with respect to δ_{ij}, with respect to the summand $N_i(\delta)$ the integrability condition is fulfilled

$$\frac{\partial N_i(\delta)}{\partial \delta_k} = \frac{\partial N_k(\delta)}{\partial \delta_i} (\forall i \neq k); \tag{8}$$

T - the transient period is considered given.

The equations system (7) is supplemented by the initial conditions

$$\delta_i(0) = \delta_{i0}, \quad S_i(0) = S_{i0}, \quad i = 1,...,l. \tag{9}$$

Terminal values $\delta(T), S(T)$ are not known in advance, so they are also subject to definition.

The following theorem is valid.

Theorem 1. In order that the controls $v_i^0(S_i) = -\frac{1}{w_{v_i}} S_i, \quad i = \overline{1,l}$ and the corresponding solution of the system (7)-(9) be optimal, it is necessary and sufficient that

$$\Lambda(\delta(T), S(T)) = K(\delta(T), S(T)),$$
$$w_{S_i}(t) = 2D_i + \frac{1}{w_{v_i}} > 0, \quad i = \overline{1,l},$$

Where

$$K(\delta, S) = \frac{1}{2} H_i S_i^2 + \sum_{i=1}^{l} \int_0^{\delta_i} f_i(\delta_i) d\delta_i + \sum_{\substack{i=1, \\ \delta_j = 0, \\ j>i}}^{l} \int_0^{\delta_i} N_i(\delta_1, ..., \delta_{i-1}, \delta_{i+1}, ..., \delta_l) d\xi_i$$

the Bellman-Krotov function, and

$$J(v^0) = \min_v J(v) = K(\delta^0, S^0). \tag{10}$$

Proof. For the continuously differentiable function $K(\delta(t)), S(\delta(t))$ the functional (6) has the representation:

$$J(u) = J(\delta(t), S(t), v(t)) = \int_0^T R[\delta(t), S(t), v(t)] dt + m_0(\delta(0)S(0)) + m_i(\delta(T)S(T)) \tag{11}$$

Where

$$R(\delta, S, v) = \sum_{i-1}^{l} \left[K_{\delta_i} S_i + \frac{1}{H_i} K_{S_i}(-D_i S_i - f_i(\delta_i) - N_i(\delta) + v_i) + \right] + \frac{1}{2}(w_{si} S_i^2 + w_{vi} v_i^2)] \tag{12}$$

$$K_{\delta_i} = \frac{\partial K}{\partial \delta_i}, \quad K_{S_i} = \frac{\partial K}{\partial S_i} \tag{13}$$

$$m_0(\delta, S) = K(\delta, S), \quad m_1(\delta, S) = -K(\delta, S) + \Lambda(\delta, S).$$

We use the Cauchy-Bellman problem to find the Bellman function $K(\delta, S)$:

$$\begin{aligned} &\inf_v R[\delta, S, v] = 0, \quad t \in [0, T], \\ &K(\delta(T), S(T)) = \Lambda(\delta(T), S(T)). \end{aligned} \tag{14}$$

From the necessary condition of extremum of the function $R(\delta, S, v)$ we obtain:

$$R_{vj} \equiv \frac{1}{H_i} K_{Si} + w_{Si} v_i = 0, i = \overline{1,l}$$

$$\nu_i^0 \equiv -\frac{1}{H_i w_{\nu i}} K_{Si},\ i=\overline{1,l}, \quad (15)$$

We find the function $K(\delta, S)$ and weight coefficients $w_{Si}, w_{\nu i}$ from the condition (14) i.e.

$$\overline{R} = \min R(\delta, S, \nu) = \sum_{i=1}^{l} K_{\delta_i} S_i - \frac{1}{H_i} K_{S_i}(D_i S_i + f_i(\delta_i) +$$

$$+ N_i(\delta)) - \frac{1}{2H_i^2 w_{\nu i}} K_{Si}^2 + \frac{1}{2} w_{Si}^2 S_i^2) = 0 \quad (16)$$

For this, we set

$$K_{\delta_i} S_i = \frac{K_{S_i}}{H_i}(f_i(\delta_i) + N_i(\delta)), i = \overline{1,l}$$

i.e.

$$K_{S_i} = H_i S_i, K_{\delta_i} = f_i(\delta_i) + N_i(\delta)), i = \overline{1,l} \quad (17)$$

Then taking into account (17), we obtain from (16) that

$$\sum_{i-1}^{l}\left[-D_i S_i - \frac{1}{w_{\nu i}} S_i^2 + \frac{1}{2} w_{Si} \nu_i^2\right] = 0$$

or

$$w_{si} = 2D_i + \frac{1}{2w_{\nu i}} > 0, w_{\nu i} > 0, i = \overline{1,l} \quad (18)$$

In addition, from (15) we find that the optimal controls $\nu_i^0, i = \overline{1,l}$ have the form

$$\nu_i^0(S_i) = -\frac{1}{w_{\nu i}} S_i, i = \overline{1,l} \quad (19)$$

2.2 Numerical solution

The implicit Adams method (Adams-Moulton) has been used for the numerical solution of the differential equations system (7)-(9).

Table 1: Adams-Moulton formulas in different orders

Order	Formula	Errororder
1	$y_{n+1} = y_n + hf_{n+1}$	$-\frac{h^2}{2} y''(\eta)$
2	$y_{n+2} = y_{n+1} + \frac{h}{2}[f_{n+2} - f_{n+1}]$	$-\frac{h^3}{12} y'''(\eta)$
3	$y_{n+3} = y_{n+2} + \frac{h}{12}[5f_{n+3} + 8f_{n+2} - f_{n+1}]$	$-\frac{h^4}{24} y^{(4)}(\eta)$
4	$y_{n+4} = y_{n+3} + \frac{h}{24}[9f_{n+4} + 19f_{n+3} -$ $-5f_{n+2} + f_{n+1}]$	$-\frac{19h^5}{720} y^{(5)}(\eta)$
5	$y_{n+5} = y_{n+4} + \frac{h}{720}[251f_{n+5} + 646f_{n+4} -$ $-264f_{n+3} + 106f_{n+2} - 19f_{n+1}]$	$-\frac{3h^6}{160} y^{(6)}(\eta)$

We have used the Adams-Moulton method of the 4th order to solve this problem:

$$y_{n+4} = y_{n+3} + \frac{h}{24}(9f(t_{n+4}, y_{n+4}) + 19f(t_{n+3}, y_{n+3}) -$$

$$-5f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1})) - \frac{19}{720}h^5(\eta).$$

And we also have used the Runge-Kutta method of the 4-th order:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where

$$k_1 = f(t, y),$$

$$k_2 = f(t + \frac{h}{2}, y + \frac{hk_1}{2}),$$

$$k_3 = f(t + \frac{h}{2}, y + \frac{hk_2}{2}),$$

$$k_4 = f(t + h, y + hk_3).$$

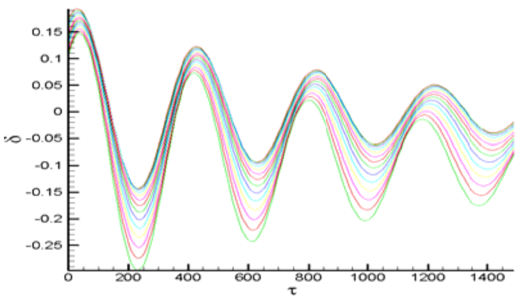

Figure 1 - Angle of rotor rotation at l = 15.

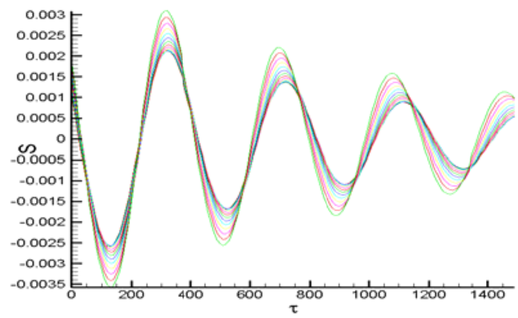

Figure 2 - Generator slide at l=15.

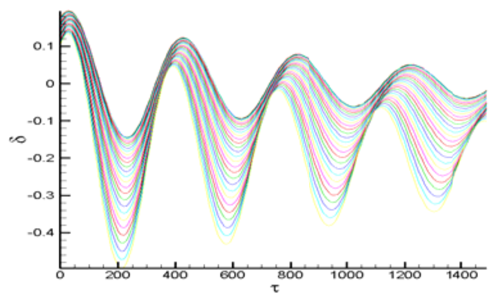

Figure 3 - Angle of rotor rotation at l=30.

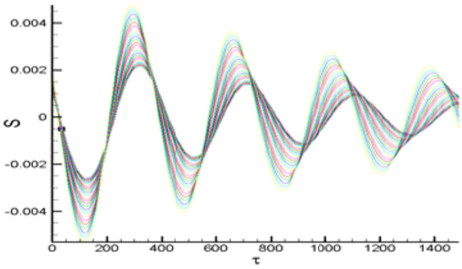

Figure 4 - Generator slide at l=30

3 CONCLUSIONS

The process of optimal control of complex electric power systems is described in this paper. The continuous Bellman-Krotov function, that has continuous partial derivatives everywhere by all its arguments, has been found. The numerical solution of this problem is obtained at l = 15 and l = 30. When solving it the implicit Adams method (Adams-Moulton) has been used.

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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ЭЛЕКТРОЭНЕРГЕТИЧЕСКИМИ СИСТЕМАМИ

Аннотация:В данной статье рассматривается задача оптимального управления системами нелинейных обыкновенных дифференциальных уравнений. Рассматриваемая математическая модель описывает переходные процессы в электроэнергетической системе. Проблема оптимального управления электроэнергетическими системами рассматривается более подробно. Численные эксперименты показали, что найденное управление является оптимальным для данной задачи.

Ключевые слова: математическая модель, электроэнергетическая система, оптимальное управление.

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ЭЛЕКТР ЭНЕРГЕТИКАЛЫҚ ЖҮЙЕЛЕРДІ ТИІМДІ БАСҚАРУ МӘСЕЛЕСІН МАТЕМАТИКАЛЫҚ МОДЕЛЬДЕУ

Аннотация: Бұл мақалада **с**ызықты емес қарапайым дифференциалдық теңдеулер жүйелерін тиімді басқару мәселесі қарастырылады. Қарастырылған математикалық модель электроэнергетикалық жүйедегі өтпелі процестерді сипаттайды. Электроэнергетикалық жүйелерді тиімді басқару мәселесі толығырақ қарастырылады. Сандық эксперименттер табылған басқарудың бұл мәселе үшін тиімді екендігін көрсетті.

Түйін сөздер: математикалық модель, электроэнергетикалық жүйе, тиімдібасқару.

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PROBLEM FROM THE THEORY OF BRIDGE EROSION

Abstract. In this paper, we represent the exact solution of a two phase Stefan problem. Radial heat polynomialsand integral error function are used for solving bridge problem. The recurrent expressions for the coefficients of these series are presented. The mathematical models describe the dynamics of contact opening and bridging.
Keywords: radial heat polynomials, Stefan problem.

Introduction

In considerence the heat transfer, the shape of the liquid bridge plays an important role. The overwhelming majority of researchers proceed from the fact that the visible part of the bridge has the shape of a cylinder whose axis is directed perpendicular to the plane of the electrodes [1]. In the general case, in the result of the action of surface tension and the pinch effect, the bridge takes the form of a certain surface of revolution about the z- axis. In this problem, we consider a symmetric model of the bridge, where the shape of the bridge is described by a surface $y(z,t) = z^{\nu/2}$. For a liquid bridge in we consider the generalized heat equation for and for solid contact we use the spherical Holm model [2].

Preliminaries

The fundamental solution for the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\nu}{x}\frac{\partial\theta}{\partial x}\right) \tag{1}$$

can be obtained by the solution of this equation with the initial condition containing delta-function using the Laplace transform in the form [3]

$$G(x,y,t) = \frac{C_\nu}{2t}(xy)^{-\beta} e^{-\frac{x^2+y^2}{4t}} I_\beta\left(\frac{xy}{2t}\right), \qquad \beta = \frac{\nu-1}{2}, \qquad C_\nu = 2^{-\beta}\Gamma(\beta+1) \tag{2}$$

If we consider the corresponding heat potentials for this solution

$$Q_{n,\nu}(x,t) = 2^{-\beta}\Gamma(\beta+1)^{-1}\int_0^\infty G(x,y,t)y^{2n+\nu}dy \tag{3}$$

and integrating by parts we obtain the explicit formula for the heat polynomials

$$Q_{n,\nu}(x,t) = \sum_{k=0}^{n} 2^{2k}\frac{n!\Gamma(\beta+1)}{k!(n-k)!\Gamma(\beta+1+n-k)}x^{2n-2k}t^k \tag{4}$$

It is more convenient for applications to multiply both sides of this formula by $\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}$.

Then $R_{n,\nu}(r,t)=\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}Q_{n,\nu}(x,t)=\sum_{k=0}^{n}2^{2k}\frac{n!\Gamma(\beta+1)}{k!(n-k)!\Gamma(\beta+1+n-k)}x^{2n-2k}t^{k}$ (5)

Mathematical model

The heat equations for each zone are

$$\frac{\partial\theta_1}{\partial t}=a_1^2\left(\frac{\partial^2\theta_1}{\partial x^2}+\frac{\upsilon}{x}\frac{\partial\theta_1}{\partial x}\right)\quad \alpha(t)<x<0 \tag{6}$$

$$\frac{\partial\theta_2}{\partial t}=a_1^2\left(\frac{\partial^2\theta_2}{\partial r^2}+\frac{2}{r}\frac{\partial\theta_2}{\partial r}\right)\quad r_0<r<\beta(t) \tag{7}$$

$$\frac{\partial\theta_3}{\partial t}=a_2^2\left(\frac{\partial^2\theta_3}{\partial r^2}+\frac{2}{r}\frac{\partial\theta_3}{\partial r}\right)\quad \beta(t)<r<\infty \tag{8}$$

with boundaryandinitialconditions:

$$\alpha(0)=0,\ \beta(0)=r_0,\ \theta_1(0,0)=\theta_2(r_0,0)=\theta_m,\ \theta_3(r,0)=f(r),\ f(r_0)=\theta_m \tag{9}$$

$$-\lambda\pi\alpha^{\nu}(t)\frac{\partial\theta_1}{\partial x}\bigg|_{x=\alpha(t)}=Q(t) \tag{10}$$

$$\theta_1(0,t)=\theta_2(r_0,t) \tag{11}$$

$$\lambda_1\frac{\partial\theta_1(0,t)}{\partial x}=2\lambda_2\frac{\partial\theta_2(r_0,t)}{\partial r} \tag{12}$$

$$\theta_2(\beta(t),t)=\theta_m \tag{13a}$$

$$\theta_3(\beta(t),t)=\theta_m \tag{13b}$$

The Stefan's condition

$$-\lambda_1\frac{\partial\theta_2}{\partial r}\bigg|_{r=\beta(t)}=-\lambda_2\frac{\partial\theta_3}{\partial r}\bigg|_{r=\beta(t)}+L\gamma\frac{d\beta}{dt} \tag{14}$$

$$\theta_3(\infty,t)=0 \tag{15}$$

Method of solution

We represent solution of the problem (6)-(15) ii the form

$$\theta_1(x,t)=\sum_{n=0}^{\infty}A_n\sum_{k=0}^{n}\zeta_{n,k}x^{2n-2k}t^k \tag{16}$$

$$\theta_2(r,t)=\sum_{n=0}^{\infty}C_n\sum_{k=0}^{n}\zeta_{n,k,2}x^{2n-2k}t^k+\sum_{n=0}^{\infty}D_n\frac{(2a_1\sqrt{t})^{2n+1}}{r}\left(i^{2n+1}erfc\frac{-(r-r_0)}{2a_1\sqrt{t}}-i^{2n+1}erfc\frac{(r-r_0)}{2a_1\sqrt{t}}\right) \tag{17}$$

$$\theta_3(r,t)=\sum_{n=0}^{\infty}E_n\sum_{k=0}^{n}\zeta_{n,k,2}x^{2n-2k}t^k+\sum_{n=0}^{\infty}G_n\frac{(2a_2\sqrt{t})^{2n+1}}{r}\left(i^{2n+1}erfc\frac{-(r-r_0)}{2a_2\sqrt{t}}-i^{2n+1}erfc\frac{(r-r_0)}{2a_2\sqrt{t}}\right) \tag{18}$$

Where $\zeta_{n,k} = \dfrac{2^{2k} n!\Gamma\left(\frac{\nu-1}{2}+n+1\right)}{k!(n-k)!\Gamma\left(\frac{\nu-1}{2}+n+1-k\right)}$ and $\zeta_{n,k,2} = \dfrac{2^{2k} n!\Gamma\left(\frac{3}{2}+n\right)}{k!(n-k)!\Gamma\left(\frac{3}{2}+n-k\right)}$,

andcoefficients A_n, C_n, D_n, E_n, G_n have to be determined. Free boundary $\beta(t)$ represent in the form [4] $\beta(t) = \sum_{n=0}^{\infty} \beta_n t^{n/2}$.

Using the boundary condition (10) we get

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \zeta_{n,k} (n-k)\alpha(t)^{2n-2k-1} t^k = F(t) \tag{19}$$

where

$$F(t) = -\frac{\mathrm{Q}(t)}{2\lambda\pi\alpha^{\nu}(t)}$$

Taking into account that $\alpha(t)$ is given and using properties of raising the power series to a power[5]

$$\alpha(t)^{2n-2k-1} = \sum_{m=0}^{\infty} \beta(\alpha)_m t^m \tag{20}$$

Where coefficients $\beta_m(\alpha)$ determined by

$$\beta_0(\alpha) = \alpha_0^{2n-2k-1}$$

$$\beta(\alpha)_i = \frac{1}{i\alpha_0} \sum_{m=1}^{i} \left[2m(n-k)-i\right]\alpha_m \beta(\alpha)_{i-m} \qquad i \geq 1$$

Substituting the formula (20) into (19) we obtain

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^{n} \zeta_{n,k} (n-k) \sum_{m=0}^{\infty} \beta(\alpha)_m t^{m+k} = F(t) \tag{21}$$

F(t) is given, can be expanded in Maclaourin series thus to derive A_m, we take both sides of (21), l - times derivative at $t = 0$ we have

$$\sum_{n=0}^{l} A_{n+1}\hbar_{1,n} + \sum_{n=l}^{\infty} A_{n+1}\hbar_{2,n} = l!F_n \tag{22}$$

where

$$\hbar_{2,n} = \sum_{i=0}^{l} \zeta_{n+1,i} \beta(\alpha)_{l-i,i}\, l!(n-i+1)$$

$$\hbar_{1,n} = \sum_{i=0}^{n} \zeta_{n+1,i} \beta(\alpha)_{l-i,i}\, l!(n-i+1)$$

from (22) we can find A_n.

Satisfying the boundary conditions of conjugations of temperature (11) and heat flux (12) we get

$$\sum_{n=0}^{\infty} A_n \zeta_{n,n} t^m = \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} r_0^{2n-2k} t^k \tag{23}$$

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} r_0^{2n-2k-1} t^k + \frac{2}{r_0} \sum_{n=0}^{\infty} D_n (2a_1\sqrt{t})^{2n} i^{2n} erfc0 = 0 \tag{24}$$

from taking the m - times derivative of (23) and (24) we get

$$A_m \zeta_{\mathrm{m,m}} = \sum_{i=m}^{\infty} C_i \cdot \zeta_{\mathrm{i,m,2}} r_0^{2n-2m} \tag{23*}$$

$$\sum_{i=0}^{\infty} C_n \zeta_{i+m+1,m,2} r_0^{2i+1}(i+1) = -\frac{1}{r_0} D_m (2a_1)^{2m} i^{2m} erfc0 \tag{25}$$

From expression (13a) when we put $r = \beta(t)$ we have

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} + \sum_{n=0}^{\infty} \frac{1}{\beta(\tau)} (2a_1\tau)^{2n+1} (i^{2n+1} erfc(-\gamma(\tau)) - i^{2n+1} erfc(\gamma(\tau))) = \theta_m \tag{26}$$

where $\tau = \sqrt{t}$ and $\frac{\beta(\tau)-\beta_0}{2a_1\tau} = \frac{1}{2a_1} \sum_{n=0}^{\infty} \gamma_{n+1} \tau^n$

Multiplying both sides of (26) by $\beta(\tau)$ we get

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} \sum_{m=0}^{\infty} \eta(\beta)_m \tau^{m+2k} + \sum_{n=0}^{\infty} D_n (2a_1\tau)^{2n+1} (i^{2n+1} erfc(-\gamma(\tau)) - i^{2n+1} erfc(\gamma(\tau))) = \theta_m \beta(\tau) \tag{27}$$

To comparing coefficients in (27) we applyLeibniz, Faa Di Bruno’s formula and Bell polynomials. Using Leibniz we have

$$\left.\frac{\partial^l \left[(2a_1)^{2n+1} \tau^{2n+1} i^{2n+1} erfc(\gamma) \right]}{\partial \tau^l}\right|_{\tau=0} = \begin{cases} 0 \quad , \quad for \quad l < 2n+1 \\ \dfrac{(2a_1)^{2n+1} l!}{(l-2n-1)!} \left[i^{2n+1} erfc(\delta) \right]^{(l-2n-1)} \end{cases}$$

Using Faa Di Bruno’s formula and Bell polynomials for a derivative of a composite function we have

$$\left.\frac{\partial^{l-2n-1}}{\partial \tau^{l-2n-1}} \left\{ i^{2n+1} erfc(\pm\delta) \right\}\right|_{\tau=0} = \sum_{m=1}^{l-2n-1} \left.\left(i^{2n+1} erfc(\pm\delta) \right)^{(m)}\right|_{\delta=0} B_{l-2n-1,m}$$

where

$$B_{l-2n-1,m} = \sum \frac{(l-2n-1)!}{j_1! j_2! \ldots j_{l-2n-m}!} \varsigma_1^{j_1} \varsigma_2^{j_2} \varsigma_3^{j_3} \ldots \varsigma_{l-2n-m}^{j_{l-2n-m}}$$

and $j_1, j_2, \ldots$ satisfy the following equations

$$j_1 + j_2 + \ldots + j_{l-2n-m} = m$$

$$j_1 + 2j_2 + \ldots + (l-2n-m) j_{l-2n-m} = l-2n-1$$

by taking both sides of (27) l-times derivative at $\tau = 0$ we get

$$\sum_{n=0}^{\left[\frac{l}{2}\right]-1} C_n \sum_{i=0}^{n} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + \sum_{n=\left[\frac{l}{2}\right]}^{\infty} C_n \sum_{i=0}^{\left[\frac{l}{2}\right]} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} +$$

$$+ D_n \frac{2^{2n+1} l!}{(l-2n-1)!} \sum_{m=1}^{l-2n-1} \left(i^{2n+1-m} erfc(-\gamma_1) + (-1)^{2n+1-m} erfc(\gamma_1) \right) \beta_{l-2n-1,m} = \theta_m l! \beta_l \tag{28}$$

from (28), (23*) and (25) we can find C_n *and* D_n.

By the properties of Integral error functions [6] and condition (9) we get

$$\sum_{n=0}^{\infty} \left\{ E_n r^{2n} + \frac{G_n}{r} \frac{2}{(2n+1)!} r^{2n+1} \right\} = f(r)$$

Suggesting that the initial function can be expanded in $f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n$

we have

$$E_n + G_n \frac{2}{(2n+1)!} = \frac{f^{(2n)}(0)}{2n!} \tag{29}$$

From condition (13b) we have

$$\sum_{n=0}^{\infty} E_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} + \sum_{n=0}^{\infty} G_n \frac{(2a_1\tau)^{2n+1}}{\beta(\tau)} \left(i^{2n+1} erfc(-\xi(\tau)) - i^{2n+1} erfc(\xi(\tau)) \right) = \theta_m \tag{30}$$

where

$$\xi(\tau) = \frac{\beta(\tau)}{2a_2}$$

As previously by taking by both sides of (30), l – times derivatives at $\tau = 0$ by using Leibniz, Faa Di Bruno's formulas and Bell polynomials we have

$$\sum_{n=0}^{\left[\frac{l}{2}\right]-1} E_n \sum_{i=0}^{n} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + \sum_{n=\left[\frac{l}{2}\right]}^{\infty} E_n \sum_{i=0}^{\left[\frac{l}{2}\right]} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} +$$

$$+G_n \frac{2^{2n+1} l!}{(l-2n-1)!} \sum_{m=1}^{l-2n-1} \left(i^{2n+1-m} erfc(-\xi_1) + (-1)^{2n+1-m} erfc(\xi_1) \right) \beta_{l-2n-1,m} = \theta_m l! \beta_l \tag{31}$$

From expression (29) and (31) we can find E_n and G_n

Satisfying Stefan's condition (14) and substituting $\sqrt{t} = \tau$ we have

$$-\lambda_1 \left\{ \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k-1} \tau^{2k} - \frac{1}{\beta^2(\tau)} \sum_{n=0}^{\infty} D_n (2a_1\tau)^{2n+1} \left(i^{2n+1} erfc(-\gamma(\tau)) \right. \right.$$

$$\left. \left. -i^{2n+1} erfc(\gamma(\tau)) \right) - \frac{\lambda_1}{\beta(\tau)} \sum_{n=0}^{\infty} D_n (2a_1\tau)^{2n} \left(i^{2n} erfc(-\gamma(\tau)) + i^{2n} erfc(\gamma(\tau)) \right) \right\} =$$

$$= -\lambda_2 \left\{ \sum_{n=0}^{\infty} E_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k-1} \tau^{2k} - \frac{1}{\beta^2(\tau)} \sum_{n=0}^{\infty} G_n (2a_2\tau)^{2n+1} \left(i^{2n+1} erfc(-\xi(\tau)) - \right. \right.$$

$$\left. \left. -i^{2n+1} erfc(\xi(\tau)) \right) + \frac{1}{\beta(\tau)} \sum_{n=0}^{\infty} D_n (2a_2\tau)^{2n} \left(i^{2n} erfc(-\xi(\tau)) + i^{2n} erfc(\xi(\tau)) \right) \right\} + L\gamma\beta'(\tau) \tag{32}$$

If multiply both sides of (32) by $\beta(\tau)$ and using conditions (13a), (13b) we have the following expression

$$-2\lambda_1 \sum_{n=0}^{\infty} C_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} - \lambda_1 \sum_{n=0}^{\infty} D_n (2a_1\tau)^{2n} \left(i^{2n} erfc(-\gamma(\tau)) + i^{2n} erfc(\gamma(\tau)) \right)$$

$$= -2\lambda_2 \sum_{n=0}^{\infty} E_n \sum_{k=0}^{n} \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} - \lambda_2 \sum_{n=0}^{\infty} G_n (2a_2\tau)^{2n} \left(i^{2n} erfc(-\xi(\tau)) + i^{2n} erfc(\xi(\tau)) \right)$$

$$+\left(\lambda_2 - \lambda_1\right)\theta_m + L\gamma'\psi(\tau) \tag{33}$$

where

$$\psi(\tau) = \beta'(\tau)\beta(\tau) = \frac{1}{2}\frac{d}{d\tau}\beta^2(\tau)$$

$$\left(\beta(\tau)\right)^2=\sum_{n=0}^{\infty}\mu(\beta)_n\tau^n$$

$$\mu(\beta)=\beta_0^2 \qquad \mu(\beta)_m=\frac{1}{m\beta_0}\sum_{k=1}^{m}(3k-m)\beta_k\cdot\mu(\beta)_{m-k} \qquad m\ge 1$$

Previously by taking both sides of (33)l – times derivatives $\tau=0$ and for $l\ge 1$ we have

$$-2\lambda_1\left(\sum_{n=0}^{\left[\frac{l}{2}\right]-1}C_n\chi_{1,n}+\sum_{n=\left[\frac{l}{2}\right]}^{\infty}C_n\chi_{2,n}\right)-\lambda_1 D_n\frac{2^{2n}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left(i^{2n-m}erfc(-\gamma_1)+(-1)^m i^{2n-m}erfc(\gamma_1)\right)\beta_{l-2n,m}$$

$$=-2\lambda_2\left(\sum_{n=0}^{\left[\frac{l}{2}\right]-1}E_n\chi_{21n}+\sum_{n=\left[\frac{l}{2}\right]}^{\infty}E_n\chi_{2,n}\right)-\lambda_2 G_n\frac{2^{2n}l!}{(l-2n)!}\sum_{m=1}^{l-2n}\left(i^{2n-m}erfc(-\xi_1)+(-1)^m i^{2n-m}erfc(\xi_1)\right)\beta_{l-2n,m}$$

$$+\frac{L\gamma'}{2}l!\mu(\beta)_{l+1} \quad (34)$$

For $l=0$ we have

$$\theta_m-D_0\left(erfc(-\gamma_1)+erfc(\gamma_1)\right)-2\sum_{n=0}^{\infty}C_n\zeta_{n,0,2}r_0^{2n}=\frac{\lambda_2}{\lambda_1}\left(\theta_m-G_0\left(erfc(-\xi_1)+erfc(\xi_1)\right)-2\sum_{n=0}^{\infty}E_n\zeta_{n,0,2}r_0^{2n}\right)$$

where

$$\chi_{2,n}=\sum_{i=0}^{\left|\frac{l}{2}\right|}\zeta_{n,i,2}\upsilon(\beta)_{l-2i,i,2}$$

$$\chi_{1,n}=\sum_{i=0}^{n}\zeta_{n,i,2}\upsilon(\beta)_{l-2i,i,2}$$

From this recurrent formula we can express β_n.

Conclusion

To summarize, the coefficients A_n, C_n, D_n, E_n, G_n are determined from equations (22),(23*),(28),(29),(31) and (25), the moving boundary $\beta(t)$ obtained from equation (34). For the convergence of temperature functions $\theta_1,\theta_2,\theta_3$, it is possible to follow the idea proposed in [6].

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КӨПІР ЭРОЗИЯСЫНЫҢ ТЕОРИЯ ЕСЕБІ

Аннотация. Осы мақалада біз екі фазалық Стефан мәселесінің дәл шешімін ұсынамыз. Көпір мәселесін шешу үшінрадиалды жылу полиномы және интегралдық қателілфункциясы қолданылады. Осы қктарлардың коэффициенттері үшін қайталанатын өрнектер ұсынылған. Математиқалық модельдер байланыстың ашылу және көпіршікті динамикасын сипаттайды

Түйін сөздер: радиалды жылу полиномы,Стефанпроблемасы, интеграл

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ЗАДАЧА ИЗ ТЕОРИИ МОСТИКОВОЙ ЭРОЗИИ

Аннотация. В настоящей работе мы представляем точное решение двухфазной задачи Стефана. Для решения данной задачи использовали решение в виде радиальных тепловых полиномов и интегральной функции ошибок. Приводятся рекуррентные выражения для коэффициентов ряда. Математические модели описывают динамику размыкания металлических контактов.

Ключевые слова: радиальные тепловые полиномы, проблема Стефана.

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THE SPECTRAL DECOMPOSITION OF CAUCHY PROBLEM'S SOLUTION FOR LAPLACE EQUATION

Abstract: The spectral decomposition of Cauchy problem for Laplace equation is obtained in Krein space, and is made a regularization of a problem, using the resolvent of the corresponding operator.
Keywords: spectrum, spectral decomposition, equation with deviating argument, Hilbert-Schmidt theorem, Cauchy Problem, Laplaceequation, incorrect, r ange.

1. Introduction.

Currently, there are different approaches to the solution of Cauchy problem for elliptic equations, which is a classic example of an ill-posed problem. All approaches basically can be divided on two large groups. One group consists of methods based on the introduction of the problem into the class of correctness by Tikhonov [1] - [3], the other are the methods using the universal regularizing algorithms, obtained by means of the parametric functional of Tikhonov [4].

It should be noted that the second group of methods received the most spread and major achievements in the practical application. In this approach, are used different variants of regularized algorithms that reduce the problem or to the solution of integral equations of the first kind, or to the representation of the desired field in the region beside or to the construction of finite-difference regularized algorithms [4] — [6].

In view of a great importance of the problem, which has applications in many fields of science and technology, and constantly rising requirements for the reliability of the results, the search for other approaches to its solution is continued. Iterative methods in recent years are becoming more widely used in the practice of solutions of various ill-posed problems of mathematical physics [7] - [9]. These methods have a number of undoubted advantages, which include simple computational schemes, their uniformity for applications with linear and nonlinear operators, the high accuracy of the solution, and so on.

An important advantage is the fact that they allow simple accounting of the essential restrictions for tasks on the solution directly in the scheme of the iterative algorithm (e.g., restrictions on non-negativity of solutions, monotonicity, and so on). In [10] was proposed a new method for solving the problem in question, based on the alternating iterative procedure, which is a consistent solution of the correct mixed boundary value problems for the original equation.

It is prooved the convergence of the method and its regularizing properties. This method is general and can be extended to a wide range of similar ill-posed boundary value problems of mathematical physics. The weak point of the method is the requirement for the smoothness of the boundary, which is not always fulfilled, in particular, in our case. In this paper we propose a spectral method [11-14].

Let $\Omega = [-1,1] \times [0, \pi]$ be a rectangle with sides
$AB: y = 0, -1 \le x \le 1; BC: x = 1, 0 \le y \le \pi; CD: y = \pi, -1 \le x \le 1; DA: x = -1, 0 \le y \le \pi$
(see fig.1)

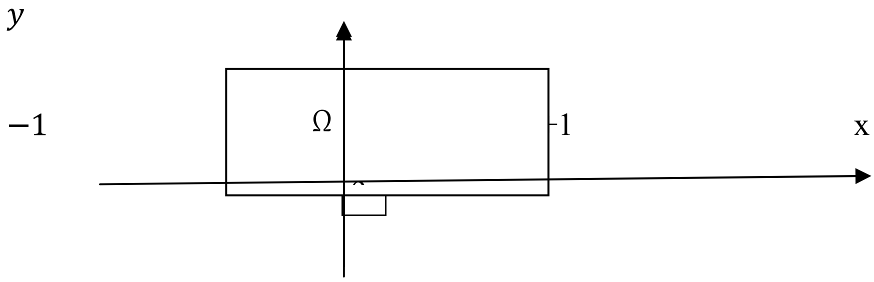


Fig 1.

Let's consider the following Cauchy - Dirichlet problem for Poisson's equation in the region Ω:

$$Lu = u_{xx} - u_{yy} = f(x,y), \tag{1}$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0, \tag{2}$$

$$u|_{x=-1} = 0, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0, \tag{3}$$

where$f(x,y) \in L^2(\Omega)$. This problem has been investigated previously in [11], [12] and is found that inverse operator L^{-1} exists, but is unlimited, in particular, it was shown that the "smallest" eigenvalue of the operator $A = SL$has asymptotics

$$\lambda_{m0} = 4m^2 e^{-2m}[1 + 0(1)], \quad m \to \infty,$$

where the operator S has the form $Su(x,y) = u(-x,y)$.

This work complements and refines the results of these studies.

2. Research Methods

The main idea of the method belongs to T.Sh. Kalmenov [11], and consists in the following. The operator $A = SL$ is symmetric in the space $L^2(\Omega)$, so with the original problem the boundary value problem is studied

$$SLu = Sf, \tag{1'}$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0, \tag{2'}$$

$$u|_{x=-1}, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0, \tag{3'}$$

where the operator S has the form, see [13] - [15].

$$Su(x,y) = u(-x,y),$$

and resembles an involution of M.G. Krein, see [16].

The following spectral problem corresponds to this boundary value problem (1 ') - (3')

$Au = \lambda u,$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0,$$

where

$$A = SL,$$

or in expanded form

$$u_{xx} + u_{yy} = \lambda u(-x, y), \quad (4)$$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \ \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0, \quad (5)$$

We solve this spectral problem (4) - (5) using the method of separation of variables, assuming

$$u(x, y) = v(x)w(y),$$

and as a result we get two spectral problems

a) $-w''(y) = \mu w(y),$ (6)

$w(0) = 0, \ w(\pi) = 0;$ (7)

$b) \ v''(x) - \mu v(x) = \lambda v(-x),$ (8)

$v(-1) = 0, v'(-1) = 0.$ (9)

The solution of (6) - (7) is well known and has the form $w_m(y) = \sin my, \ \ m = 1,2 \dots$,; an analogue of the spectral problem (8) - (9) was investigated in detail in [14], however, we will give a full and detailed study of this problem in the fourth section of the article. As a result, we have

$$Au_{mn} = \lambda_{mn} u_{mn}, \qquad m = 1,2, \dots; \quad n = 0,1,2, \dots$$

where$\{u_{mn}\}, m = 1,2, \dots; \quad n = 0,1,2, \dots$is complete and orthonormal system of functions in the space $L^2(\Omega)$.

Further, from the equation (1') we have
$\bar{A}u = Sf,$

Where $\bar{A}$ is a closure of the operator A in the space $L^2(\Omega)$.

Hence,

$$u = (\bar{A})^{-1}Sf = \sum_{m,n}^{\infty} \langle (\bar{A})^{-1}Sf, u_{mn} \rangle u_{mn} = |(\bar{A})^{*} = \bar{A}| = \langle Sf, (\bar{A})^{-1}u_{mn} \rangle u_{mn}$$
$$= \left| Au_{mn} = \lambda_{mn}u_{mn}, \Rightarrow \bar{A}u_{mn} = \lambda_{mn}u_{mn}, \qquad A \subset \bar{A}, \Rightarrow (\bar{A})^{-1}u_{mn} = \frac{u_{mn}}{\lambda_{mn}} \right|$$
$$= \sum_{m,n}^{\infty} \frac{(Sf, u_{mn})}{\lambda_{mn}} u_{mn} = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn},$$

where$[f, u_{mn}] = (Sf, u_{mn})$ is the inner product of Krein's space , and $(.,.)$ is the usual inner product of the space $L^2(\Omega)$, i.e.,

$$(f, g) = \iint_{\Omega} f(x, y) \cdot \overline{g(x, y)} dxdy.$$

Therefore, we need to show a closability of operator A, an essential self-adjointness: $\bar{A} = A^*$, and reversibility: $\ker \bar{A} = \{0\}$, because all of these properties are used in derivation of the last formula

$$u(x,y) = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn}(x,y).$$

In addition, it is necessary to examine the spectrum of the operator $\bar{A}$.

3. Results of research.

Let $D(A)$ is a domain of definition of operator A, and $R(A)$ is a domain of its values, $kerA$ is a kernel of the operator A, where

$$A = SL$$

$$Lu = u_{xx} + u_{yy}, \qquad Su(x,y) = u(-x,y),$$

$$D(A) = \left\{u(x,y) \in C^2(\Omega) \cap C^1(\bar{\Omega}) : u|_{y=0} = 0, \quad u|_{y=\pi} = 0, u|_{x=-1}, \ \frac{\partial u}{\partial x}\Big|_{x=-1} = 0\right\};$$

We denote through $\bar{A}$ the closure of the operator A in the space $L^2(\Omega)$.

The following theorem holds

Theorem 1.

(a) A is closable, i.e. its closure exists;

(b) A is essentially self-adjoint in the space $L^2(\Omega)$, i.e. the equality holds $(\bar{A})^* = \bar{A}$;

(c) A is invertible, i.e. $ker\bar{A} = \{0\}$, but the inverse operator $(\bar{A})^{-1}$ is unlimited, and has the form

$$u(x,y) = (\bar{A})^{-1} Sf(x,y) = \sum_{m,n}^{\infty} \frac{(Sf, u_{mn})}{\lambda_{mn}} u_{mn}(x,y) = \sum_{m,n}^{\infty} \frac{[f, u_{mn}]}{\lambda_{mn}} u_{mn}(x,y),$$

where$\{u_{mn}\}, m = 1,2,\dots; n = 0,1,2,...$ are the orthonormal eigenvectors of A, and $\lambda_{mn}$are the corresponding eigenvalues;

d) $\overline{R(A)} = H = L^2(\Omega) \neq R(\bar{A})$;

i.e. the operator equation

$$\bar{A}u = Sf$$

is densely solvable in the space $L^2(\Omega)$, but not everywhere solvable. The following theorem 2 reveals the spectral properties of the operator $\bar{A}$.

Theorem 2. The spectrum of operator $\bar{A}$ consists of four parts

a) The negative part:

$$-m^2 - (n\pi)^2 < \lambda_{mn}^- < -m^2 - \left(n\pi + \frac{\pi}{4}\right)^2, \quad m,n = 1,2,\dots;$$

b) the "zero" part:

$$m^2 e^{-2\sqrt{2}m cos u_0(m^2)} < \lambda^0(m^2) < 2\left(1+\sqrt{2}\right) m^2 e^{-2\sqrt{2}m \sin u_0(m^2)},$$

where

$$u_0(m^2) \geq u_0(1) > u_0^*(1) > 0,$$

and

$$\lim_{m\to\infty} u_0(m^2) = \frac{\pi}{4}; \quad m = 1,2 \dots$$

c) the positive part

$m^2 + (n\pi)^2 < \lambda_{mn}^+ < m^2 + (n\pi + \frac{\pi}{2})^2, \quad m,n = 1,2,...$

d) the limit part, i.e. point $\lambda = 0$ belongs to the limit spectrum of the operator $\bar{A}$, i.e. the equality holds

$$\overline{\{\lambda_{mn}\}} \ni \{0\}$$

g) theinequalities hold $\lambda_{mn} \neq 0, \ m = 1,2,\dots; \ n = 0,1,2,\dots$

Theorem 3. The boundary value problem

$u_{xx} + u_{yy} = \lambda u(-x, y), \quad x \in (-1,1], \qquad y(x, \pi)$

$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$

$$u|_{x=-1}, \ \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0,$$

has a complete and orthogonal system of eigenvectors:

$u_{mn}(x, y) = \sin my * v_{mn}(x), \qquad m = 1,2, \dots; n = 0,1,2, \dots$

$v_{mn}(x) = K_{mn}\left[ch\sqrt{m^2 + \lambda_{mn}} sh\sqrt{m^2 - \lambda_{mn}} x sh\sqrt{m^2 - \lambda_{mn}} ch\sqrt{m^2 + \lambda_{mn}} x\right],$

$$m = 1,2, \dots; n = 0,1,2, \dots$$

where K_{mn} are the normalization coefficients, and λ_{mn} are the roots of the equation

$$th\sqrt{m^2 - \lambda} th\sqrt{m^2 + \lambda} = \frac{\sqrt{m^2 - \lambda}}{\sqrt{m^2 + \lambda}},$$

for each fixed value m = 1,2, ... All the eigenvalues are simple, real and not equal to zero.

4. The proofs and discussion.

4.1. On the solvability.

Lemma 1. If the eigenvectors of a symmetric operator T, corresponding to non-zero eigenvalues, form an orthonormal basis of the Hilbert space H, then

a) this operator is essentially self-adjoint;

b) the operator $\overline{T}$ is reversible;

c) $\overline{R(\overline{T})} = H$;

d) $\overline{R(\overline{T})} = R(\overline{T})$,

if and only if the inequality holds

$$|\lambda_n| \geq \varepsilon > 0, n = 1,2, \dots$$

where $\lambda_n (n = 1,2, \dots)$ are the eigenvalues of the operator T acting in a Hilbert space H.

4.2. On the spectrum of the operator B.

Consider in the space $L^2(-1,1)$ the following spectral problem

$v''(x) - \mu v(x) = \lambda v(-x), x \in (-1,1]$ (8)

$v(-1) = 0, \ v'(-1) = 0$ (9)

where μ is a fixed real quantity, λ is a spectral parameter.

Let $B = S\hat{L}$, where

$$\hat{L}v = v''(x) - \mu v(x), \qquad Su(x) = u(-x),$$

then the spectral problem (8) - (9) takes the form

$$Bv = \lambda v; \ v(-1) = 0, v'(-1) = 0.$$

Notice that

$$D(B) = \{v(x) \in C^2(-1,1) \cap C^1[-1,1]: v(-1) = 0, v'(-1) = 0\}.$$

Obviously $C_0^\infty(-1,1) \subset D(B)$. In addition, the equality holds

$$(Bu, v) = (u, Bv), \qquad \forall u, v \in D(B)$$

If $\lambda = 0$, then $v(x) = 0$ by virtue of the uniqueness of the solution of Cauchy problem, so $kerB = \{0\}$, i.e. the inverse operator B^{-1} exists, which has the following form

$$v(x) = B^{-1}f(x) = \int_{-1}^{x} \frac{sh\sqrt{\mu}(x-t)Sf(t)}{\sqrt{\mu}} dt,$$

for any continuous function $f(x) \in C[-1,1]$. By means of the extension theorem (see [19]., C.154), we will continue this operator on the whole space $L^2(-1,1)$ as a continuous operator

$$\overline{B^{-1}}f(x) = \int_{-1}^{x} \frac{sh\sqrt{\mu}(x-t)Sf(t)}{\sqrt{\mu}}dt, \ \ \forall f(x) \in L^2(-1,1).$$

It is obvious that the operator $\overline{\mathrm{B}^{-1}}$ is completely continuous and self-adjoint in the space $\mathrm{L}^2(-1,1)$. By the formula,

$$\overline{\mathrm{B}^{-1}} = (\overline{\mathrm{B}})^{-1}$$

we have that the operator $\overline{\mathrm{B}}$ is reversible. Operator $(\overline{\mathrm{B}})^{-1}$ is completely continuous and self-adjoint in $L^2(-1,1)$.

By Hilbert-Schmidt Theorem (see [17], p. 226), for any $f(x) \in L^2(-1,1)$ the formula holds

$$(\bar{B})^{-1}f = \sum_{n=1}^{\infty}\langle(\bar{B})^{-1}f, v_n(\mu;x)\rangle v_n(\mu;x) = \sum_{n=1}^{\infty}\langle f, (\bar{B})^{-1}v_n(\mu;x)\rangle v_n(\mu;x)$$
$$= \sum_{n=1}^{\infty}\langle f, \overline{B^{-1}} \ \ v_n(\mu;x)\rangle \, v_n(\mu;x) = \left|\overline{B^{-1}}v_n = B^{-1}v_n = \frac{v_n(\mu;x)}{\lambda_n(\mu)}\right|$$
$$= \sum_{n=1}^{\infty}\langle f, v_n(\mu;x)\rangle \frac{v_n(\mu;x)}{\lambda_n(\mu)},$$

where$\lambda_n(\mu)$ are the eigenvalues of the operator B, and $v_n(\mu;x)$ are the corresponding eigenvectors.

If $\langle f, v_n(\mu;x)\rangle = 0$, for $n = 1,2,...,$ then $(\bar{B})^{-1}f = 0$,hence $f = 0$, i.e. the system $\{v_n(\mu;x)\}, n = 1,2,...$ is complete and orthogonal in the space $L^2(-1,1)$. We formulate the obtained results as following lemma.

Lemma 2. If $\mu = \bar{\mu}$, i.e. it is a real value, then

a) the operator $(\bar{B})^{-1}$ is completely continuous and self-adjoint;

b) the spectrum of $\bar{B}$ is discrete, i.e., it has no the condensation points;

c) the normalized eigenvectors of $\bar{B}$ form the orthonormal basis of the space $L^2(-1,1)$.

Let's find the eigenfunctions of the problem (8) - (9). The general solution of equation (8) has the form

$$v(\mu,\lambda;x) = a(\mu,\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)ch\sqrt{\mu+\lambda}x, \quad (10)$$

where$a(\mu,\lambda)$, $b(\mu,\lambda)$ are arbitrary constants.

Indeed,

$$v'(\mu,\lambda;x) = a(\mu,\lambda)\sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda}x + b(\mu,\lambda)\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda}x, \quad (11)$$
$$v''(\mu,\lambda;x) == a(\mu,\lambda)(\mu-\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)(\mu+\lambda)ch\sqrt{\mu+\lambda}x$$
$$= \mu\lambda(\mu,\lambda;x)\left[-a(\mu,\lambda)sh\sqrt{\mu-\lambda}x + b(\mu,\lambda)ch\sqrt{\mu+\lambda}x\right]\lambda = \mu v(\mu,\lambda;x) + \lambda v(\mu,\lambda;-x),$$
$$\Rightarrow v''(\mu,\lambda;x) - \mu v(\mu,\lambda;x) = \lambda v(\mu,\lambda;-x).$$

Substituting (10) - (11) into the boundary conditions (9), we have

$$\begin{cases} -ash\sqrt{\mu-\lambda} + bch\sqrt{\mu+\lambda} = 0 \\ a\sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda} - b\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda} = 0 \end{cases} (*)$$

This system of equations has a nontrivial solution only if its determinant $\Delta(\mu,\lambda)$ equals 0, where

$$\Delta(\mu,\lambda) = \begin{vmatrix} -sh\sqrt{\mu-\lambda} & ch\sqrt{\mu+\lambda} \\ \sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda} & -\sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda} \end{vmatrix}.$$

Expanding this determinant, we obtain

$$\Delta(\mu,\lambda) = \sqrt{\mu+\lambda}sh\sqrt{\mu+\lambda}sh\sqrt{\mu-\lambda} - \sqrt{\mu-\lambda}ch\sqrt{\mu-\lambda}ch\sqrt{\mu+\lambda} \quad (12)$$

If $\lambda = 0$, then

$$\Delta(\mu,0) = \sqrt{\mu}sh^2\sqrt{\mu} - \sqrt{\mu}ch^2\sqrt{\mu} = -\sqrt{\mu}[ch^2\sqrt{\mu} - sh^2\sqrt{\mu}] = -\sqrt{\mu},$$

If in addition $\mu = 0$, then from (8) - (9) we have that $v(x) = 0$. Consequently, the value of $\lambda = 0$ is not an eigenvalue of the problem (8) - (9).

If $\lambda = \mu$, then $\Delta(\mu,\mu) = 0$, therefore, the value of $\lambda = \mu$ is probably the eigenvalue of the problem (8) - (9), to which the following eigenfunction corresponds

$$v(\mu,\mu;x) = b(\mu,\mu)ch\sqrt{2\mu}x.$$

But this function satisfies the boundary condition (9) only when the $b(\mu,\mu) = 0$, so there is no eigenvalues of the boundary value problem (8) - (9) in the segment $[-\mu, 0]$, $\mu \geq 0$.

Assuming $\mu > 0$ for definiteness, let us study the distribution of zeros of functions (12).

Lemma 3. If $\mu > \mu_0$, then the function $F(\mu,u)$has a unique simple zero u_0, located in the interval$0 < u_0^* < u_0 < \frac{\pi}{4}$, where

$$1 - \sqrt{2\mu_0}th\sqrt{2\mu_0} = 0, \ \frac{\partial F}{\partial u}\Big|_{u=u_0^*} = 0.$$

Lemma 4. If $z = u$ a real quantity, the

a) for $0 < \mu < \mu_0$ segment $[-\mu,\mu]$no eigenvalues;

b) when $\mu \geq \mu_0$in interval $(0,\mu)$will be exactly one eigenvalue $\lambda_0^+(\mu)$, which satisfies the estimate

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu\left(1+\sqrt{2}\right) \cdot e^{-2\sqrt{2\mu}\sin u_0(\mu)},$$

where

$$u_0(\mu) > u_0^*(\mu) > 0, \ \ \forall\mu > \mu_0, \qquad \lim_{\mu\to+\infty} u_0(\mu) = \frac{\pi}{4},$$

$$\frac{\partial F}{\partial u}\Big|_{u=u_0^*} = 0, \quad 1 - \sqrt{2\mu_0}th\sqrt{2\mu_0} = 0.$$

a) $0 < \mu \leq \mu_0$

$-\infty + \infty$

$-\mu 0 \qquad + \mu$

б) $\mu > \mu_0 > 0$

$-\infty\lambda_0^+ \qquad + \infty$

$-\mu 0 \qquad + \mu$

fig 5.

Consequence. Forall $\mu \geq 1$, the inequality holds

$$\mu \cdot e^{-2\sqrt{2\mu}} < \lambda_0^+(\mu) < 2\mu(1+\sqrt{2})e^{-2\sqrt{2\mu}\sin u_0^*(1)}$$

where

$$\frac{\partial u}{\partial u}\Big|_{u=u_0^*(1)} = 0, \ u_0^*(1) > 0.$$

Lemma 5.

a) If $0 < \mu < \mu_0$, then the eigenvalues of the spectral task

$$v''(x) - \mu v(x) = \lambda v(-x), \quad x \in (-1,1] \quad (8)$$

$$v(-1) = 0, \quad v'(-1) = 0 \quad (9)$$

consists of two series: negative

$$-\left[\mu + \left(n\pi + \frac{\pi}{2}\right)^2\right] < \lambda_n^- < -\left[\mu + \left(n\pi + \frac{\pi}{4}\right)^2\right], \; n = 0,1,2,...,$$

and positive

$$\mu + (n\pi)^2 < \lambda_n^+(\mu) < \mu + (n\pi + \frac{\pi}{2})^2, \quad n = 0,1,2 ...$$

b) If $\mu > \mu_0$, then the third "zero" series will appear, that lays in the interval $(0, \mu)$, for which a two-sided estimate is valid

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu(1+\sqrt{2})e^{-2\sqrt{2\mu}\sin u_0(\mu)}$$

where

$$u_0(\mu) > u_0^*(\mu) > 0, \quad \forall \mu > \mu_0, \quad \lim_{\mu\to+\infty} u_0(\mu) = \frac{\pi}{4}$$

$$\left.\frac{\partial F}{\partial u}\right|_{n=u_0^*} = 0, \quad 1 - \sqrt{2\mu_0}th\sqrt{2\mu_0} = 0$$

b) we will name the quantity μ_0 the threshold, it is the root of the equation

c)

$$1 - \sqrt{2\mu}th\sqrt{2\mu} = 0$$

for it the assessment holds: 0,5 <μ_0 <0,72, see. ([18].,p.33)

From the system of equations (*) and formula (10), see. P.10 assuming

$$a_n = K_n ch\sqrt{\mu + \lambda_n}, b_n = K_n sh\sqrt{\mu - \lambda_n},$$

we will find the eigenfunctions of the boundary value problem (8) - (9)

$$v_n(\mu, \lambda_n, x) = K_n\left[ch\sqrt{\mu+\lambda_n}sh\sqrt{\mu-\lambda_n}x + sh\sqrt{\mu-\lambda_n}ch\sqrt{\mu-\lambda_n}x\right].$$

Lemma 6. If $\mu \geq 1$, then the spectral task

$$v''(x) - \mu v(x) = \lambda v(-x), \quad x \in (-1,1] \quad (8)$$

$$v(-1) = 0, \quad v'(-1) = 0 \quad (9)$$

has complete and orthogonal system of eigenvectors:

$$v_n(\mu, \lambda_n, x) = K_n\left[ch\sqrt{\mu+\lambda_n}sh\sqrt{\mu-\lambda_n}x + sh\sqrt{\mu-\lambda_n}ch\sqrt{\mu-\lambda_n}x\right], \qquad n = 0,1,2,...$$

corresponding to the real eigenvalues $\lambda_n(\mu)$, $n = 0,1,2,...$, which are distributed as follows:
a) negative:

$$-[\mu + (n\pi + \tfrac{\pi}{2})^2] < \lambda_n^-(\mu) < -\left[\mu + \left(n\pi + \frac{\pi}{4}\right)^2\right], \quad n = 0,1,2 ...$$

b) zero:

$$\mu \cdot e^{-2\sqrt{2\mu}\cos u_0(\mu)} < \lambda_0^+(\mu) < 2\mu(1+\sqrt{2})e^{-2\sqrt{2\mu}\sin u_0(\mu)}$$

where

$u_0(\mu) > u_0^*(\mu) > 0, \quad \forall\mu > \mu_0, \quad \lim_{\mu\to+\infty} u_0(\mu) = \frac{\pi}{4}$ (monotonically)

$$\left.\frac{\partial F}{\partial u}\right|_{u=u_0^*(\mu)} = 0, \quad 1 - \sqrt{2\mu_0} th\sqrt{2\mu_0} = 0, \quad 0{,}5 < \mu_0 < 0{,}72$$

c) positive:

$$\mu + (n\pi)^2 < \lambda_n^+(\mu) < \mu + (n\pi + \frac{\pi}{2})^2, \quad n = 1,2 \ldots$$

g) the interval $[-\mu, 0]$ remains free of eigenvalues where $\mu \geq 0$;

4.3. The proofs of theorems.

Let's start with Theorem 3. Assuming

$$u_{mn}(x, y) = \sin my \cdot v_{mn}(x), \quad m = 1,2, \ldots.$$

We divide the variables of the equation

$$u_{xx} + u_{yy} = \lambda u(-x, y), \quad x \in (-1,1], \quad y\epsilon(0, \pi)$$

result for $u_{mn}(x, y)$ we obtain the spectral problem

$$v''_{mn} - m^2 v_{mn}(x) = \lambda_{mn} v_{mn}(-x), \quad m = 1,2, \ldots$$
$$v_{mn}(-1) = 0, \quad v'_{mn}(-1) = 0.$$

By virtue of the proven Lemma 6, eigenfunctions of the spectral problem :$\{v_{mn}(x)\}$, $m = 1,2, \ldots, n = 0,1,2, \ldots$ form a complete orthogonal system in the space $L^2(-1,1)$. Therefore, after normalization, they form an orthonormal basis of the space.

Lemma 7. If the system $\{\varphi_m(y)\}$, $m = 1,2, \ldots$ is an orthonormal basis of the space $L^2(0, \pi)$, and the system $\{\psi_{mn}(x)\}, m = 1,2, \ldots;$ $n = 0,1,2, \ldots$ for each fixed value of m is an orthonormal basis of the space $L^2(-1,1)$, then the system

$$u_{mn}(x, y) = \psi_{mn}(x) \cdot \varphi_m(y), \quad m = 1,2, \ldots; n = 0,1,2, \ldots$$

is an orthonormal basis of the space $L^2(\Omega)$, where $\Omega = [-1,1] \times [0, \pi]$. See the proof [14].

In our case,

$$\varphi_m(y) = \sqrt{\frac{2}{\pi}} \sin my, \quad \psi_{mn}(x) = K_{mn} \cdot v_{mn}(x), \quad m = 1,2, \ldots.; n = 0,1,2, \ldots,$$

whereK_{mn} are the normalization coefficients, hence eigenfunctions

$$u_{mn}(x, y) = K_{mn} \sin my v_{mn}(x), \qquad m = 1,2, \ldots; n = 0,1,2, \ldots$$

of the boundary value problem

$$u_{xx} + u_{yy} = \lambda u(-x, y), \quad x \in (-1,1], \qquad y \in (0, \pi)$$
$$u|_{y=0} = 0, \quad u|_{y=\pi} = 0,$$
$$u|_{x=-1}, \quad \left.\frac{\partial u}{\partial x}\right|_{x=-1} = 0,$$

After normalization form an orthonormal basis of the space $L^2(\Omega)$. Theorem 3 is proved. Theorem 2 is a consequence of Lemma 6, when $\mu = m^2$, $m = 1,2, \ldots$ Theorem 1 follows from Lemma 1 and Theorem 3.

5. Conclusions.

The operator $\overline{\mathrm{A}}$ has a spectral hatch

$(-\mu_0, 0)$, where$0{,}5 < \mu_0 < 0{,}72$;

1) If $f(x,y) \in L^2(\Omega)$, then the solution of (1) - (3) exists if and only if
2)

$$\sum_{m=1}^{\infty}\sum_{n=0}^{+\infty}\frac{|(Sf,u_{mn})|^2}{\lambda_{mn}^2} < +\infty,$$

where$Sf(x,y) = f(-x,y)$.

2) There is regularizing family of tasks, which has the form
$\bar{A}u + \mu u = Sf$, где $0 < \mu < \mu_0$.

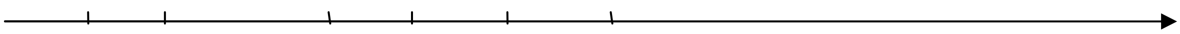
Proof of paragraph 3).
Let $0 < \mu < \mu_0$, we estimate the resolution of $(\bar{A} + \mu I)^{-1}$. It's obvious that
$\bar{A}u_{mn} + \mu u_{mn} = (\lambda_{mn} + \mu)u_{mn}$,therefore$(\bar{A} + \mu I)^{-1}u_{mn} = \frac{u_{mn}}{\lambda_{mn}+\mu}$.

$$\begin{aligned} u_\mu(x,y) &= [\bar{A} + \mu I]^{-1}Sf \\ &= \sum\nolimits_{m,n}([\bar{A} + \mu I]^{-1}Sf, u_{mn})u_{mn} \\ &= \sum\nolimits_{m,n}(Sf, [\bar{A} + \mu I]^{-1}, u_{mn})u_{mn} = \sum_{m,n}\frac{(Sf,u_{mn})}{\mu + \lambda_{mn}}u_{mn}. \end{aligned}$$

Next, we estimate the distance from the point $-\mu$ to the spectrum of the operator$\bar{A}$.
a) For the negative eigenvalues of the operator$\bar{A}$ thefollowing inequality holds

$$-m^2 - (n\pi)^2 < \lambda_{mn}^- < -m^2 - (n\pi - \frac{\pi}{2})^2, m,n = 1,2,...,$$

Therefore
$-\infty + \infty$



$-\infty + \infty$

$\lambda_{mn}^- - 1 - (\frac{\pi}{2})^2 - 1 - \mu_0 - \mu \quad 0$

since, $0 < \mu < \mu_0 < 0{,}72$,then $0{,}72 < -\mu_0 < -\mu < 0$;

$$|\lambda_{mn}^- + \mu| > \left|-1 - (\frac{\pi}{2})^2 + \mu\right| > \left|-1 - (\frac{\pi}{2})^2 + \mu_0\right| = \left|1 + (\frac{\pi}{2})^2 - \mu_0\right| = 1 - \mu_0 + (\frac{\pi}{2})^2;$$

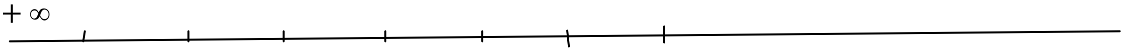
b) to "zero" eigenvalues $\lambda_{mn}^+ > 0$, we have

$$|-\mu - \lambda_{mn}^+| > |-\mu - 0| = |-\mu| = \mu > 0;$$

For positive eigenvalues the following inequalities hold

$$m^2 + (n\pi)^2 < \lambda_{mn}^+ < m^2 + (n\pi + \frac{\pi}{2})^2, \quad m,n = 1,2,...$$

$-\infty + \infty$



$-1 - \mu_0 - \mu \quad 0 \quad 1 \quad 1 + \pi^2 \quad \lambda_{mn}^+$

$$|-\mu - \mu_{mn}^+| \geq \mu + \lambda_{mn}^+ > \mu + 1 + \pi^2 > 1 + \pi^2.$$

Based on these inequalities, we estimate the resolventof$\bar{A}$, at $\lambda = -\mu_0$.
From the equation,

$$(\bar{A} + \mu I)u_\mu = Sf,$$

we have

$$u_\mu(x,y) = [\bar{A} + \mu I]^{-1} Sf = \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn} + \mu} u_{mn} = \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn}^- + \mu} u_{mn} + \sum_{m=1}^{\infty} \frac{(Sf, u_{mn}) u_{mn}}{\lambda_{mn}^- + \mu} += \sum_{m,n} \frac{(Sf, u_{mn})}{\lambda_{mn} + \mu} u_{mn};$$

$$\|u_\mu\|^2 = \sum_{m,n} \frac{(Sf, u_{mn})^2}{(\lambda_{mn}^- + \mu)^2} + \sum_{m=1}^{\infty} \frac{(Sf, u_{m0})^2}{(\lambda_{mn}^+ + \mu)^2} + \sum_{m,n} \frac{(Sf, u_{mn})^2}{(\lambda_{mn}^+ + \mu)^2} \le \frac{1}{\left[1 - \mu_0 + (\frac{\pi}{2})^2\right]^2} \sum_{m,n} |(Sf, u_{mn}|^2 + \frac{1}{\varepsilon^2} \sum_{m=1}^{\infty} |(Sf, u_{m0})|^2 + \frac{1}{1+\pi^2} \sum_{m,n} |(sf, u_{mn})|^2 \le K^2(\varepsilon) \|sf\|^2 \le K^2(\varepsilon) \cdot \|f\|^2,$$

where

$$K(\varepsilon) = \max\left\{\frac{1}{1 - \mu_0 + (\frac{\pi}{2})^2}, \frac{1}{\varepsilon^2}, \frac{1}{1+\pi^2}\right\}.$$

Assume that for a given $f \in L^2(\Omega)$ there exists a solution of equation
$\bar{A}u = Sf$.
then

$$\lim_{\mu \to 0} \|u_\mu - u\| = 0.$$

Indeed,

$$\|u_\mu(x,y) - u(x,y)\|^2 = \sum_{m,n} \left(\frac{1}{\lambda_{mn} + \mu} - \frac{1}{\lambda_{mn}}\right)^2 |(Sf, u_{mn})|^2 \le \sum_{m,n} \frac{\mu^2}{(\lambda_{mn} + \mu)^2} |(Sf, u_{mn})|^2 \le \sum_{m,n} \frac{\mu^2}{(\lambda_{mn}^- + \mu)^2 (\lambda_{mn}^-)^2} |(Sf, u_{mn})|^2 + \sum_{m,n} \frac{\mu^2}{(\lambda_{m0}^+ + \mu)^2 (\lambda_{m0}^+)^2} |(Sf, u_{m0})|^2 + \sum_{m,n}^{+\infty} \frac{\mu^2}{(\lambda_{mn}^+ + \mu)^2} |(Sf, u_{mn})|^2 \le \frac{\mu^2}{\left[1 - \mu_0 + (\frac{\pi}{2})^2\right]^2} \sum_{m,n} \frac{|(Sf, u_{mn}|^2}{(\lambda_{mn}^-)^2} + \frac{\mu^2}{(1+\pi^2)^2} \sum_{m,n} \frac{|(Sf, u_{mn}|^2}{(\lambda_{mn}^+)^2} + \sum_{m=1}^{\infty} \frac{\mu^2}{(\lambda_{m0}^+ + \varepsilon)^2 (\lambda_{m0}^+)^2} |(Sf, u_{m0})|^2 <$$

$$< \mu^2 \left[\frac{1}{\left[1-\mu_0+\left(\frac{\pi}{2}\right)^2\right]^2} + \frac{1}{(1+\pi^2)^2} \right] ||f||^2$$

$$+ \sum_{m=1}^{\infty} \frac{\mu^2 |(Sf, u_{m0}|^2}{\left(\lambda_{m0}^{+}\right)^2 (\lambda_{m0}^{+}+\mu)} < \frac{2\mu^2}{\left[1-\mu_0+\left(\frac{\pi}{2}\right)^2\right]^2} + \sum_{m=1}^{\infty} \frac{\mu^2}{\left(\left(\lambda_{m0}^{+}\right)^2 \lambda_{m0}^{+}+\mu\right)^2} |(Sf, u_{m0})|^2$$

$$\leq \left| \begin{matrix} \lambda_{10}^{+} > \lambda_{20}^{+} > \cdots > \lambda_{N0}^{+} \\ \frac{\mu}{\lambda_{m0}^{+}+\mu} < 1 \end{matrix} \right|$$

$$\leq \frac{2\mu^2}{\left[1-\mu_0+\left(\frac{\pi}{2}\right)^2\right]^2} ||f||^2$$

$$+ \frac{\mu^2}{\left(\lambda_N^{+}\right)^{+}} \sum_{m=1}^{N} |(Sf, u_{m0})|^2$$

$$+ \sum_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^{+}\right)^2}$$

$$< \frac{2\mu^2 ||f||^2}{\left[1-\mu_0+\left(\frac{\pi}{2}\right)^2\right]^2} + \frac{\mu^2}{\left(\lambda_N^{+}\right)^{+}} ||f||^2$$

$$+ \sum_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^{+}\right)^2} \leq \frac{3\mu^2}{\left(\lambda_N^{+}\right)^2} ||f||^2 + \sum\nolimits_{N+1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^{+}\right)^2}.$$

By our assumption$\sum_{m=1}^{\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^{+}\right)^2} < +\infty$, therefore for any $\varepsilon > 0$there exists a number $N(\varepsilon)$such that for all $M > N(\varepsilon)$ the inequality holds
$\sum_{M}^{+\infty} \frac{|(Sf, u_{m0})|^2}{\left(\lambda_{m0}^{+}\right)^2} < \frac{\varepsilon^2}{2}$.

Then, at a fixed $N(\varepsilon)$, there exists a number $\delta > 0$such that for all $0 \leq \mu < \delta$the following inequality holds

$$\frac{3\mu^2 ||f||^2}{(\lambda_N^{+})^2} < \frac{\varepsilon^2}{2}$$

Consequently, for any $\varepsilon > 0$ there exists $\delta > 0$ such that
$\left\|u_\mu - u\right\|^2 < \varepsilon^2$ i.e. $\left\|u_\mu - u\right\| < \varepsilon$for all $0 \leq \mu < \delta$, as required.

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ЛАПЛАС ТЕҢДЕУІНІҢ КОШИЛІК ЕСЕБІНІҢ СПЕКТРӘЛДІК ТАРАЛЫМЫ

Аннотация. Лаплас теңдеуіне қойылған Кошидің есебінің шешімі бар екені анықталып,оның Крейннің кеңістігіндегі спектрәлді таралымы алынды. Сонан соң резәлвентасы арқылы есептің жөнге келетіні көрсетілді.

Түйін сөздер: Кошидің есебі, Лапластың теңдеуі, ауытқыған аргумент, жалылық, әсіре үзіксіздік, Гилберт пен Шмидтің теоремасы.

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СПЕКТРАЛЬНОЕ РАЗЛОЖЕНИЕ РЕШЕНИЯ ЗАДАЧИ КОШИ ДЛЯ УРАВНЕНИЯ ЛАПЛАСА

Аннотация.В пространстве Крейна, получено спектральное разложение решении задачи Коши уравнения Лапласа,и произведена регуляризация задачи, с помощью резольвенты соответствующего оператора.

Ключевые слова: Задача Коши, уравнение Лапласа, некорректность, спектр, отклояющиеся аргумент, самосопряженность, компактность, теорема Гильберта-Шмидта.

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