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# Х А Б А Р Л А Р Ы

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# INVERSE PROBLEM OF THE STORM-LIUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

**Abstract.** Under the inverse tasks of spectral analysis understand tasks reconstruction of a linear operator from one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Ambartsumian. He proved the following theorem.

We denote by $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ the eigenvalues of the Sturm - Liouville problem

$$-y'' + q(x)y = \lambda y, \quad (1.1)$$

$$y'(0) = 0,\ y'(\pi) = 0; \quad (1.2)$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2 (n = 0,1,2,\dots) \text{ то } q(x) \equiv 0.$$

The first mathematician who drew attention to the importance of this result of Ambartsumian was the Swedish mathematician Borg. He performed the first systematic study of one of the important inverse problems, namely, the inverse problem for the classical Sturm - Liouville operator of the form (1.1) with respect to spectra. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the result of Ambartsumian is an exception to the general rule. In the same work, Borg shows that two spectra of the Sturm - Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

**Borg's theorem.**

Let the equations

$$-y'' + q(x)y = \lambda y,$$
$$-z'' + p(x)z = \lambda z,$$

have the same spectrum under boundary conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases}$$

and under boundary conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0,\pi]$ if

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Soon after Borg's work, important studies on the theory of inverse problems were carried out by Levinson, in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm-Liouville operator

$$-y'' + q(x)y = \lambda y, \quad (1.1)$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.3}$$

is restored by one spectrum.

The inverse problems for differential operators with decaying boundary conditions are fairly well understood. More difficult inverse problems for Sturm - Liouville operators with unseparated boundary conditions have also been studied. In particular, the periodic boundary-value problem was considered in a number of papers. I.V. Stankevich proposed the formulation of the inverse problem and proved the corresponding uniqueness theorem.

The present work is devoted to a generalization of the theorems of Ambartsumian and Levinson, in particular, our results contain the results of these authors. In the paper, a uniqueness theorem is proved, for one spectrum, for the Sturm-Liouville operator with unseparated boundary conditions, a real continuous and symmetric potential. The research method differs from all previously known methods, and is based on the internal symmetry of the operator generated by invariant subspaces.

Note that the operator we are considering is non-self-adjoint, although the potential is real and symmetric, this moment plays an essential role for our method, because we construct a pair of Borg operators through the operator and its adjoint one. Other authors use the Leibenzon mapping method.

**Keywords:** Sturm-Liouville operator, spectrum, inverse Sturm-Liouville problem, Borg theorem, Hambardzumyan theorem, Levinson theorem, non-separated boundary value conditions, symmetric potential, invariant subspaces, differential operators, inverse spectral problems.

## 1. Introduction

We study the inverse spectral problem for the Sturm – Liouville operator:

$$Ly := y'' + q(x)y, \; x\epsilon(0,1),$$

on the finite interval (0,1) with non-separated boundary value conditions. Inverse problems consist in restoring the coefficients of differential operators by their spectral characteristics. Such problems often arise in mathematics and its applications.

Inverse problems for differential operators with decaying boundary value conditions have been thoroughly studied (see monographs [1–5] and references). More difficult inverse problems for Sturm – Liouville operators with non-separated boundary value conditions were studied in [6–9] and other works. In particular, periodic boundary-value problem was considered in [6, 7]. I. V. Stankevich [6] proposed formulation of the inverse problem and proved the corresponding uniqueness theorem. V. A. Marchenko and I. V. Ostrovsky [7] gave a characteristic of the spectrum of a periodic boundary-value problem in terms of special conformal mapping. The conditions proposed in [7] are difficult to verify. Another method used in [8] made it possible to obtain necessary and sufficient conditions for solvability of the inverse problem in the periodic case that are more convenient for verification. Similar results were obtained in [8] for another type of boundary conditions, namely

$$y'(0) - a y(0) + b y(\pi) = y'(\pi) + d y(\pi) - b y(0) = 0.$$

Later similar results were obtained in [9]. In [10], the case when the potential is $q -$ symmetric with respect to the middle of the interval, that is, $q(x) = q(\pi - x)$ a.e. on (0, π), was investigated, and for this case, solution of the inverse spectral problem was constructed and the spectrum was characterized. The symmetric case requires nontrivial changes in the method and allows us to specify less spectral information than in the general case. Some results for the symmetric case were obtained in [11] - [13].

By inverse problems of spectral analysis, we understand the problems of reconstructing a linear operator by one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Hambardzumyan [14]. He proved the following theorem.

By $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ we denote eigenvalues of the Sturm - Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0, \; y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2 \; (n = 0,1,2,\dots) \; \text{ then } q(x) \equiv 0.$$

The first mathematician who drew attention to the importance of this Hambardzumyan result was the Swedish mathematician Borg. He performed the first systematic study of one of the important inverse problems, namely, the inverse problem for the classical Sturm – Liouville operator of the form (1.1) by the spectra [15]. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the Hambartsumyan result is an exception to the general rule. In the same paper [15], Borg showed that two spectra of the Sturm – Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Soon after the Borg work, important studies on the theory of inverse problems were carried out by Levinson [18], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm – Liouville operator

$$-y'' + q(x)y = \lambda y,$$
$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases}$$

is reconstructed by one spectrum.

A number of works by B.M. Levitan [19] are devoted to the reconstruction the Sturm – Liouville operator by one and two spectra.

This work is devoted to a generalization of the theorems of Hambartsumian [14] and Levinson [16], in particular, our results contain the results of these authors. The research method of this work appeared under influence of [18] - [20], and differs from all previously known methods.

## 2. Research Methods.

Idea of this work is very simple. Having studied in detail contents of [14, 16], we realized that both of these operators have invariant subspaces. Generalization of this property of the Sturm - Liouville operator led us to the results presented below.

## 3. Research Results.

In the Hilbert space $H = L^2(0,\pi)$ we consider the Sturm - Liouville operator:

$$Ly = -y'' + q(x)y, \ x \in (0,\pi); \tag{1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{2}$$

where $q(x)$ is a continuous complex function, $a_{ij}$ $(i = 1,2; \ j = 1,2,3,4)$ are arbitrary complex coefficients, and by $\Delta_{ij}$ $(i = 1,2; \ j = 1,2,3,4)$ we denote minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Assume that $\Delta_{13} \neq 0$, then the Sturm - Liouville operator (1) – (6) takes the following form:

$$Ly = -y'' + q(x)y, \ x \in (0,\pi); \tag{1}$$

$$\begin{cases} \Delta_{13}y(0) - \Delta_{32}y'(0) - \Delta_{34}y'(\pi) = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0, \end{cases} \tag{2'}$$

and its conjugate operator $L^+$ takes the form

$$L^+z = -z'' + \overline{q(x)}z, \ x \in (0,\pi); \tag{1$^+$}$$

$$\begin{cases} \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0. \end{cases} \tag{2$'^+$}$$

Let $P$ and $Q$ be projections, defined by the formulas

$$Pu(x) = \frac{u(x) + u(\pi - x)}{2}, \ Qv(x) = \frac{v(x) - v(\pi - x)}{2}$$

The main result of this work is the following theorem.

**Theorem 3.1.** If $\Delta_{13} \neq 0$, then

1) $PL = L^+P$; (3)
2) $LQ = QL^+$; (4)
3) $\Delta_{12} = -\Delta_{34}$; (5)

and the Sturm - Liouville operator (1) – (2$'$) is reconstructed by one spectrum.

**4. Discussion.**

In this section we prove Theorem 3.1, and discuss the obtained results. The following Lemmas 3.1 and 3.2 have and independent meaning.

**Lemma 4.1.** If for a linear discrete operator $L$ we have

1) $PL = L^{+}P$; (3)
2) $LQ = QL^{+}$; (4)
3) $P + Q = I$; (6)

where $P, Q$ are orthogonal projections, and $I$ is unit operator, then all its eigenvalues are real.

The following lemma shows that the spectrum $\sigma(L)$ of the operator $L$ consists of two parts, so the operator $L$, apparently, splits into two parts. In the future, we will see that this is exactly what happens, and moreover, under certain conditions, these parts form a Borg pair.

**Lemma 4.2.** If $L$ is a linear discrete operator satisfying the conditions:

1) $PL = L^{+}P$; (3)
2) $LQ = QL^{+}$; (4)
3) $P + Q = I$; (6)

where $P, Q$ are orthogonal projections, and $I$ is unit operator, then

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2),$$

where $L_1 = PL,\ L_2 = LQ,\ \sigma(L)$ is a spectrum of the operator $L$.

**Lemma 4.3.** If

a) $\Delta_{13} \neq 0$; (7)

b) $PL = L^{+}P$; (3)

then for the Sturm - Liouville operator (1) – (6′) we get

1) $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$; (8)

2) $\overline{\left(\frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}\right)} = \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} == \frac{\Delta_{34}-\Delta_{14}}{\Delta_{13}}$;

3) $q(\pi - x) = q(x), \overline{q(x)} = q(x)$. (9)

Moreover, operators $L$ and $L^{+}$ take the following forms:

a) $Ly = -y'' + q(x)y,\ x \in (0,\pi)$; (1)

$$\begin{cases} y(0) + \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} y'(0) + y(\pi) - \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} y'(\pi) = 0, \\ \frac{\Delta_{12}}{\Delta_{13}} y'(0) + y(\pi) + \frac{\Delta_{14}}{\Delta_{13}} y'(\pi) = 0. \end{cases} \quad (10)$$

b) $L^{+}z = -z'' + \bar{q}(x)z,\ x \in (0,\pi)$; (1)$^{+}$

$$\begin{cases} z(0) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}} z'(0) - z(\pi) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}} z'(\pi) = 0, \\ z(0) - \frac{\overline{\Delta_{32}}}{\overline{\Delta_{13}}} z'(0) - \frac{\overline{\Delta_{12}}}{\overline{\Delta_{13}}} z'(\pi) = 0. \end{cases} \quad (10)^{+}$$

Similar lemma holds with the projector $Q$.

**Lemma 4.4.** If

a) $\Delta_{13} \neq 0$; (7)

б) $LQ = QL^{+}$,

then for the Sturm - Liouville operator (1) – (6′) we get

1) $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$; (8)

2) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right)} = \left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right) = \frac{\Delta_{32}+\Delta_{34}}{\Delta_{13}}$;

3) $q(\pi - x) = q(x), \bar{q}(x) = q(x)$. (9)

In this case operators $L$ and $L^+$ take the following forms:

**a**) $Ly = -y'' + q(x)y,\ x \in (0,\pi)$; (1)

$$\begin{cases} y(0) + y(\pi) + \dfrac{\Delta_{12} - \Delta_{32}}{\Delta_{13}}[y'(0) - y'(\pi)] = 0, \\ \dfrac{\Delta_{12}}{\Delta_{13}}y'(0) + y(\pi) + \dfrac{\Delta_{14}}{\Delta_{13}}y'(\pi) = 0; \end{cases}$$

**b**) $L^+z = -z'' + \bar{q}(x)z,\ x \in (0,\pi)$; $(1)^+$

$$\begin{cases} z(0) - z(\pi) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0) + z'(\pi)] = 0, \\ z(0) - \frac{\overline{\Delta_{32}}}{\overline{\Delta_{13}}}z'(0) - \frac{\overline{\Delta_{12}}}{\overline{\Delta_{13}}}z'(\pi) = 0. \end{cases}$$

**Lemma 4.5.** If $\Delta_{13} \neq 0$, and

**a)** $PL = L^+P$,

**b)** $LQ = QL^+$;

then the operators $L$ and $L^+$ take the following forms:

$$Ly = -y'' + q(x)y,\ x \in (0,\pi);$$

$$\begin{cases} y(0) + y(\pi) + \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}[y'(0) - y'(\pi)] = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0. \end{cases}$$

$$L^+z = -z'' + \bar{q}(x)z,\ x \in (0,\pi);$$

$$\begin{cases} z(0) - z(\pi) - \frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}[z'(0) + z'(\pi)] = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0; \end{cases}$$

where

1) $q(\pi - x) = q(x)$;

2) $\bar{q}(x) = q(x)$,

3) $\overline{\left(\frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}\right)} = \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} = \frac{\Delta_{34}-\Delta_{14}}{\Delta_{13}}$;

4) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right)} = \left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right) = \frac{\Delta_{32}+\Delta_{34}}{\Delta_{13}}$.

Further, from the formula $PL = L^+P$ we note that the operator $L_1 = PL$ maps from the subspace $H_1 = PH$, where $H = L^2(0,\pi)$. Assuming

$$u(x) = Py(x) = \frac{y(x) + y(\pi - x)}{2},$$

we have

$$u'(x) = \frac{y'(x) - y'(\pi - x)}{2}.$$

Then from Lemma 4.5 it follows that

$$\begin{cases} \Delta_{13}u(0) + (\Delta_{12} - \Delta_{32})u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13} u(0) + (\Delta_{12} - \Delta_{32}) u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

Similarly, assuming that

$$v(x) = \frac{z(x) - z(\pi - x)}{2}, \text{ we get } v'(x) = \frac{z'(x) + z'(\pi - x)}{2}.$$

Then Lemma 4.5 implies that

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \overline{\Delta}_{13} v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{14}}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

Due to the lemmas,

$$\overline{\left(\frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}}\right)} = \frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}},$$

therefore, the last boundary condition has the form

$$\begin{cases} \Delta_{13} v(0) - (\Delta_{12} + \Delta_{14}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

Thus, the operators $L_1$ and $L_2$ take the following forms

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13} u(0) + (\Delta_{12} - \Delta_{32}) u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \overline{\Delta}_{13} v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{14}}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

where $\frac{\Delta_{12} - \Delta_{32}}{\Delta_{13}}$ and $\frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}}$ are real quantities.

From the condition (10) of the proved Theorem 3.1 it follows that

$$\Delta_{12} - \Delta_{32} = -(\Delta_{12} + \Delta_{14}).$$

Assuming $\alpha = \Delta_{13}$, $\beta = \Delta_{12} - \Delta_{32}$, we rewrite the operators $L_1$ and $L_2$ as follows:

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \alpha u(0) + \beta u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$
$$\begin{cases} \alpha v(0) + \beta v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

If a spectrum of the operator $L$ is known, then by Lemma 2 spectra of the operators $L_1$ and $L_2$ are known. It is obvious that they form Borg pair in the interval $\left[0, \frac{\pi}{2}\right]$. By the Borg theorem spectra of these two operators uniquely determine the Sturm - Liouville operator on the segment $\left[0, \frac{\pi}{2}\right]$, and due to the formula $q(x) = q(\pi - x)$ on the whole segment $[0, \pi]$. Theorem 3.1 is proved.

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## ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

**Аннотация**. Спектрәлдік анализдің кері есептері ретінде сызықтық операторды, оның спектрәлдік сыйпаттары арқылы қалпына келтіру есептері танылады. Бұл бағытта елеулі нәтиежеге, 1929 жылы В.А. Амбарцумян қол жеткізді. Ол келесі, теореманы дәлелдеді.

Мына,
$$-y'' + q(x)y = \lambda y, \tag{1.1}$$
$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$
Штурм-Лиувилл есебінің меншікті мәндерін былай, $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ белгілейік, мұндағы $q(x)$ −дегеніміз, нақты әрі үзіліссіз функция.

Егер
$$\lambda_n = n^2\ (n = 0,1,2, \dots) \text{ болса, онда } q(x) \equiv 0.$$

Амбарцумянның бұл еңбегінің маңыздылығына алғаш рет көңіл аударған швед математигі Борг еді. Штурм-Лиувиллдің (1.1) түріндегі өте маңызды байырғы операторының кері есебін спектрі арқылы жүйелі, әрі мақсатты түрде, алғаш рет зерттеген-де осы автордың өзі.

Борг бір спектрдің Штурм-Лиувилл операторын анықтауға жалпы жағдайда жетпейтінін,ал Амбарцумянның нәтиежесі сәтті бір кездейсоқтық екенін көрсетті.Дәл сол еңбегінде, Борг Штурм-Лиувилл операторын екі спектр арқылы (әртүрлі шекаралық шарттар бойынша) бірмәнді анықтауға болатынын көрсетті. Іле-шала, Боргтың еңбегінен соң, кері есептер теориясының маңызды есептерін Левинсон зерттеді, мысалы, егер $q(\pi - x) = q(x)$ болса, онда, Штурм-Лиувиллдің, мына,
$$-y'' + q(x)y = \lambda y, \tag{1.1}$$
$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.3}$$
операторының, бір спектр арқылы, бірмәнді анықталатынын көрсетті.

Бұл еңбек Амбарцумян мен Левинсонның теоремаларын дамытуға арналған және өз бойында олардың теоремаларын алып жатыр.Бұл еңбекте потенциалы нақты үзіксіз әрі симметриялы,ал шекаралық шарттары әртарапты Штурм-Лиувилл операторын, бір спектр арқылы, бірмәнді анықтауға болатыны көрсетілді.Зерттеу әдісі бұрынғы әдістердің бәрінен өзгеше,және ол оператордың инвариантты іш кеңістіктерінің туындатқан симметриясына негізделген.

**Түйін сөздер**: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, әртарапты шекаралық шарттар, симметриялы потенциал, инвариантты ішкеңістіктері, дифференциәлдік операторлар, кері спектрәлді есептер.

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# ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕРАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

**Аннотация.** Под обратными задачами спектрального анализа понимают задачи восстановления линейного оператора по тем или иным его спектральным характеристикам. Первый существенный результат в этом направлении был получен в 1929 году В.А. Амбарцумяном. Он доказал следующую теорему.

Обозначим через $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ собственные значения задачи Штурма-Лиувилля

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$

где $q(x) -$ действительная непрерывная функция. Если

$$\lambda_n = n^2 (n = 0,1,2,\dots) \text{ то } q(x) \equiv 0.$$

Первым из математиков, кто обратил внимание на важность этого результата Амбарцумяна, был шведский математик Борг. Он же выполнил первое систематическое исследование одной из важных обратных задач, а именно, обратной задачи для классического оператора Штурма-Лиувилля вида (1.1) по спектрам. Борг показал, что в общем случае один спектр оператора Штурм-Лиувилля его не определяет, так что результат Амбарцумяна является исключением из общего правила. В той же работе Борг показывает, что два спектра оператора Штурма-Лиувилля (при различных граничных условиях) однозначно его определяют. Точнее, Борг доказал следующую теорему.

**Теорема Борга.**

Пусть уравнения

$$-y'' + q(x)y = \lambda y,$$

$$-z'' + p(x)z = \lambda z,$$

имеют одинаковый спектр при краевых условиях

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases}$$

и при краевых условиях

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases}$$

Тогда $q(x) = p(x)$ почти всюду на отрезке $[0, \pi]$, если

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Вскоре после работы Борга важные исследования по теории обратных задач были выполнены Левинсоном, в частности, им доказано, что если $q(\pi - x) = q(x)$, то оператор Штурма-Лиувилля

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.3}$$

восстанавливается по одному спектру.

Обратные задачи для дифференциальных операторов с распадающимися краевыми условиями достаточно полно изучены. Более трудные обратные задачи для операторов Штурма-Лиувилля с неразделенными краевыми условиями также изучались. В частности, периодическая краевая задача рассматривалась в ряде работ. И. В. Станкевич предложил постановку обратной задачи и доказал соответствующую теорему единственности.

Настоящая работа посвящена обобщению теорем Амбарцумяна и Левинсона, в частности, наши результаты содержат в себе результаты этих авторов. В работе доказана теорема единственности, по одному спектру для оператора Штурма-Лиувилля с неразделенными краевыми условиями, вещественным

непрерывным и симметричным потенциалом. Метод исследования отличается от всех ранее известных методов и основан на внутренней симметрии оператора, порожденного инвариантными подпространствами.

Отметим, что рассматриваемый нами оператор является несамосопряженным, хотя потенциал вещественный и симметричный, этот момент играет существенную роль для нашего метода, ибо мы через оператора и его сопряженного строим пару операторов Борга. Другие авторы используют метод отображений Лейбензона.

**Ключевые слова:** оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства, дифференциальные операторы, обратные спектральные задачи.

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# UNIVERSAL COMPLEX OF PSYCHOPHYSIOLOGICAL TESTING

**Abstract.** An experimental version of the system of psychological testing with fixation in real time of physiological parameters of the tested person has been developed. The Data of photoplethysmogram (PPG) and galvanic skin reaction (GSR) have been determined as sources of physiological data. The soft & hardware complex of psychophysiological testing allows in relation to each question of the test to capture and evaluate the psychophysiological state of the testing person, which provides additional information for the psychologist. For experimental tests two methods were chosen, the first one was the Buss-Durkee technique (BDHI), which allows to diagnose the aggressiveness of the individual, and the second was the method of assessing the neuropsychological stability. These tests are recommended for use in psychological selection for military service [1-2]. The tests are adapted to the Kazakh language and tested on cadets of military institutions and students of civil Universities in Almaty.

**Key words:** electrocardiogram, photoplethysmogram, galvanic skin response, microprocessor, signal processing, psychological tests, Bass-Darki technique, personality, self-esteem level, neuropsychic stability, intellectual regulation, mental adaptation systems, validity criterion, retest method, maladaptive form of behavior, programmatically hardware complex, graphical interface.

**Introduction.** In the era of scientific and technological progress with its stressful rhythms andnew specific conditions of human activity significantly increasing requirements for humans intellectual, emotional and volitional resources [3-4]. In this regard, especially there is a requirement from the human resources departments of the organizations in objective psycho-physiological portrait of a person. The primery tool of psychologists are psychological tests. However, as practice shows, in connection with the public accessibility of tests, the effect of subjectivism will recently increase.

The rapid development of computer technology contributed to automate conducting and processing of psychological testing [5] and the use of new methods of mathematic treatment of biomedical data [6]. Modern possibilities for the development of various sensors [7] and price reduction of the microprocessors opened a wide opportunity for impementation of the software&hardware for assessing the psycho-physiological portrait of the individual [8-10]. The paper [11] describes a software&hardware complex of psychophysiological testing based on the processing of electrocardiogram (ECG) data. Experimental research have shown the inconvenience of using ECG sensors, because they have to be placed on the body of the test subject. This circumstance entails some discomfort for the testing person. In this regard, the decision on replacement of the ECG sensor to PPG sensor. The photoplethysmogram sensor clip on to the hand finger of the testing person and provides completeness of information comparable to the ECG data. GSR sensors are clip on two free hand fingers and do not create inconvenience for the testing person.

**Research methods.** A methods of psychological testing has developed with using software and hardware control of the psychophysiological state of the tested person. This circumstance significantly increases the systems objectivity of professional selection of person.

**Research results.** For the system of professional selection two methods were chosen, the first one was the Buss-Durkee test which allows to diagnose the personality aggressivenessand the second was the neuro-psychological stability assessment test. A system of psychological testing in Kazakh and Russian languages has been developed with recording the physiological parameters of the testing person in real time. As sources of physiological data, the data of PPG and GSR has been determined.

**Software and hardware implementation.** On Arduino platform [12-13] has developed a system for receiving and processing data from PPG and GSR sensors. To connect the sensors used chip AD8232 (from AnalogDevices), which is an integrated signal processing unit for ECG and other biopotential tasks [14].

A distinguishing feature of this module is its compact size and external connection to computers, its allows create a mobile diagnostic equipment. The device connects to the computer via a USB port.

The software&hardware complex of psycho-physiological testing allows capture and evaluate the psycho-physiological state of the testing person, when answering each test question. Its provides additional information for the psychologist.

At processing physiological data, the following parameters PPG and GSR are calculated, which are necessary for a mathematical model for assessing the state of the testing person: the minimum and maximum amplitude; mean value of the root mean square deviation of the amplitude. For PPG, the minimum and maximum values of the RR-interval and the minimum and maximum values of the T-peak amplitud, and also the minimum and maximum T-peak offset are additionally calculated.

The PPG sensor is analog device, based on the method of photoplethysmography - the change in the optical density of the blood volume in the finger, due to changes in blood flow through the vessels depending on the phase of the cardiac cycle. The sensor contains a light source (green LED) and a photodetector (figure 1), the voltage at which varies depending on the blood volume during cardiac pulsations.

Figure 1 - Sensor PPG

Data from the PPG sensor is received as a number characterizing the amplitude of the signal, with an intensity (frequency) of 160 samples per second. Denote by $x_i^{(k)}$- the i-th signal of the PPG related to the k-th effect (sample). Figure 2 shows a graph of the variation of the PPG over time. Figure 3 presents a general view of one period of the PPG signal.


Figure 2 - Graph of changes in the signal of the PPG

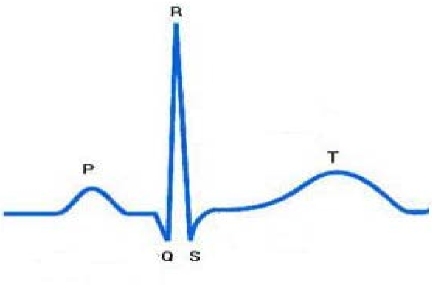


Figure 3 - General view of one period of the PPG signal

$$x_{\text{cp}}^{(k)} = \frac{1}{n_k}\sum_{i=1}^{n_k} x_i^{(k)},$$

$$x_{min}^{(k)} = \min_{i=1,n_k} x_i^{(k)},$$

$$x_{max}^{(k)} = \max_{i=1,n_k} x_i^{(k)} \qquad (1)$$

$$\sigma^{(k)} = \frac{1}{n_k}\sqrt{\sum_{i=1}^{n_k}\left(x_i^{(k)} - x_{\text{cp}}^{(k)}\right)^2},$$

$$z^{(k)} = max\left\{x_{max}^{(k)} - x_{min}^{(k)}, \sigma^{(k)}\right\}$$

For the remaining characteristics calculate following procedures are used. By reason of the signal have a periodic pattern, each separately selected RR intervalallocate. During the processing of background data, an averaged shape of the PPG signal is formed, which is typical of a particular person being tested in a calm environment. For the background averaged RR-interval denote by Yi, i = 1,100. Thus, the vector Y characterizes the shape of the individual background RR interval. In-process of the PPG data, coming in during subsequent impacts (question and answer), RR intervals are allocated respectively. Denote by Zi, i = 1, Lr. Here, Lr denotes the length of the next RR-interval. When processing the vector Z, a T-wave is distinguished, which is characterized by a shift of Lt relative to beginning of the RR interval and amplitude. The area of the RR interval is calculated - S. The value of the shift function F is calculated:

$$S = \int z(t)dt = \sum_{i=1}^{Lr} z_i\,,$$
$$F = \sum_{i=1}^{100}(y_i - z_i)^2 \qquad (2)$$

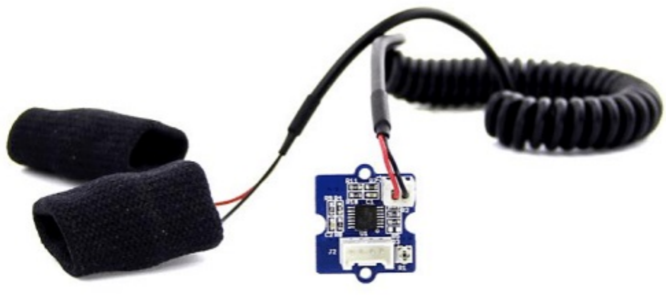

Figure 4 - Sensor GSR

As an additional parameter, the response time for each test question is analyzed.

The GSR sensor allows measuring the galvanic skin response by measuring the electrical conductivity of the skin (figure 4). Skin conduction varies with the amount of sweat on the skin. Sweaty glands are controlled by the sympathetic nervous system, which can be affect by strong emotions. Thus, strong emotions will lead to more sweating on the skin, which will lead to changes in the electrical conductivity of the skin. GSR parameters are calculated by the formulas (1).

**Adaptation of psychological tests. Buss-Durkee test.** Aggression - individual or collective behavior, action aimed at causing physical or psychological harm, damage, or the destruction of another person or group of people. Aggressive behavior in this case is defined as one of the forms of response to various adverse physical and mental life situations that cause stress and frustration. Aggressive actions in aggressive behavior act as a way to achieve any meaningful goal, a method of psychological relaxation, a way to satisfy the need for self-realization and self-affirmation [15].

Considering the concept of “frustration” in the framework of the psychodiagnostic approach and from the point of view of interpersonal relationship, we mean situations in which individual's surrounding persons intentionally or unintentionally infringe upon his interests, which leads to blocking significant needs or hurt feelings, negatively affecting him self-rating.

In a frustrating situation emotional state is manifested:

1) as a reaction of fear, anxiety, refusal of self-realization, may be accompanied by a sense of guilt, a desire to escape from conflict;

2) as offensive, accusing others, active or even aggressive behavior, hostile statements or actions;

3) as the desire to suppress certain reactions, to be passive or indifferent to the incident, to try to balance the severity of the conflict.

For research the level of students aggression the test by American psychologists Buss A. and Durkee A. (developed in 1957 and adapted in 1989 by Soviet and Russian psychologist S. N. Enikolopov [16]) was used.

The test was conducted among students of Kazakhstan universities. 105 people passed the Buss-Durkee test in Russian. The age of the subjects 18-30 years. Of these, 54 men and 51 women. The results obtained on 8 scales:

1) Physical aggression - the use of physical force against another person;

2) Indirect aggression - directed in a roundabout way at another person or directed at no one;

3) Irritation - readiness for the manifestation of negative feelings at the slightest excitement (short temper, rudeness);

4) Negativism - an oppositional behavior from passive resistance to active struggle against established customs and laws;

5) Resentment - envy and hatred to others for real and fictional actions;

6) Suspicion - ranging from distrust and caution towards people to the belief that other people plan and cause harm;

7) Verbal aggression - is the expression of negative feelings both through form (screaming, squeal) and through the content of verbal responses (curses, threats);

8) Guilt - expresses the subject's possible belief that he is a bad person, that evil is being done, as well as the remorse he feels.

The questionnaire consists of 75 statements, to which the subject answers “yes” or “no”.

Testing was conducted on the basis of voluntary consent. The method of Buss-Durkee in processing was very long and difficult. The obtained indices of aggressiveness and hostility can be found in the following tables:

Table 1 – Aggression index

| | Below normal | Norm | Above normal |
|---|---|---|---|
| The number of tested women | 7 | 44 | 0 |
| % ratio | 13,7% | 86,2% | 0% |
| The number of test men | 10 | 38 | 6 |
| % ratio | 18,5% | 70,3% | 11% |

The table shows that, on average, the index of aggressiveness in most subjects is normal, but the aggressiveness that is below the norm is observed in 17 people and mainly in mens. Indicators above the norm were found in 6 men.

Table 2 – Hostility index

| | Below normal | Norm | Nbove normal |
|---|---|---|---|
| The number of tested women | 4 | 35 | 12 |
| % ratio | 7,8% | 68,6% | 23,5% |
| The number of test men | 0 | 31 | 23 |
| % ratio | 0 | 57,4% | 42,5% |

The table shows that a low index of hostility is observed only in 4 womens from the total number of subjects. The norm is inherent in the majority of the subjects: 23.5% of womens and 42.5% of mens were distinguished with an indicator of hostility above the norm.

The hostility index is within the normal range of 3-6 (H.I. is normal - 6-7 ± 3);

The hostility index includes the 5th and 6th scales, Hostility = Offense + Suspicion;

The index of aggression is also in the normal range of 15-20. (The norm of aggressiveness is the value of its index, equal to 21 ± 4).

**Test of neuropsychological stability.** Neuropsychological stability is a features that characterizes a person in the process of a complex activity, some of his emotional mechanisms, closely interacting with each other, lead to the successful achievement of goals.

The primary elements here are: the level of self-rating, emotional stability, social approval of the people around them. In the understanding of stability included the concept of reliability and functionality of reality. The stability of psychological stability depends on the realization of the individual in society, it affects the satisfaction with life, the success of professional activity and the world outlook as a whole. Decrease in neuropsychological stability leads to stressful situations with negative consequences for health and extinction of personality development in the process of life. Among the diversity of factors, there are personality traits and factors related to the social environment.

Factors of neuropsychological stability are:

- environmental factors maintaining self-rating;
- support in self-realization;
- adaptation assistance;
- reliable assistance of the social world, including from friends, relatives, colleagues.

These factors have a positive effect on neuropsychological stability in a person. Their presence forms favorable behavior in the process of professional activity and personal development of the individual.

Psychological stability is a variety of personality qualities and selected aspects of character, which are determined by endurance, poise, resilience. These qualities help to resist a person in the process of life difficulties, unfavourable circumstances, while maintaining health and effectiveness of work [17].

One of the most important criteria for entering the military service is the assessment of the level of neuropsychological stability. Assessment of neuropsychological stability and identification of persons with neuropsychological instability is an important direction in the psychological (psychophysiological) maintenance of conscripts and contract service mans in military units.

A mentally healthy is considered serviceman who is mentally qualified, staid, able to master a military specialty, to be in an organized military collective and undergo increased mental and physical stress without effect their health. There are no people absolutely immune to stress. Everyone has a strictly individual limit of resistance, after which psychoemotional stress, overwork or violation of body functions leads to a breakdown of mental activity.

To determine the "propensity for nervous breakdown in the activity of the nervous system with considerable mental and physical stress" in 1978 Spivakproposed to consider the concept of “neuropsychic instability” (NPI) [18]. Changing the rhythm of life, separation from home and family, daily routine according to military regulations, the need to obey, no privacy, increased responsibility, certain household inconveniences, unusual climatic and geographical conditions, various occupational hazards that accompany one or another type of military labor – all of this places increased demands on the mental and physical health of military personnel. Based on the foregoing, an extremely important role in the practical work of military psychologists and specialists in professional psychological selection is assigned to an assessment of the level of neuropsychological stability of military personnel. Based on the study of servicemen serving on conscription, it was established that healthy - 61%, with some signs of neuro-mental instability - 25%, with pronounced signs of neuro-mental instability - 10%, patients - 4%.

Thus, revealing a high neuropsychic stability, we can talk about the high functional ability of the system of mental adaptation for maintaining stability and high efficiency of mental activity both under ordinary conditions and under the influence of extreme stressful environmental factors. Conversely, unsatisfactory neuropsychic stability and neuropsychic instability indicate a low functional capacity of the mental adaptation system, an increased risk in terms of the development of maladaptive mental disorders not only in extreme, but even in normal conditions of professional activity when its individual parameters change.

The technique developed in S. M. Kirov Military Medical Academy and is intended for the initial selection of persons with signs of neuropsychic instability. It allows to identify individual pre-painful signs of personality disorders, as well as to assess the probability of their evelopment and manifestations in human behavior and activity [19].

The test was conducted among cadets of military institutions and students of civil universities of Almaty. 145 people passed the test "Neuropsychic Stability (NPS)" in Russian. The subjects were 18–20 years old. All test men. The indicator on the scale of NPS is obtained by simply summing up the positive and negative answers that coincide with the “key”.

The data obtained can be found in the following table:

Table 3 – Test Results

| Amount / ratio in% | Low | Average | High |
|---|---|---|---|
| The number of subjects NPS | 52 | 78 | 15 |
| % ratio NPS | 35,8% | 53,7% | 10,3% |

The test “Neuropsychic Stability” is adapted and translated into Kazakh.

When adapting the test “Neuropsychic Stability” to the Kazakh language, a certain algorithm was observed:

1) The validity of the methodology on a sample that yields statistically significant results between test indicators and the validity criterion is verified. The first results were unsatisfactory, since the correlation coefficient of -0.560 and the sample build-up did not improve it, therefore, the criterion was validated and verified by the results, the internal consistency of the test items. With the exclusion of uninformative and socially significant tasks in this situation, the desired validity was found.

2) Reliability by retest method checked. Without information on retest reliability, the test cannot be used to build a psychological forecast.

3) Analysis of correlation with relevant external criteria, with the author's criteria has been performed.

4) Test standards after checking the sustainability of the obtained distribution of test scores were checked.

As a result, it was found that it is personal and biological factors that influence the development of neurotic disorders. It should be noted that individuals, military personnel with signs of neuro-psychic instability require special attention of a psychologist. Mental states can have the opposite effect on personality, its development and dynamics, the formation of some properties and the weakening of others, changes in the structure of motives, goals and activities. Individuals with neuropsychic instability are at risk group. There is a high probability of disadaptive forms of behavior.

**Conclusion.** A software&hardware complex for psychophysiological testing has been developed, which allows recording and assessing the psychophysiological state of the test person when answering each test question. The graphical user interface of the application is implemented in the Kazakh and Russian languages. The techniques of Buss-Durkee and assessment of neuropsychic stability are automated and adapted to the Kazakh language.

It is expected to use a software&hardware complex for obtaining a psychophysiological portrait of a person when hiring in state and private organizations, as well as for service in law enforcement agencies.

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## ПСИХОФИЗИОЛОГИЯЛЫҚ ТЕСТІЛЕУДІҢ ӘМБЕБАП КЕШЕНІ

**Аннотация.** Мақалада психофизиологиялық тестілеудің аппараттық-бағдарламалық кешенін (АБК) қолданудың өзекті мәселесі қарастырылады. Нақты уақытта тестілеудің физиологиялық параметрлерін анықтай отырып, психологиялық тестілеу жүйесінің тәжірибелік нұсқасы жасалды. Физиологиялық деректердің көзі ретінде тері-гальваникалық реакцияның (ТГР) фотоплетизмасының (ФП) мәліметтері анықталды. Arduino платформасында фотоплетизма және тері-гальваникалық реакция сенсорларынан мәліметтерді қабылдау және өңдеу жүйесі жасалды. Фондық деректерді өңдеу кезінде тыныш жағдайда нақты сынақ жүргізушіге тән ФП сигналының ортаңғы формасы қалыптасады. Тақырыптың психофизиологиялық жағдайының өзгеруімен (мысалы, стресс жағдайында) тыныс алу жылдамдайды немесе азаяды (ол RR интервалдарының мөлшерін өзгерту арқылы белгіленеді), R шыңының амплитудасы төмендейді («таралады» сигналы), T шыңының амплитудасы мен позициясы өзгереді. Осы белгілердің барлығы бағдарламалық түрде анықталады және тақырыпты диагностикалауда қолданылады. Психофизиологиялық тестілеудің аппараттық-бағдарламалық кешені тесттің әр сұрағына жауап берген кезде психологқа қосымша ақпарат беретін тестілеуші адамның психофизиологиялық жағдайын тіркеуге және бағалауға мүмкіндік береді. Қойылған міндет психологиялық тестілеу арқылы шешілетін міндеттердің біріне қатысты. Агрессивті мінез-құлықпен агрессивті әрекеттер кез-келген маңызды мақсатқа жетудің тәсілі, психологиялық релаксация, өзін-өзі тану және өзін-өзі растау қажеттілігін қанағаттандыру тәсілі ретінде әрекет етеді. Тұлғалық ерекшеліктер эмоционалды күйзеліс жағдайында айқын көрінеді. Сондықтан психологтар ашушаңдық жағдайындағы адамның реакциясын мұқият зерттейді. Авторлар осы тақырып бойынша талдау жүргізді, эксперименттік тесттің әртүрлі әдістерін, мысалы, нейропсихикалық төзімділікті бағалау әдістемесін қарастырды. Бұл жағдай кәсіби тұлғаны таңдау жүйесінің объективтілігін едәуір арттырады. Тесттің физиологиялық параметрлерін белгілейтін психологиялық тестілеу жүйесінің тәжірибелік нұсқасы жасалды. Эксперименттік сынақтар ретінде жеке басының агрессивтілігін диагностикалауға мүмкіндік беретін Басс-Дарка әдісі және нейропсихикалық төзімділікті бағалау әдістемесі таңдалды. Американдық психологтар А. Басс және А. Дарка сынақтарын қолдана отырып студенттердің агрессиялық деңгейіне зерттеу жасалды. Психологиялық тұрақтылықтың орнықтылығы қоғамдағы жеке тұлғаның жүзеге асырылуына байланысты, ол өмірге қанағаттануға, кәсіби қызметтің жетістіктеріне және тұтастай алғанда дүниетанымға әсер етеді. Нейропсихикалық төзімділіктің төмендеуі денсаулыққа теріс әсер ететін стресстік жағдайларға және өмір процесінде жеке тұлғаның дамуының жойылуына әкеледі. Әр түрлі факторлардың ішінде жеке тұлғалық ерекшеліктер мен әлеуметтік ортаға байланысты факторлар бар. Бұл мәселені шешу үшін жауаптарды жазып, тесттің психофизиологиялық жағдайын бағалайтын аппараттық-бағдарламалық кешенді қолдана отырып психофизиологиялық тестілеуді қолдану ұсынылады. Осылайша, жоғары нейропсихикалық тұрақтылықты анықтай отырып, біз психикалық бейімделу жүйесінің қалыпты жағдайдағы сияқты, сонымен қатар экстремалды факторлардың әсерінен де тұрақтылықты және ақыл-ой әрекетінің жоғары тиімділігін сақтай алатын жоғары функционалды қабілеті туралы да айта аламыз. Қосымша параметр ретінде әр тест сұрақтарына жауап беру уақыты талданады. Қосымшаның графиктік интерфейсі қазақ және орыс тілдерінде жүзеге асырылады.

Берілген мысал психофизиологиялық тестілеудің аппараттық-бағдарламалық кешенін қолдану тесттің әр сұрағына жауап бергенде нақты уақыт режимінде тестілеуші адамның физиологиялық параметрлерін тіркеуге және бағалауға мүмкіндік беретіндігін көрсетеді. Жоғарыда аталған жағдайлардың әдіснамасын қолдана отырып, кәсіпқой жеке тұлғаны таңдаудың объективті жүйесін құру ұсынылады. Бұл сынақтар әскери қызметке психологиялық таңдау кезінде пайдалануға ұсынылады [1-2]. Тесттер қазақ тіліне бейімделіп, Алматы қаласындағы әскери оқу орындарының курсанттары мен студенттері арасында сынақтан өткізілді. Ұсынылған аппараттық-бағдарламалық кешен негізінде жасалған алгоритмдермен әзірленген бағдарламалық жасақтама тестілеуші адамның психофизиологиялық жағдайын бағалау үшін инвариантты екенін көрсетті.

**Түйін сөздер:** электрокардиограмма, фотоплетизм, тері-гальваникалық реакция, микропроцессор, сигналды өңдеу, психологиялық тесттер, Басс-Дарки әдісі, жеке тұлға, өзін-өзі бағалау деңгейі, нейропсихикалық тұрақтылық, интеллектуалды реттеу, психикалық бейімделу жүйелері, жарамдылық критерийі, ретест әдісі, зиянды мінез-құлық, аппараттық-бағдарламалық кешен, графиктік интерфейс.

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## УНИВЕРСАЛЬНЫЙ КОМПЛЕКС ПСИХОФИЗИОЛОГИЧЕСКОГО ТЕСТИРОВАНИЯ

**Аннотация.** В статье рассматривается актуальная проблема применения программно-аппаратного комплекса (ПАК) психофизиологического тестирования. Разработан экспериментальный вариант системы психологического тестирования с фиксированием физиологических параметров тестируемого в реальном времени. В качестве источников физиологических данных определены данные фотоплетизмограммы (ФП) кожно-гальванической реакции (КГР). На платформе Arduino разработана система приема и обработки данных с датчиков фотоплетизмограммы и кожно-гальванической реакции. Во время обработки фоновых данных формируется усредненная форма сигнала ФП, свойственная конкретному тестируемому в спокойной обстановке. При изменении психофизиологического состояния исследуемого (например, при стрессе) учащается или становится реже дыхание (что фиксируется изменением размаха RR-интервалов), уменьшается амплитуда R-пика (сигнал “размазывается”), изменяется амплитуда и положение T-пика. Все перечисленные признаки определяются программно и используются при диагностике исследуемого. Программно-аппаратный комплекс психофизиологического тестирования позволяет при ответе на каждый вопрос теста фиксировать и оценивать психофизиологическое состояние тестируемого, что дает дополнительную информацию для психолога. Поставленная задача относится к одной из задач, решаемых при помощи психологического тестирования. Агрессивные действия при агрессивном поведении выступают как способ достижения какой-либо значимой цели, способ психологической разрядки, способ удовлетворения потребности в самореализации и самоутверждении. Особенности личности проявляются ярче в состоянии эмоционального напряжения. Поэтому психологи внимательно изучают реакции индивида в ситуации фрустрации. Авторам проведен анализ по данной тематике, рассмотрены различные методы экспериментального теста, например, методика оценки нервно-психической устойчивости. Данное обстоятельство существенно повышает объективность системы профессионального отбора личности. Разработан экспериментальный вариант системы психологического тестирования с фиксированием физиологических параметров тестируемого. В качестве экспериментальных тестов выбраны методика Басса-Дарки, позволяющая диагностировать агрессивность личности, и методика оценки нервно-психической устойчивости. Проведено исследование уровня агрессии студентов с использованием теста американских психологов А. Басса и А.Дарки. Предложено решение этой проблемы с использованием ПАК психофизиологического тестирования, которое фиксирует ответы и оценивает психофизиологическое состояние тестируемого. Стабильность психологической устойчивости зависит от реализации личности в социуме, она оказывает влияние на удовлетворение жизнью, на успешность профессиональной деятельности и мировоззрение в целом. Снижение нервно-психической устойчивости ведет к появлению стрессовых ситуаций с отрицательными последствиями для здоровья и угасанию развития личности в процессе жизни. Среди разнообразия факторов существуют личностные особенности и факторы, связанные с социальной средой. Так, выявляя высокую нервно-психическую устойчивость, можно говорить о высокой функциональной способности системы психической адаптации по сохранению устойчивости и высокой эффективности психической деятельности как в обычных условиях, так и в условиях воздействия экстремальных стрессовых факторов внешней среды. В качестве дополнительного параметра анализируется время ответа на каждый вопрос теста. Графический интерфейс пользователя приложения реализован на казахском и русском языках.

Приведенный пример показывает, что использование программно-аппаратного комплекса психофизиологического тестирования позволяет при ответе на каждый вопрос теста фиксировать и оценивать физиологические параметры тестируемого в реальном режиме времени. Предложено решение этой проблемы с использованием методики вышеизложенных обстоятельств, которые диктуют необходимость в создании объективной системы профессионального отбора личности. Указанные тесты рекомендуются для применения при психологическом отборе на военную службу [1-2]. Тесты адаптированы на казахский язык и апробированы на курсантах военных заведений и студентах гражданских вузов г. Алматы. Предложенные автором алгоритмы, созданные на основе предложенного ПАК, показали, что разработанная программная система обладает инвариантностью к оцениванию психофизиологического состояния тестируемого.

**Ключевые слова:** электрокардиограмма, фотоплетизмограмма, кожно-гальваническая реакция, микропроцессор, обработка сигналов, психологические тесты, методика Басса-Дарки, личность, уровень самооценки, нервно-психическая устойчивость, интеллектуальная регуляция, системы психической адаптации, критерий валидности, метод ретеста, дезадаптивная форма поведения, программно-аппаратного комплекс, графический интерфейс.

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# ON THE DYNAMICS OF THREE AXISYMMETRIC BODIES

**Abstract.** It explores the translational and rotational movement of the three free non-stationary axisymmetric celestial bodies with variable mass, size and variable compression interacting according to Newton's law. Newtonian force interaction is characterized by an approximate expression of the force function, which takes into account the second harmonic. Differential equations of translational-rotational motion of three non-stationary axisymmetric bodies with variable mass and size in the relative coordinate system, with the beginning in the center of a more massive body, are given. The axes of inertia of own coordinate system of non-stationary axisymmetric three bodies coincide with the main axes of inertia of the bodies, and it is assumed that their relative orientation remains unchanged during evolution. The mass of bodies are varied isotropically in the different rates. Canonical equations of translational-rotational motion of three non-stationary axisymmetric bodies with variable masses and sizes are obtained in the osculating analogues of the elements of Delaunay-Andoyer. Canonical equations of unperturbed motion and their integrals are given.

**Keywords:** Translational-rotational movement, Variable mass, Three-body problem, Axisymmetric celestial body, Osculating elements, Delaunay-Andoyer elements.

## 1. Introduction

In classical celestial mechanics, real celestial bodies are modeled by a material point (a spherically symmetric body). In cases when such a description of physical phenomena inadequately reflects the essence of a real celestial-mechanical problem, celestial bodies are modeled by a solid body of constant size, mass and unchanged structure [1, 2]. Observational astronomy shows that real celestial bodies are non-point and unsteady. Celestial bodies are non-stationary, in the process of evolution their masses, sizes, shapes and structures will change [1, 3]. In this connection, the creation of mathematical models of the motion of celestial bodies with variable masses, sizes, and shapes becomes relevant.

The purpose of this work is to obtain differential equations of translational-rotational motion of non-stationary three axisymmetric bodies with variable masses, sizes and variable compression in osculating elements based on the equations of motion obtained in our previous work [3]

## 2. Equations of motion in a relative coordinate

Under certain assumptions for the physical problem, the equations of translational-rotational motion of three axisymmetric bodies in the relative coordinate system were obtained in our work [3]. The beginning of the relative coordinate system $G_0xyz$ coincides with the barycenter of the body $T_i$, and the coordinate axes are parallel to the respective axes of coordinate of the absolute coordinate system (see figure 1).

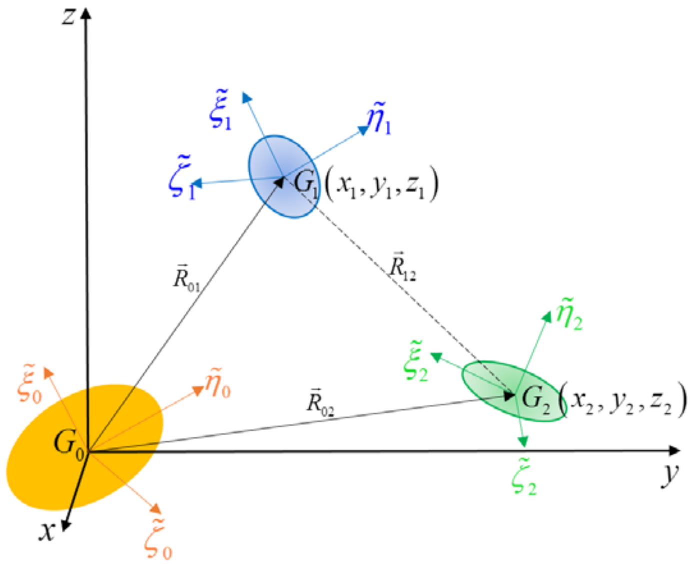


Figure 1 - Bodies in a relative coordinate system $G_0xyz$.

$G_i\tilde{\xi}_i\tilde{\eta}_i\tilde{\zeta}_i$ – own coordinate systems

**2.1. Rotational motion**

Differential equations of the rotational motion of bodies around their own center of mass in the relative coordinate system are written as follows

$$\begin{aligned}
&\frac{d}{dt}(A_i p_i)-(A_i-C_i)q_i r_i=\left[\frac{\partial U}{\partial \psi_i}-cos\theta_i\frac{\partial U}{\partial \varphi_i}\right]\frac{sin\varphi_i}{sin\theta_i}+cos\varphi_i\frac{\partial U}{\partial \theta_i},\\
&\frac{d}{dt}(A_i q_i)-(C_i-A_i)p_i r_i=\left[\frac{\partial U}{\partial \psi_i}-cos\theta_i\frac{\partial U}{\partial \varphi_i}\right]\frac{cos\varphi_i}{sin\theta_i}-sin\varphi_i\frac{\partial U}{\partial \theta_i},\\
&\frac{d}{dt}(C_i r_i)=\frac{\partial U}{\partial \varphi_i}=0. \qquad\qquad i=0,1,2
\end{aligned} \tag{2.1}$$

where $p_i=\dot{\psi}_i\sin\theta_i\sin\varphi_i+\dot{\theta}_i\cos\varphi_i,\ q_i=\dot{\psi}_i\sin\theta_i\cos\varphi_i-\dot{\theta}_i\sin\varphi_i,\ r_i=\dot{\psi}_i\cos\theta_i+\varphi_i.$ (2.2)

$p_i, q_i, r_i$ – the projections of the angular velocity of the rotational motion of the bodies $T_i$ on the axes of its own coordinate system $G_i\tilde{\xi}_i\tilde{\eta}_i\tilde{\zeta}_i$, $\varphi_i, \psi_i, \theta_i$ - Euler angles.

In General, the Newtonian force function of the problem of three non-stationary bodies has the form [1-3].

$$U=U_{01}+U_{12}+U_{02} \tag{2.3}$$

where

$$U_{01}=f\int\limits_{(T_0)}\int\limits_{(T_1)}\frac{dm_0dm_1}{R_{01}},\ U_{12}=f\int\limits_{(T_1)}\int\limits_{(T_2)}\frac{dm_1dm_2}{R_{12}},\ U_{02}=f\int\limits_{(T_0)}\int\limits_{(T_2)}\frac{dm_0dm_2}{R_{02}} \tag{2.4}$$

$$R_{ij}=R_{ji}=\sqrt{(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2}\quad i,j=0,1,2,\quad i\neq j \tag{2.5}$$

there is a mutual distance between the centers of inertia $G_i$ and $G_j$ of the bodies $T_i$ and $T_j$, and $f$ is the gravitational constant.

$U_{ij}$ – the force function of the mutual attraction of two bodies $T_i$ and $T_j$ is defined as follows.

$$U_{ij} = U_{ij}^{(0)} + U_{ij}^{(2)} \tag{2.6}$$

Here

$$U_{ij}^{(0)} = f\frac{m_i m_j}{R_{ij}},\ U_{ij}^{(2)} = fm_i\frac{A_j + B_j + C_j - 3I_j^{(i,j)}}{2R_{ij}^{\ 3}} + fm_j\frac{A_i + B_i + C_i - 3I_i^{(i,j)}}{2R_{ij}^{\ 3}} \tag{2.7}$$

$$I_i^{(i,j)} = A_i\alpha_{ij}^2 + B_i\beta_{ij}^2 + C_i\gamma_{ij}^2 \qquad I_j^{(i,j)} = A_j\alpha_{ji}^2 + B_j\beta_{ji}^2 + C_j\gamma_{ji}^2 \tag{2.8}$$

$I_i^{(i,j)}$ and $I_j^{(i,j)}$ - moments of inertia of bodies $T_i$ and $T_j$ relatively straight $R_{ij}$ – connecting the centers of mass of two bodies $G_iG_j$, respectively. $i, j = 0,1,2, \quad i \neq j$.

### 2.2. Translational motion

We will not consider the equations of translational motion of body $T_0$, since the beginning of the relative coordinate system coincides with the barycenter of body $T_0$, so we will only consider its rotational motion.

The equations of translational motion of the body $T_1$ in the field of gravity of the "Central" body $T_0$ in the presence of disturbances from the body $T_2$, in the relative coordinate system is written as follows [1-3]

$$\ddot{x}_1 = \frac{1}{\mu_1(t)}\frac{\partial U_{10}}{\partial x_1} + \frac{\partial V_1^*}{\partial x_1}\quad \ddot{y}_1 = \frac{1}{\mu_1(t)}\frac{\partial U_{10}}{\partial y_1} + \frac{\partial V_1^*}{\partial y_1}\quad \ddot{z}_1 = \frac{1}{\mu_1(t)}\frac{\partial U_{10}}{\partial z_1} + \frac{\partial V_1^*}{\partial z_1} \tag{2.9}$$

where

$$\mu_1(t) = \frac{m_0 m_1}{m_0 + m_1} \text{ – reduced mass,} \tag{2.10}$$

$$V_1^* = \frac{1}{m_1}U_{12} + \frac{1}{m_0}\left[x_1\frac{\partial U_{20}}{\partial x_2} + y_1\frac{\partial U_{20}}{\partial y_2} + z_1\frac{\partial U_{20}}{\partial z_2}\right] \text{ – perturbance from body } T_2 \tag{2.11}$$

The equations of translational motion of the body $T_2$ in the field of gravity of the "Central" body $T_0$ in the presence of disturbances from the body $T_1$, in the relative coordinate system is written as follows [1-3]

$$\ddot{x}_2 = \frac{1}{\mu_2(t)}\frac{\partial U_{20}}{\partial x_2} + \frac{\partial V_2^*}{\partial x_2},\ \ddot{y}_2 = \frac{1}{\mu_2(t)}\frac{\partial U_{20}}{\partial y_2} + \frac{\partial V_2^*}{\partial y_2},\ \ddot{z}_2 = \frac{1}{\mu_2(t)}\frac{\partial U_{20}}{\partial z_2} + \frac{\partial V_2^*}{\partial z_2} \tag{2.12}$$

where

$$\mu_2(t) = \frac{m_0 m_2}{m_0 + m_2} \text{ – reduced mass,} \tag{2.13}$$

$$V_2^* = \frac{1}{m_2}U_{21} + \frac{1}{m_0}\left[x_2\frac{\partial U_{10}}{\partial x_1} + y_2\frac{\partial U_{10}}{\partial y_1} + z_2\frac{\partial U_{10}}{\partial z_1}\right] \text{ – perturbance from body } T_1 \tag{2.14}$$

For our purposes, it is preferable to use canonical equations of perturbed motion in the osculating analogues of the elements of Delaunay-Andoyer [1].

**3. Equations of motion in the osculating elements delaunay-andoyer**

The translational motion of the center of mass of axisymmetric bodies $T_1$ and $T_2$ is further described in the osculating elements. Rewrite the equations of translational motion of the body $T_1$ (2.9) as

$$\ddot{x}_1 + f\frac{m_0+m_1}{R_{10}^{\;3}}x_1 - b_1x_1 = \frac{\partial V_1}{\partial x_1},$$
$$\ddot{y}_1 + f\frac{m_0+m_1}{R_{10}^{\;3}}y_1 - b_1y_1 = \frac{\partial V_1}{\partial y_1}, \qquad (3.1)$$
$$\ddot{z}_1 + f\frac{m_0+m_1}{R_{10}^{\;3}}z_1 - b_1z_1 = \frac{\partial V_1}{\partial z_1}.$$

where $V_1 = V_1^* + \frac{1}{\mu_1}U_{10}^{(2)} - \frac{1}{2}b_1R_{10}^2$ – force function of the perturbing force (3.2)

$$b_1 = b_1(t) = \frac{\ddot{\nu}_1}{\nu_1} = (m_0+m_1)\frac{d^2}{dt^2}\left(\frac{1}{m_0+m_1}\right),\; \nu_1 = \frac{m_0(t_0)+m_1(t_0)}{m_0(t)+m_1(t)} \qquad (3.3)$$

We will write the equations of translational motion of the body $T_2$ in the following form

$$\ddot{x}_2 + f\frac{m_0+m_2}{R_{20}^{\;3}}x_2 - b_2x_2 = \frac{\partial V_2}{\partial x_2},$$
$$\ddot{y}_2 + f\frac{m_0+m_2}{R_{20}^{\;3}}y_2 - b_2y_2 = \frac{\partial V_2}{\partial y_2}, \qquad (3.4)$$
$$\ddot{z}_2 + f\frac{m_0+m_2}{R_{20}^{\;3}}z_2 - b_2z_2 = \frac{\partial V_2}{\partial z_2}.$$

Where $V_2 = V_2^* + \frac{1}{\mu_2}U_{20}^{(2)} - \frac{1}{2}b_2R_{20}^2$ – force function of the perturbing force (3.5)

$$b_2 = b_2(t) = \frac{\ddot{\nu}_2}{\nu_2} = (m_0+m_2)\frac{d^2}{dt^2}\left(\frac{1}{m_0+m_2}\right),\; \nu_2 = \frac{m_0(t_0)+m_2(t_0)}{m_0(t)+m_2(t)} \qquad (3.6)$$

And the equation of rotational motion remains unchanged

$$\frac{d}{dt}(A_jp_j) - (A_j - C_j)q_jr_j = \left[\frac{\partial U}{\partial \psi_j} - cos\theta_j\frac{\partial U}{\partial \varphi_j}\right]\frac{sin\varphi_j}{sin\theta_j} + cos\varphi_j\frac{\partial U}{\partial \theta_j},$$
$$\frac{d}{dt}(A_jq_j) - (C_j - A_j)p_jr_j = \left[\frac{\partial U}{\partial \psi_j} - cos\theta_j\frac{\partial U}{\partial \varphi_j}\right]\frac{cos\varphi_j}{sin\theta_j} - sin\varphi_j\frac{\partial U}{\partial \theta_j}, \qquad (3.7)$$
$$\frac{d}{dt}(C_jr_j) = \frac{\partial U}{\partial \varphi_j} = 0, \qquad j = 0,1,2$$

**3.1 Equations of translational-rotational motion in osculating analogues of the Delaunay-Andoyer elements.**

Consider the analogues of the Delaunay-Andoyer elements

$$L_i,\quad G_i,\quad H_i,\quad l_i,\quad g_i,\quad h_i \quad \text{– Delaunay elements} \qquad (3.8)$$

$$L'_i, \quad G'_i, \quad H'_i, \quad l'_i, \quad g'_i, \quad h'_i \text{ – Andoyer elements} \tag{3.9}$$

The equations of motion in the osculating elements (3.8), (3.9) have the form

$$\dot{L}_i = \frac{\partial F_i}{\partial l_i}, \quad \dot{G}_i = \frac{\partial F_i}{\partial g_i}, \quad \dot{H}_i = \frac{\partial F_i}{\partial h_i}, \quad \dot{l}_i = -\frac{\partial F_i}{\partial L_i}, \quad \dot{g}_i = -\frac{\partial F_i}{\partial G_i}, \quad \dot{h}_i = -\frac{\partial F_i}{\partial H_i} \tag{3.10}$$

$$F_i = \frac{1}{\nu_i^2}\frac{\mu_0^2}{2L_i^2} - H_{1i}^{trans} \ (3.11) \quad H_{1i}^{trans} = -\left(\frac{m_0+m_i}{m_0 m_i}U^{(2)} - \frac{1}{2}b_i R_{0i}{}^2\right), \ i=1,2 \ (3.12)$$

$$\dot{L}'_j = \frac{\partial F'_j}{\partial l'_j}, \quad \dot{G}'_j = \frac{\partial F'_j}{\partial g'_j}, \quad \dot{H}'_j = \frac{\partial F'_j}{\partial h'_j}, \quad \dot{l}'_j = -\frac{\partial F'_j}{\partial L'_j}, \quad \dot{g}'_j = -\frac{\partial F'_j}{\partial G'_j}, \quad \dot{h}'_j = -\frac{\partial F'_j}{\partial H'_j} \tag{3.13}$$

$$F'_j = \frac{1}{2}\left(G'^2_j - L'^2_j\right)\frac{1}{A_j} + \frac{L'^2_j}{2C_j} = \frac{1}{2}\frac{G'^2_j}{A_j} + \frac{1}{2}\left(\frac{1}{C_j} - \frac{1}{A_j}\right)L'^2_j - H_{1j}^{rot} \tag{3.14}$$

$$H_{1j}^{rot} = -\left(U^{(2)} - \frac{1}{2}b_j R_{0j}{}^2\right), \ j = 0,1,2 \tag{3.15}$$

Note that the perturbing functions (3.12), (3.15) must be expressed in terms of the osculating elements (3.8), (3.9). These procedures are time-consuming and cumbersome analytical calculations. To do this, we use the system of symbolic calculations MATHEMATICA [10].

**3.2 Equations of unperturbed translational-rotational motion in osculating analogues of the Delaunay-Andoyer elements.**

**3.2.1 Unperturbed translational motion**.

If $H_{1i}^{trans} = 0,$ in the Hamiltonian (4.5) then

$$F_i = F_i^{unpert} = \frac{1}{\nu_i^2(t)} \cdot \frac{\mu_0^2}{2L_i^2}. \tag{3.16}$$

Equations of unperturbed translational motion of the center of inertia of bodies $T_1$ and $T_2$ in analogues of Delaunay elements (3.8) have the form

$$\dot{l}_i = -\frac{\partial F_i^{unpert}}{\partial L_i}, \quad \dot{g}_i = 0, \quad \dot{h}_i = 0, \quad \dot{L}_i = 0, \quad \dot{G}_i = 0, \quad \dot{H}_i = 0, \quad i = 1,2 \tag{3.17}$$

Integrals of the system (3.17) can be written as follows

$$L_i = L_0 = const, \quad G_i = G_0 = const, \quad H_i = H_0 = const, \tag{3.18}$$

$$l_i = \frac{\mu_0^2}{L_i^3}\int_{t_0}^{t}\frac{dt}{\nu_i^2(t)} + l_0, \quad l_0 = const, \quad g_i = g_0 = const, \quad h_i = h_0 = const, \tag{3.19}$$

**3.2.2 Unperturbed rotational motion.**

If $H_{1j}^{rot} = 0$ in the Hamiltonian (3.15), then

$$F'_j = \frac{1}{2}\left(G'^2_j - L'^2_j\right)\frac{1}{A_j} + \frac{L'^2_j}{2C_j} = \frac{1}{2}\frac{G'^2_j}{A_j} + \frac{1}{2}\left(\frac{1}{C_j} - \frac{1}{A_j}\right)L'^2_j \tag{3.20}$$

Equation of unperturbed rotational motion in the analogues of Andoyer variables (3.9) has the form

$$\dot{L}'_j = 0, \quad \dot{G}'_j = 0, \quad \dot{H}'_j = 0, \quad \dot{l}'_j = -\frac{\partial F'_j}{\partial L'_j}, \quad \dot{g}'_j = -\frac{\partial F'_j}{\partial G'_j}, \quad \dot{h}'_j = 0, \qquad (3.21)$$

From equation (3.9) it follows

$$L'_j = L'_0 = const, \quad G'_j = G'_0 = const, \quad H'_j = H'_0 = const, \qquad (3.22)$$

$$l'_j = L'_0 \int_{t_0}^{t} \frac{A_j(t) - C_j(t)}{A_j(t) C_j(t)} dt + l'_0, \quad l'_0 = const, \quad g'_j = G'_0 \int \frac{dt}{A_j(t)} + g'_0, \quad g'_0 = const, \quad h'_j = h'_0 = const, \quad (3.23)$$

The geometric meaning of the analogues of Andoyer variables is given in [1, 4].

**4. Conclusion**

The article deals with the translational-rotational motion of non-stationary bodies that are interacting according to Newton's law. Based on the equation in the relative coordinate system with the beginning at the center of inertia of the most massive body, the equations of translational-rotational motion of three non-stationary axisymmetric bodies in the osculating analogues of the Delaunay-Andoyer elements are obtained. In the future, it is planned to Express the perturbing function through the osculating Delaunay-Andoyer variables and numerical analysis of the obtained equations.

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## ӨСТІК СИММЕТРИЯЛЫ ҮШ ДЕНЕНІҢ ДИНАМИКАСЫНА

**Аннотация.** Классикалық аспан механикасында табиғи аспан денелері нүкте ретінде қарастырылады (сфералық симметриялы дене). Егер мұндай модель физикалық құбылыстың қасиеттерін нақты сипаттап бере алмаған жағдайда табиғи аспан денелерін өлшемдері мен массалары тұрақты және құрылымы өзгермейтін қатты дене ретінде модельдейді. Аспан денелерінің нүкте (шар) де емес, қатты дене де емес екеніне бүгінде бақылаушы астрономия куәлік етеді. Табиғи аспан денелері негізінен бейстационар. Уақыт өте олардың массалары, өлшемдері және массаның таралу құрылымдары өзгереді. Сәйкесінше, олардың гравитациялық тартылыс күші және өзара Ньютондық әсерлесу күші уақытқа тәуелді болады. Бұл факторлар дененің динамикалық эволюциясына айтарлықтай әсер етеді. Ең көп тараған бейстационарлық-гравитациялық денелер массаларының және өлшемдерінің өзгермелілігі. Жоғарыда сипатталған физикалық жүйелердің ілгерілемелі-айналмалы қозғалысын зерттеу қазіргі заманғы теориялық және аспан механикасының маңызды есебі болып табылады. Мысалға, айнымалы жұлдыздардың радиусының периодты өзгеруі осыған дәлел. Әр түрлі жұлдыздардың өсу этапында олардың қысылуы мен кеңеюі жұлдыздың ішкі эволюция теориясынан шығады. Мысалы, жұлдыздың өлшемі қызыл гигант стадиясына өткен кезде, массасына тәуелді 10-нан 100-ге дейін өзгереді. Әсіресе диссипация процессінің интенсиві және массаның ауысуы тығыз екі жүйеде болады. Жұлдыздардың шоғырлануы, газдық фондағы эволюциялануы пульсация жағдайында болады. Соған қатысты олардың тартылыс байланыстары айнымалы болады, Ньютондық өзара потенциалды байланысы тікелей уақыттан тәуелді болады. Осы факторлар олардың динамикалық эволюциясына әсер етеді. Жұлдыздардың эволюциясын сипаттау үшін массаның және радиусының өзгеруіне жұлдыздың реакциясын беретін, сипаттама функциясы енгізіледі. Жұлдыздың массасы өзгерген кездегі, оның өлшемінің өзгеруі зерттеледі. Масса, өлшем және пішіннің өзгеруі физикалық айнымалы жұлдыздар – пульсацияланатын жұлдыздарда айқын көрінеді.

Классикалық үш дене мәселесі - аспан механикасының өте қызықты, күрделі және өзекті мәселесі, ғалымдар бұл мәселенің жалпы шешімін әлі күнге дейін таба алмай келеді. Егер осы мәселеде біз денелердің массалары айнымалы екенін ескеретін болсақ, бұл есепті қиындатады. Осыған байланысты айнымалы массалы үш дене мәселесінің жалпы және дербес бірде-бір нақты шешімі жоқ.

Гравитациялаушы дененің массасы мен өлшемінің өзгеру салдары бейстационар нақты ғарыштық жүйе динамикалық эволюциясының негізгі факторларының бірі болып табылады. Осы жұмыста массасы, өлшемі және пішіні айнымалы үш дене мәселесінің айнымалы-ілгерлемелі қозғалысы қарастырылған. Бұл жұмыстың

негізгі мақсаты біздің алдыңғы жұмысымызда алған қозғалыс теңдеулерінің негізінде үш өстік симметриялы, массалары мен өлшемедері айнымалы, айнымалы сығылатын денелердің ілгерілемелі-айналмалы қозғалысының дифференциалдық теңдеулерін оскуляциялаушы элементтерде алу.

Мақалада өзара гравитацияланушы бейстационар үш дене қарастырылады: бірінші дене – «центрлік», яғни басқа екі денеге қарағанда үлкенірек. Үш дененің де динамикалық құрылымы және пішіні өстік симметриялы. Ньютонның өзара әсерлесу күші екінші гармониканы ескергендегі күштік функцияның жуық өрнегімен сипатталған. Массасы және өлшемі айнымалы өстік симметриялы денелердің ілгерілемелі-айналмалы қозғалысының дифференциалдық теңдеулері салыстырмалы координаталар жүйесінде өткен жұмысымызда қорытылып шығарылған болатын. Бейстационар үш дене үшін меншікті координаталар жүйесінің өстері дененің бас инерция өстерімен сәйкес келеді және бұл күй эволюция барысында өзгеріссіз қалады. Денелердің массалары әртүрлі қарқында изотропты өзгереді. Есепте ұйытқу теориясының тәсілдері пайдаланылған. Оскуляциялаушы Делоне-Андуайе элементтерінің аналогтарында серіктің ілгерілемелі-айналмалы қозғалысының теңдеулері алынды. Ұйытқымаған қозғалыстың конондық теңдеулерінің интегралдары келтірілді.

**Түйін сөздер**: өстік симметриялы дене, ілгерілемелі-айналмалы қозғалыс, айнымалы масса, үш дене есебі, оскуляциялаушы элементтер.

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## К ДИНАМИКЕ ТРЕХ ОСЕСИММЕТРИЧНЫХ ТЕЛ

**Аннотация.** В классической небесной механике реальные небесные тела моделируются материальной точкой (сферически симметричного тела). В случаях, когда такое описание физических явлений неадекватно отражает суть реальной небесно-механической проблемы, небесные тела моделируются твердым телом постоянного размера, массы и неизменной структурой. Наблюдательная астрономия свидетельствует, что реальные небесные тела неточечные и нетвердые. Реальные космические тела по существу нестационарные. Со временем меняются их массы, размеры, формы и структура распределения массы внутри тел. Соответственно, становится переменной их гравитирующая связь и ньютоновский потенциал взаимодействия оказывается явно зависящим от времени. Эти факторы существенно влияют на динамическую эволюцию тел. На некоторых этапах эволюции гравитирующих систем эффекты нестационарности тел, входящих в систему в конце этого этапа. Наиболее часто распространенная нестационарность – переменность масс гравитирующих тел. Исследование поступательно-вращательного движения выше описанных физических систем является актуальной задачей современной теоретической и небесной механики. В связи с этим становится актуальным создание математических моделей движения небесных тел с переменными массами, размерами, и формами. Целью настоящей работы является на основе уравнения движения, полученные в предыдущей нашей работе вывести дифференциальные уравнения поступательно-вращательного движения нестационарных трех осесимметричных тел с переменными массами, размерами и переменного сжатия в оскулирующих элементах.

Исследуется поступательно-вращательное движение трех свободных нестационарных осесимметричных небесных тел с переменными массами, размерами и переменного сжатия взаимодействующих по закону Ньютона, из которых никакие два не имеют общие части. Ньютоновская сила взаимодействия характеризуется приближенным выражением силовой функции, учитывающая вторую гармонику. Пусть эллипсоид инерции тел $T_0, T_1, T_2$ различные, осесимметричные и имеют собственную экваториальную плоскость симметрии и в ходе эволюции эти свойства сохраняются. Так же допустим, что сжатия тел относительно экваториальной плоскости переменные. Исходные расположения главных осей инерции и центр инерции в теле осесимметричных тел в ходе эволюции остаются неизменными и направлены вдоль линии пересечения трех взаимоперпендикулярных плоскостей.

Приведены дифференциальные уравнения поступательно-вращательного движения трех нестационарных осесимметричных тел с переменными массами и размерами в относительной системе координат, с началом в центре более массивного тела. Оси инерции собственной системы координат нестационарных осесимметричных трех тел совпадают с главными осями инерции тел и предполагается, что в ходе эволюции их относительная ориентация остаются неизменными. Массы тел изменяются изотропно в различных темпах. Получены канонические уравнения поступательно-вращательного движения трех нестационарных осесимметричных тел с переменными массами и размерами в аналогах оскулирующих элементов Делоне-Андуайе. Приведены канонические уравнения невозмущенного движения, и их интегралы.

**Ключевые слова:** поступательно-вращательное движение, переменная масса, задача трех тел, осесимметричные небесные тела, оскулирующие элементы.

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# BOUNDEDNESS OF THE HILBERT TRANSFORM FROM ONE ORLICZ SPACE TO ANOTHER

**Abstract.** In this paper, we investigate boundedness of the Hilbert transform from one Orlicz space to another. D.W. Boyd gave examples (see [1]) showing that reflexivity of $X$ is both unnecessary and insufficient for boundedness of the Hilbert transform from $X$ to itself. This may be somewhat surprising, since the condition for $L_p$ to be reflexive $(1 < p < \infty)$ is the same as the condition for the Hilbert transform to be bounded from $L_p$ to $L_p$. However, D.W.Boyd considered only the cases when the domain and the range of the Hilbert transform coincide. Since $L_p$ spaces are examples of rearrangement-invariant (r.i.) Banach function spaces, we consider boundedness of the Hilbert transform from one reaarangement-ivariant Banach function space to another. To be precise, we generalise the Boyd's results allowing the domain of the Hilbert transorm to be a particular Orlicz space defined on (0,1), and the range is different from the domain an other Orlicz space defined on (0,1). Moreover, we also consider boundedness of the Hilbert transform from one Lorentz space on (0,1) (which is also rearrangement invariant) to another Lorentz space (0,1). In case when the domain of the Hilbert transform is a Lorentz space $\Lambda_{\phi,p}(\mathbb{R}_+)$ which coincides with its range, the problems was fully resolved by D.W.Boyd. He showed (see [1]) that uniform convexity of $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ is necessary and sufficient condition for boundedness of the Hilbert transform. However, for the most important case when $p = 1$ the result was proved recently [2, Theorem 4.2]. Moreover, applying the main theorem D.W.Boyd obtained the following [1]:

*Let $L_\phi$ be an Orlicz space. Then $\mathcal{H} \in B(L_\phi, L_\phi)$ if and only if $L_\phi$ os reflexive.*

Towards these goals we also investigate boundedness of the Calderón operator from one rearrangement-invariant Banach function space to another. Such questions have been attracting a great deal of attention for many years, in particular in connection with embeddings of Sobolev spaces. In the present paper we discuss such boundedness problems for classical operators of great interest in analysis and its applications, namely the Hilbert transform and the Calderón operator. The action of these operators on specific classes of function spaces has been extensively studied over the several decades. Classical results are available for example in connection with familiar function spaces. Besides the importance of these operators is very well known, and their properties have been deeply studied.Classical Lorentz spaces which originated in 1950s and have been occurring occasionally later became extremely fashionable in 1990s when the fundamental papers appeared.

In this paper we study the boundedeness of such classical operators on rearrangement-invariant spaces, a class of function spaces that includes for example all Lebesgue, Lorentz, Orlicz, Marcinkiewicz spaces and more. Our focus is mainly on boundedness of the Hilbert transform from one Orlicz space to another. We also give examples of particular rearrangement-invariant spaces on which the Hilbert transform acts boundedly.

**Key words:** Hilbert transform, Calderón operator, Rearrangement invariant Banach functon spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces.

## 1 Introduction

The purpose of this paper is to determine Orlicz spaces such that the Hilbert transform defines a bounded linear operator from one Orlicz space to another. Besides the Orlicz space, we deal with other spaces having the property of rearrangement-invariance.

Let $\mathcal{H}$ be the classical (singular) Hilbert transform (for measurable functions on $\mathbb{R}$), given by the formula

$$(\mathcal{H}x)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{x(s)}{t-s} ds.$$

If $X$ is a Banach space, let $B(X,X)$ denote the space of bounded linear operators from $X$ into itself. A classical result of M.Riesz states that $\mathcal{H} \in B(L_p, L_p)$ if and only if $1 < p < \infty$. The main result of the paper by D.W.Boyd [1] generalises this as follows.

**Theorem 1.** *Let $X$ be a rearrangement-invariant space. Define the operator $E_s$ for $0 < s < \infty$ by $(E_s f)(x) = f(sx), f \in X$. Denote the norm of $E_s$ as a member of $B(X,X)$ by $h(s;X)$. Then, $\mathcal{H} \in B(X,X)$ if and only if*

$$sh(s;X) \to 0 \text{ as } s \to 0+, \text{and } h(s;X) \to 0 \text{ as } s \to \infty.$$

Using this result, D.W.Boyd gives examples showing that reflexivity of $X$ is both unnecessary and insufficient in order that $\mathcal{H} \in B(X,X)$. This may be somewhat surprising, since the condition for $L_p$ to be reflexive (i.e. $1 < p < \infty$) is the same as the condition for $\mathcal{H} \in B(L_p, L_p)$. However, $1 < p < \infty$ also ensures that $L_p$ is uniformly convex, so D.W.Boyd [1] obtained the following result:

*Let $X$ be the Lorentz space $\Lambda(\phi, p)$,$1 < p < \infty$. Then $\mathcal{H} \in B(X,X)$ if and only if $X$ is uniformly convex.*

In [1], D.W. Boyd characterized Lorentz spaces $\Lambda_{\phi,p}(\mathbb{R}_+)$ in which the Hilbert transform $\mathcal{H}$ is bounded from $\Lambda_{\phi,p}(\mathbb{R}_+)$ into itself in the case when $1 < p < \infty$. However, for the most important case when $p = 1$ the result was proved recently [2, Theorem 4.2]. Moreover, applying the main theorem D.W.Boyd obtained the following [1]:

*Let $L_\phi$ be an Orlicz space. Then $\mathcal{H} \in B(L_\phi, L_\phi)$ if and only if $L_\phi$ os reflexive.*

While D.W.Boyd investigated the boundedness of the Hilbert transform acting from a rearrangement-invariant space into itself, we rather consider the boundedness of the Hilbert transform from one (particular) Orlicz space, defined on (0,1), to another, i.e. we provide the Orlicz functions $G_1$ and $G_2$ such that (see Corollary 10) the Hilbert transform

$$\mathcal{H}: L_{G_2}(0,1) \to L_{G_1}(0,1).$$

We also give examples of Lorentz spaces such that the Calderón operator

$$S: \Lambda_\phi(0,1) \to \Lambda_\varphi(0,1).$$

**2 Preliminaries**

Let $(I,m)$ denote the measure space, where throughout this paper $I = (0,1)$, equipped with Lebesgue measure $m$. Let $L(I,m)$ be the space of all measurable real-valued functions on (0,1) equipped with Lebesgue measure $m$, i.e. functions which coincide almost everywhere are considered identical. Let $L(I,m)$ be the space of all measurable real-valued functions on (0,1) equipped with Lebesgue measure $m$. Define $L_0(0,1)$ to be the subset of $L(0,1)$ which consists of all functions $x$ such that $m(\{t: |x(t)| > s\})$ is finite for some $s > 0$.

**2.1 Rearrangement invariant Banach Function Spaces**

**Definition 2.** *[3, Definition I. 1.1, p. 2] A mapping $\rho: L(I)^+ \to [0,\infty]$ is called a Banach function norm if, for all $x, y, x_n$, $(n = 1,2,3,\dots)$, in $L(I)^+$, for all $m$-measurable subsets $\Delta$ of $\mathbb{R}$, the following properties hold:*

1. $\rho$ is a norm
2. $0 \leq y \leq x$ a.e. $\Rightarrow \rho(y) \leq \rho(x)$
3. $0 \leq x_n \uparrow x$ a.e. $\Rightarrow \rho(x_n) \uparrow \rho(x)$
4. $\rho(\Delta) < \infty \Rightarrow \rho(\chi_\Delta) < \infty$
5. $\rho(\Delta) < \infty \Rightarrow \int_\Delta xdm \leq c_\Delta \rho(x)$

for some constant $c_\Delta$, $0 < c_\Delta < \infty$, depending on $\Delta$ and $\rho$ but independent of $x$.

Let $\rho$ be a function norm. The set $E = E(\rho)$ of all functions $x$ in $L(I)$ for which $\rho(|x|) < \infty$ is called a Banach function space. For any $x \in E$, define

$$\| x \|_E = \rho(|x|).$$

Define $L_0(I)$ to be the subset of $L(I)$ which consists of all functions $x$ such that $m(\{t: |x(t)| > s\})$ is finite for some $s > 0$. Two functions $x$ and $y$ are called equimeasurable, if

$$m(\{t: |x(t)| > s\}) = m(\{t: |y(t)| > s\}).$$

For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t,x) = \inf\{s \geq 0: m(\{|x| > s\}) \leq t\}, t > 0.$$

**Definition 3.** *[3, Definition 4.1, p. 59] A Banach function space $E$ is called rearrangement-invariant if, whenever $x$ belongs to $E$ and $y$ is equimeasurable with $x$, then $y$ also belongs to $E$ and $\| y \|_E = \| x \|_E$.*

For the general theory of rearrangement invariant Banach function spaces, we refer the reader to [3, 4, 5, 6].

### 2.2 Köthe dual of Rearrangement invariant Function spaces

Next we define the Köthe dual space of rearrangement invariant Banach function spaces. Given a rearrangement invariant Banach function space $E$ on (0,1), equipped with Lebesgue measure $m$, the Köthe dual space, denoted by $E^{\times}(0,1)$, is defined by

$$E(0,1)^{\times} = \left\{y \in S(0,1): \int_0^1 |x(t)y(t)|dt < \infty, \forall x \in E(0,1)\right\}.$$

$E^{\times}$ is a Banach space with the norm

$$\| y \|_{E(0,1)^{\times}} := \sup\left\{\int_0^1 |x(t)y(t)|dt: x \in E(0,1), \| x \|_{E(I)} \leq 1\right\}. \tag{2.1}$$

If $E$ is a rearrangement invariant Banach function space, then $(E^{\times}, \|\cdot\|_{E^{\times}})$ is also a rearrangement invariant Banach function space (see [3, Section 2.4]). For more details on Köthe duality see [3, 5].

### 2.3 Lorentz and Marcinkiewicz spaces

**Definition 4.** *[4, Definition II. 1.1, p. 49] A function $\varphi$ on the interval* [0,1] *is said to be quasiconcave if*

1. $\varphi(t) = 0 \Leftrightarrow t = 0$;
2. $\varphi(t)$ is positive and increasing for $t > 0$;
3. $\frac{\varphi(t)}{t}$ is decreasing for $t > 0$.

Observe that every nonnegative concave function on $[0, \infty)$ that vanishes only at origin is quasiconcave. The reverse, however, is not always true. But, we may replace, if necessary, a quasiconcave function $\varphi$ by its least concave majorant $\tilde{\varphi}$ such that

$$\frac{1}{2}\tilde{\varphi} \leq \varphi \leq \tilde{\varphi}$$

(see [3, Proposition 5.10, p. 71]).

Let $\Omega$ denote the set of increasing concave functions $\varphi: [0,1] \to [0,1]$ for which $\lim_{t\to 0+}\varphi(t) = 0$ (or simply $\varphi(0+) = 0$). For a function $\varphi$ in $\Omega$, the Lorentz space $\Lambda_\varphi(0,1)$ is defined by setting

$$\Lambda_\varphi(0,1) := \left\{x \in L_0(0,1): \int_0^1 \mu(s,x)d\varphi(s) < \infty\right\}$$

equipped with the norm

$$\| x \|_{\Lambda_\varphi(0,1)} := \int_0^1 \mu(s,x)d\varphi(s). \tag{2.2}$$

The Lorentz spaces $(\Lambda_\varphi(0,1), \|\cdot\|_{\Lambda_\varphi(0,1)})$ are examples of rearrangement invariant Banach function spaces. For more details on Lorentz spaces, we refer the reader to [3, Chapter II.5] and [4, Chapter II.5]. Let $\psi$ be a quasiconcave function on $(0,1)$. The space

$$M_\psi(0,1) = \{f \in L_1 : \| f \|_{M_\psi} < \infty\}$$

equipped with the norm

$$\| f \|_{M_\psi(0,1)} = \sup_{t\in(0,1)} \frac{1}{\psi(t)} \cdot \int_0^t \mu(s,f)dm$$

is the rearrangement invariant space with the fundamental function $\varphi(t) = \frac{t}{\varphi(t)} \cdot \mathbf{1}_{(0,1)}(t)$. The space $(M_\psi, \|\cdot\|_{M_\psi})$ is called the Marcinkiewicz space.

### 2.4 Orlicz spaces

**Definition 5.** *An Orlicz function is a function* $G: [0,1] \to [0,1]$ *such that*

(1) $G(0) = 0$, $G(\lambda_1) > 0$ for some $\lambda_1 > 0$ and $G(\lambda_2) < \infty$ for some $\lambda_2 > 0$.

(2) $G$ is increasing.

(3) $G$ is convex: $G(\alpha\lambda_1 + (1-\alpha)\lambda_2) \le \alpha G(\lambda_1) + (1-\alpha)G(\lambda_2), 0 \le \alpha \le 1$.

(4) $G$ is left-continuous.

In what follows, unless otherwise specified, we always denote by $G$ an Orlicz function.

For every Orlicz function $G$, we define a functional

$$I_G(f) = \int_0^1 G(|f|)dm \in [0,\infty]$$

and set

$$||f||_{L_G} = \inf\{a > 0: I_G\left(\frac{f}{a}\right) \le 1\} \in [0,\infty]$$

for every measurable function $f: [0,1] \to \mathbf{R}$. We put here $\inf\{\emptyset\} = \infty$. The set

$$L_G = \{f \in L_0: ||f||_{L_G} < \infty\}$$

equipped with the norm $||\cdot||_{L_G}$ is the rearrangement invariant space. The space $(L_G, ||\cdot||_{L_G})$ is called the *Orlicz space*. Orlicz spaces which generalize Lebesgue's scale in a direction essentially different from Lorentz spaces, received much attention too, see for instance [7, 8, 9, 10, 11, 12, 13].

### 2.5 Calderón operator and Hilbert transform

Let $E(0,1)$ be a r.i. Banach function space. For a function $x \in E(0,1)$, define formally the operator $S$ as follows

$$(Sx)(t) := \frac{1}{t}\int_0^t x(s)ds + \int_t^1 x(s)\frac{ds}{s}, t > 0. \tag{2.3}$$

The operator $S$ is called the Calderón operator. It is obvious that $S$ is a linear operator. If $0 < t_1 < t_2$, then

$$\min\left(1, \frac{s}{t_2}\right) \le \min\left(1, \frac{s}{t_1}\right) \le \frac{t_2}{t_1} \cdot \min\left(1, \frac{s}{t_2}\right), s > 0.$$

Let $x$ be nonnegative function on $[0,1]$. It follows from the first of these inequalities that $(Sx)(t)$ is a decreasing function of $t$. The operator $S$ is often applied to the decreasing rearrangement $\mu(x)$ of a function $x$ defined on some other measure space. Since $S\mu(x)$ is non-increasing itself, it is easy to see that $\mu(S\mu(x)) = S\mu(x)$. Throughout this paper, we write $\mathcal{A} \lesssim \mathcal{B}$ if there is a constant $c_{abs} > 0$ such that $\mathcal{A} \le c_{abs}\mathcal{B}$. We write $\mathcal{A} \approx \mathcal{B}$ if both $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$ hold, possibly with different constants.

If $x \in E(0,1)$, ($E$ is rearrangement-invariant space) then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}x)(s) := p.v. \frac{1}{\pi}\int_0^1 \frac{x(\eta)}{s-\eta}d\eta. \tag{2.4}$$

(see, e.g. [3, Chapter III. 4]). Boundedness of such classical operators on rearrangement-invariant spaces have attracted attention of leading mathematicians in the field of r.i. spaces and non-commutative analysis, see for example [14, 15, 16, 17, 18, 19, 20].

## 3 Main results

### 3.1 Boundedness of the Calderón operator on Orlicz spaces

Let $S$ be the Calderón operator acting on Lorentz spaces

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\psi}(0,1)$$

In the next lemma we find the Lorentz function $\varphi_0$ such that $S:\Lambda_{\varphi_0}(0,1)\to L_1(0,1)$ is bounded and $\Lambda_{\varphi_0}(0,1)$ is maximal.

**Lemma 6.** *If* $\varphi_0(t)=tlog\left(\frac{e}{t}\right)+t, 0<t<1$, *then* $\Lambda_{\varphi_0}(0,1)$ *is the maximal among all rearrangement invariant spaces such that* $S:\Lambda_{\varphi_0}(0,1)\to L_1(0,1)$. (Here and throughout this paper $log$ stands for the natural logarithm)

*Proof.*

$$\int_0^1 (Sx)(t)dt=\int_0^1\frac{1}{t}\int_0^t x(s)dsdt+\int_0^1\int_t^1\frac{x(s)}{s}dsdt=$$

$$=\int_0^1 x(s)\int_s^1\frac{1}{t}dtds+\int_0^1\frac{x(s)}{s}\int_0^s dtds=$$

$$=\int_0^1 x(s)\left(\log\left(\frac{1}{s}\right)+1\right)ds.$$

Hence $\varphi_0'(t)=\log\left(\frac{e}{s}\right)$. Then, by integrating we obtain

$$\varphi_0(t)=t\log\left(\frac{e}{t}\right)+t.$$

Now, we show that $\Lambda_{\varphi_0}(0,1)$ is maximal among such rearrangement invariant spaces. Indeed, let

$$S:E(0,1)\to L_1(0,1),$$

where $E(0,1)$ is arbitrary rearrangement invariant space. Then,

$$||x||_{\Lambda_{\varphi_0}}=||Sx||_{L_1}\lesssim||x||_E. \blacksquare$$

Therefore, $E(0,1)\subset\Lambda_{\varphi_0}(0,1)$.

**Lemma 7.** *Let* $\varphi(t)=tlog\left(\frac{e}{t}\right), t\in(0,1)$ *and* $\phi(t)=tlog^2\left(\frac{e}{t}\right), t\in(0,1)$. *Then*

$$S\left(\frac{\varphi(t)}{t}\right)\lesssim\frac{\phi(t)}{t}, t>0.$$

*Proof.* Indeed, integrating by parts, we obtain

$$S\left(\frac{\varphi(t)}{t}\right)=\frac{1}{t}\int_0^t\log\left(\frac{e}{s}\right)ds+\int_t^1\frac{\log\left(\frac{e}{s}\right)}{s}ds=\frac{1}{t}\left(s\log\left(\frac{e}{s}\right)\right)|_0^t+\int_0^t ds-\frac{\log^2\left(\frac{e}{s}\right)}{2}|_t^1$$

$$=\log\left(\frac{e}{t}\right)+1-\frac{1}{2}+\frac{\log^2\left(\frac{e}{t}\right)}{2}=\log\left(\frac{e}{t}\right)+\frac{1}{2}\log^2\left(\frac{e}{t}\right)+\frac{1}{2}=$$

$$=\frac{t\log\left(\frac{e}{t}\right)+\frac{1}{2}t\log^2\left(\frac{e}{t}\right)+\frac{1}{2}t}{t}\le\frac{t\log^2\left(\frac{e}{t}\right)+\frac{1}{2}t\log^2\left(\frac{e}{t}\right)+t\log^2\left(\frac{e}{t}\right)}{t}=\frac{5}{2}\frac{t\log^2\left(\frac{e}{t}\right)}{t}\ \blacksquare$$

Note that both $\varphi(t)$ and $\phi(t)$ are Lorentz weight functions for $t\in(0,1)$.

**Lemma 8.** *Let* $\varphi(t)=tlog\left(\frac{e}{t}\right), t\in(0,1)$ *and* $\phi(t)=tlog^2\left(\frac{e}{t}\right), t\in(0,1)$. *Then the Calderón operator*

$$S:\Lambda_{\phi}(0,1)\to\Lambda_{\varphi}(0,1)$$

is bounded. Moreover,

$$S:M_{\varphi}(0,1)\to M_{\phi}(0,1)$$

is also bounded.

*Proof.* First we show the boundedness of the Calderón operator from one Lorentz space to another. It was shown in [17, lemma 10] that if $\lim_{t\to 0}\log\left(\frac{1}{t}\right)\varphi(t)=0$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t}=0$, then

$$S:\Lambda_{\phi}(\mathbb{R}_+)\to\Lambda_{\varphi}(\mathbb{R}_+)$$

if and only if

$$S\left(\frac{\varphi(t)}{t}\right)\lesssim\frac{\phi(t)}{t}, t>0.$$

Here we consider the boundedness of the Calderón operator

$$S: \Lambda_{\phi}(0,1) \to \Lambda_{\varphi}(0,1).$$

It is easy to see that the first condition in [17, lemma 10], $\lim_{t\to 0}\log\left(\frac{1}{t}\right)\varphi(t) = 0$, is satisfied. The condition for a function $\phi(t)$ such that $S\left(\frac{\varphi(t)}{t}\right) \lesssim \frac{\phi(t)}{t}$ for all $t > 0$ is satisfied by lemma 7. Hence, by [17, lemma 10] we obtain that the Calderón operator

$$S: \Lambda_{\phi}(0,1) \to \Lambda_{\varphi}(0,1)$$

is bounded. It is well known fact that the associate space (Köthe dual) of the Lorentz space $\Lambda_{\varphi}$ with weight function $\varphi$ is the Marcienkiewicz space $M_{\varphi}$ and $||\cdot||_{\Lambda_{\varphi}^{\times}} = ||\cdot||_{M_{\varphi}}$. [4, Theorem 5.2, p.112]

Hence,

$$S: M_{\varphi}(0,1) \to M_{\phi}(0,1)$$

is also bounded.

**Theorem 9.** *Let* $G_1(t) = e^{|t|^{\frac{1}{2}}} - \frac{|t|}{2} - |t|^{\frac{1}{2}} - 1$ *and* $G_2(t) = e^{|t|} - 1$. *Then the Calderón operator*

$$S: L_{G_2}(0,1) \to L_{G_1}(0,1)$$

is bounded.

*Proof.* It is known from [19, see Lemma 4.3] that $L_{N_p} = M_{\psi_p}$ (with norm equivalence) holds for every $p > 0$, where

$$N_p(t) := e^{|t|^p} - \sum_{k=0}^{\lfloor\frac{1}{p}\rfloor} \frac{|t|^{kp}}{k!}, p \in (0,1);$$

$$N_p(t) := e^{|t|^p} - 1, p \geq 1, t \in \mathbf{R}.$$

and

$$\psi(t) := t\log^{1/p}\left(\frac{e}{t}\right), t \geq 0.$$

If we choose $p = \frac{1}{2}$, then $\psi(t) = t\log^2\left(\frac{e}{t}\right)$ coincides with the $\phi(t)$ above. The corresponding Orlicz function is $N_{\frac{1}{2}}(t) = e^{|t|^{\frac{1}{2}}} - \frac{|t|}{2} - |t|^{\frac{1}{2}} - 1$. For convenience let us denote $N_{\frac{1}{2}}(t) = G_1(t)$. Then

$$M_{\phi}(0,1) = L_{G_1}(0,1).$$

Similarly, we need to find the Orlicz function $G_2$ such that $M_{\varphi}(0,1) = L_{G_2}(0,1)$. For $p = 1$ we have $\psi(t) = t\log\left(\frac{e}{t}\right)$ which is equal to $\varphi(t)$. Hence, the corresponding Orlicz function is $N_1(t) = e^{|t|} - 1$. Let us denote $N_1(t) = G_2(t)$ so that $M_{\varphi}(0,1) = L_{G_2}(0,1)$ holds. Thus, we have found the Orlicz functions $G_1$ and $G_2$ such that the Calderón operator

$$S: L_{G_2}(0,1) \to L_{G_1}(0,1)$$

is bounded.

**3.2 Boundedness of the Hilbert transform from one Orlicz space to another**

In this section we use the equivalence (up to equimeasurable functions) of the Calderón operator $S$ and the Hilbert transform $\mathcal{H}$ [3, chapter 3.4].

**Corollary 10.** *The Hilbert transform*

$$\mathcal{H}: L_{G_2}(0,1) \to L_{G_1}(0,1),$$

where $G_2(t) = e^{|t|} - 1$ and $G_1(t) = e^{|t|^{\frac{1}{2}}} - \frac{|t|}{2} - |t|^{\frac{1}{2}} - 1$, is bounded.

*Proof.* An estimate of the Hilbert transform $(\mathcal{H}f)(t)$ from above by the function $S(\mu(t,f))$ is given in [3, Theorem III.4.8, p.138]. The theorem states that if $S(\mu(1,f)) < \infty$, then the Hilbert transform $(\mathcal{H}f)(t)$ exists a.e. Furthermore,

$$\mu(t, \mathcal{H}f) \leq cS(\mu(t,f)), t > 0$$

for some constant $c$ independent of $f$ and $t$. The corresponding lower estimate is false. However, there is the following substitute. If $S(\mu(1,f)) < \infty$, then there exists a function $g$ equimeasurable with $f$ such that

$$S(\mu(t,f)) \leq 2\pi\mu(t,\mathcal{H}g), t > 0.$$

[see [3, Proposition III.4.10, p. 140]].

Therefore, all results obtained in the previous section for the Calderón operator $S$ will also remain valid for the Hilbert transform $\mathcal{H}$. In particular, we also obtain boundedness of the Hilbert transform from one Lorentz space to another and from one Marcienkiewicz space to another as well.

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## ОГРАНИЧЕННОСТЬ ПРЕОБРАЗОВАНИЯ ГИЛЬБЕРТА ИЗ ОДНОГО ПРОСТРАНСТВА ОРЛИЧА В ДРУГОЕ

**Аннотация.** В этой статье исследуется ограниченность преобразования Гильберта из одного пространства Орлича в другое. Д. В. Бойд привел примеры (см. [1]), показывающие, что рефлексивность X не является как необходимым, так и достаточным условием для ограниченности преобразования Гильберта. Это может несколько удивить, поскольку условие рефлексивности $L_p$ $(1 < p < \infty)$ совпадает с условием ограниченности преобразования Гильберта из $L_p$ в $L_p$. Однако Д. В. Бойд рассматривал только случаи, когда область определения и область значений преобразования Гильберта совпадают. Поскольку пространства $L_p$ являются примерами перестановочно-инвариантных банаховых функциональных пространств, мы рассматриваем ограниченность преобразования Гильберта из одного перестановочно-инвариантного банахова пространства функций в другое. Точнее, мы обобщаем результаты Бойда, позволяя областью определения преобразования Гильберта быть конкретным пространством Орлича, определенным в (0,1); а областью значений также быть пространством Орлича, определенным в (0,1), отличным от области определения. Кроме того, мы также рассматриваем ограниченность преобразования Гильберта из одного пространства Лоренца на (0,1) (которое также является инвариантно-перестановочным) в другое пространство Лоренца (0,1).

Для достижения этих целей мы также рассмотрим ограниченность оператора Кальдерона из одного перестановочно-инвариантного банахова пространства в другое. В случае, когда областью преобразования Гильберта является пространство Лоренца $\Lambda_{\phi,p}(\mathbb{R}_+)$, совпадающее с его областью значений, задачи были полностью решены Д. В. Бойдом. Он показал (см. [1]), что равномерная выпуклость $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ является необходимым и достаточным условием ограниченности преобразования Гильберта. Однако для наиболее важного случая, когда p = 1, результат был недавно доказан [2, теорема 4.2]. Кроме того, применяя основную теорему Д.Бойд, получили следующее [1]:

Пусть $L_\phi$ - пространство Орлича. Тогда $\mathcal{H} \in B(L_\phi, L_\phi)$ тогда и только тогда, когда $L_\phi$ является рефлексивным.

Такие вопросы привлекают большое внимание математиков на протяжении многих лет, в частности, в связи с вложениями соболевских пространств. В настоящей статье мы обсуждаем такие проблемы ограниченности для классических операторов, представляющих большой интерес для анализа и его приложений, а именно для преобразования Гильберта и оператора Кальдерона. Действие этих операторов на конкретные классы функциональных пространств широко изучалось в течение нескольких десятилетий. Классические результаты доступны, например, в связи со знакомыми функциональными пространствами. Кроме того, приложения этих операторов очень хорошо известны, и их свойства были глубоко изучены. Классические пространства Лоренца, которые возникли в 1950-х годах, стали чрезвычайно модными в 1990-х годах, когда появились фундаментальные статьи.

В этой статье мы изучаем ограниченность таких классических операторов на пространствах, инвариантных относительно перестановки, то есть класс функциональных пространств, который включает, например, все пространства Лебега, Лоренца, Орлича, Марцинкевича и многие другие. Основное внимание мы уделяем ограниченности преобразования Гильберта из одного пространства Орлича в другое. Мы также приводим примеры конкретных перестановочно-инвариантных пространств, в которых преобразование Гильберта действует ограниченно.

**Ключевые слова:** преобразование Гильберта, оператор Кальдерона, перестановочно-инвариантные банаховы пространства функций, пространства Орлича, пространства Лоренца, пространства Марцинкевича.

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## ГИЛЬБЕРТ ТҮРЛЕНДІРУІНІҢ БІР ОРЛИЧ КЕҢІСТІГІНЕН ЕКІНШІСІНЕ ШЕНЕЛГЕНДІГІ

**Аннотация**. Бұл мақалада біз Гильберт түрлендіруінің бір Орлич кеңістігінен екінші Орлич кеңістігіне шенелгендігін зерттейміз. Д.В. Бойд өзінің жұмысында [1] Гильберт түрлендіруінің X кеңістігіндегі шенелгендігі үшін X кеңістігінің рефлексивтілігі қажет те жеткілікті де емес екендігін көрсетті. Бұл біршама таңқаларлық жағдай болуы мүмкін, өйткені $L_p$ $(1 < p < \infty)$ рефлексті болады сонда тек сонда ғана Гильберт түрлендіруі $L_p$-дан $L_p$-ға шенелген оператор болса. Алайда, Д.В. Бойд Гилберт түрлендіруінің аңықталу облысы мен мәндер облысы бірдей кеңістік болатын жағдайларды толықтай зерттеді. $L_p$ кеңістігі алмастырмалы-инвариантты функцияларының Банах кеңістігі болғандықтан, біз бұл жұмыста Гильберт түрлендіруінің аңықталу облысы мен мәндер облыстары әр түрлі болған жағдайды қарастырамыз. Атап айтқанда, Гилберт түрлендіруінің (0,1) аралығында анықталған өлшемді функциялардың Орлич кеңістіктерінде шенелгендігін зерттейміз, яғни бір Орлич кеңістігінен басқа Орлич кеңістігіне шенелгендігін көрсетеміз. Сонымен қатар, біз Гильберт түрлендіруінің бір Лоренц кеңістігінен басқа Лоренц кеңістігіне шенелгендік критерийін қарастырамыз. Егер Гилберт түрлендіруінің аңықталу облысы Лоренц кеңістігі $\Lambda_{\phi,p}(\mathbb{R}_+)$ болса, және оның мәндер облысы да сол Лоренц кеңістігі $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ болса, онда осы проблеманы Д.В.Бойд толықтай шешкен. Ол ([1]) Лоренц кеңістігінің $\Lambda_{\phi,p}(\mathbb{R}_+)$ $(1 < p < \infty)$ біртекті дөңестілігі Гильберт түрлендіруінің шенелгендігі үшін қажетті және жеткілікті шарт екенін көрсетті. Алайда, ең маңызды жағдай үшін p = 1 болған кездегі нәтиже тек қана жақында дәлелденді [2, Теорема 4.2]. Сонымен қатар, Д.В.Бойд негізгі теоремасын қолдана отырып, төмендегі нәтиже алды [1]:

$L_\phi$ Orlicz кеңістігі болсын. Онда $\mathcal{H} \in B(L_\phi, L_\phi)$ сонда және тек қана сонда егер $L_\phi$ рефлексивті болса. Осы мақсаттарға жету үшін біз Кальдерон операторының бір банах алмастырмалы-инвариантты функцияларының Банах кеңістігінен екіншісіне бейнелейтіндігін қарастырамыз.

Мұндай сұрақтар көптеген жылдар бойы, әсіресе Соболев кеңістігін ендіруге байланысты көпшіліктің назарын аударды. Осы жұмыста біз талдауға және оны қолдануға үлкен қызығушылық тудыратын классикалық операторлар үшін, мысалы, Гильберт түрлендіруі және Кальдерон операторы үшін мұндай шенелгендік проблемаларын талқылаймыз. Бұл операторлардың функционалдық кеңістіктердің арнайы кластарындағы әрекеті бірнеше ондаған жылдар бойы кең зерттелген. Классикалық нәтижелерге, мысалы, таныс функциялар кеңістігіне байланысты қол жетімді. Сонымен қатар, осы операторлардың маңыздылығы белгілі, және олардың қасиеттері терең зерттелген. 1950 жылдары пайда болған классикалық Лоренц кеңістігі 1990 жылдардан бастап негізгі зерттеулер пайда болған кезде өте көп қолданылады.

Бұл жұмыста біз осындай классикалық операторлардың шенелгендігін алмастырмалы-инвариантты кеңістіктерде, мысалы, барлық Лебег, Лоренц, Орлич, Марцинкевич кеңістіктерінде және басқа функционалдық кеңістіктерде зерттейміз. Біздің назарымыз, негізінен, Гильберт түрлендіруінің бір Орлич кеңістігінен екіншісіне шенелгендігінде. Сонымен қатар, біз Гилберт түрлендіруінің шектеулі болатындай инварианттық кеңістіктердің мысалдарын келтіреміз.

**Түйін сөздер:** Гильберт түрлендіруі, Кальдерон операторы, алмастырмалы-инвариантты функцияларының Банах кеңістігі, Орлич кеңістігі, Лоренц кеңістігі, Марцинкевич кеңістері.

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# TWO-PHASE STEFAN PROBLEM FOR GENERALIZED HEAT EQUATION

**Abstract.** The generalized heat equation is very important for modeling of the heat transfer in bodies with a variable cross section, when the radial component of the temperature gradient can be neglected in comparison with the axial component. Such models can be applied in the theory of the heat- and mass transfer in the electrical contacts. In particular, the temperature field in a liquid metal bridge appearing at the opening electrical contacts can be modelled by the above considered Stefan problem for the generalized heat equation. The exact solution in the case when the temperature field in a liquid bridge is modelled by the generalized heat equation, while for the temperature of the solid contact zone is modelled by the spherical model, can be represented in the form of series with radial heat polynomials and integral error functions. The recurrence formulas for the coefficients of these series are given in papers published earlier in "News of the National Academy of Sciences of the Republic of Kazakhstan, Physic-mathematical series".

The two-phase Stefan problem for the generalized heat equation is considered in this paper for the case when one of the phases collapses into a point at the initial time. That creates a serious difficulty for the solution by the standard method of reduction of the problem to the integral equations because these equations are singular. Another method is used here in the case, when the functions appearing in the initial and boundary conditions are analytical and can be expanded into Taylor series. Then the solution of the problem can be represented in the form of series for special functions (Laguerre polynomials and the confluent hypergeometric function) with undetermined coefficients. These special functions have a close link with the heat polynomials introduced by P.C. Rosenbloom and D.V. Widder.

**Keywords:** Stefan problem, special functions, Laguerre polynomial, Faa-di Bruno formula.

## 1. Introduction

These special functions have a close link with the heat polynomials introduced by P.C. Rosenbloom and D.V. Widder [1]. The similar approach was used for the solution of other free boundary problems [2], [3].

Let us consider the equation

$$x\frac{d^2\varphi}{dx^2}+(\frac{\nu+1}{2}-x)\frac{d\varphi}{dx}+\frac{\beta}{2}\varphi=0\ ,\ \nu=0\,,\ -\infty<\beta<\infty \tag{1}$$

It is well known that this equation has two linearly independent solutions

$$\varphi_1(x)=\Phi(-\frac{\beta}{2},\frac{\nu+1}{2};x)\,,\ \varphi_2(x)=x^{\frac{1-\nu}{2}}\Phi(\frac{1-\beta-\nu}{2},\frac{3-\nu}{2};x) \tag{2}$$

where $\Phi(a,b;x)$ is the confluent (degenerate) hypergeometric function . Setting $T(z)=\varphi(x)$, where $x=-z^2$, one can find that $T(z)$ satisfies the equation

$$\frac{d^2T}{dz^2}+(\frac{\nu}{z}+2z)\frac{dT}{dz}-2\beta T(z)=0$$

Using this equation one can check up that the function

$$\theta(z,t)=(2a\sqrt{t})^{\beta}T(\frac{z}{2a\sqrt{t}})$$

satisfies the equation

$$\frac{\partial\theta}{\partial t}=a^2(\frac{\partial^2\theta}{\partial z^2}+\frac{\nu}{z}\frac{\partial\theta}{\partial z}) \quad (3)$$

Hence the functions

$$S_{\beta,\nu}{}^{(1)}(z,t)=(2a\sqrt{t})^{\beta}\Phi(-\frac{\beta}{2},\frac{\nu+1}{2};-\frac{z^2}{4a^2t}), \quad (4)$$

$$S_{\beta,\nu}{}^{(2)}(z,t)=(2a\sqrt{t})^{\beta}(\frac{z^2}{4a^2t})^{\frac{1-\nu}{2}}\Phi(\frac{1-\nu-\beta}{2},\frac{3-\nu}{2};-\frac{z^2}{4a^2t})$$

satisfy the equation (3).

If $\beta$ is an even integer, $\beta=2n$, the function $S_{\beta,\nu}(z,t)$ can be expressed in terms of the generalized Laguerre polynomials

$$S_{2n,\nu}{}^{(1)}(z,t)=(4a^2t)^n\Phi\left(-n,\mu,-\frac{z^2}{4a^2t}\right)=\frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}\left(4a^2t\right)^n L_n^{(\mu-1)}\left(-\frac{z^2}{4a^2t}\right) \quad (5)$$

$$S_{2n,\nu}{}^{(2)}(z,t)=4a^2t^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{z^2}{4a^2t}\right)=\frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}\left(4a^2t\right)^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}L_n^{(\mu-1)}\left(-\frac{z^2}{4a^2t}\right) \quad (6)$$

where $\mu=\frac{\nu+1}{2}$. It should be noted that this formula is valid for $\mu>0$ only.

**Properties.** Using the integral representation for the degenerate hypergeometric function

$$\Phi(-\frac{\beta}{2},\mu;-z^2)=\frac{2\Gamma(\mu)}{\Gamma(\mu+\frac{\beta}{2})}\exp(-z^2)z^{-\mu+1}\int_0^{\infty}\exp(-x^2)x^{\mu+\beta}I_{\mu-1}(2zx)dx \quad (7)$$

and the asymptotic formula

$$\lim_{z\to\infty}\frac{e^{-z}I_{\nu}(z)}{\sqrt{2\pi z}}=1$$

it is possible to show that

$$\lim_{z\to\infty}\frac{1}{z^{\beta}}\Phi(-\frac{\beta}{2},\mu;-z^2)=\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{\beta}{2})} \quad (8)$$

In particular,

$$\lim_{z\to\infty}\frac{1}{z^{\beta}}\Phi(-\frac{\beta}{2},1;-z^{2})=\frac{1}{\Gamma(1+\frac{\beta}{2})} \tag{9}$$

For $\nu=1$ both functions ( 1 ) coincide :

$$S_{\beta,1}^{(1)}(z,t)=S_{\beta,1}^{(2)}(z,t)=(2a\sqrt{t})^{\beta}\,\Phi(-\frac{\beta}{2},1;-\frac{z^{2}}{4a^{2}t})$$

In this case, the second linearly independent solution of the equation (3) is [4]

$$\varphi_2(x)=\Phi\left(-\frac{\beta}{2},1,x\right)\ln x+\sum_{k=1}^{\infty}M_k x^k \tag{10}$$

where

$$M_k=\binom{k}{-\beta/2}\frac{1}{k!}\sum_{m=0}^{k-1}\left(\frac{1}{m-\beta/2}+\frac{2}{m+1}\right).$$

Stefan problem for spherical case when $\nu=2$ is considered in works [5]-[7]. The case when we represented spherical model as introduced in R. Holm [8] of heat transferring zones. In Stefan problem with generalized heat equation we can represent solution in heat polynomials [9], but in this work we represent in linear combination of two special functions. About special functions and their applications in heat transfer problems we can see in [10].

The generalized heat equation can be used to describe the heat transfer in a bar with the variable cross-section in the case when the radial component of the temperature gradient can be neglected in comparison with the axial component. Such mathematical model is very useful for some applied problems, in particular, for the dynamics of the heating with phase transformation in electrical contacts. Such approach was used in the papers [11] and [12] for the calculation of the temperature fields in a liquid metal bridge appearing at the contact opening, which was modelled by the generalized heat equation, and the solid contact zone modelled by the spherical heat equation. The exact solution in this case was represented in the form of radial heat polynomials and integral error functions.

**2. Problem definition**

Let us consider the two-phase Stefan problem for the equations

$$\frac{\partial\theta_1}{\partial t}=a_1^{2}\left(\frac{\partial^2\theta_1}{\partial z^2}+\frac{\nu}{r}\frac{\partial\theta_1}{\partial r}\right),\qquad 0<r<\alpha(t),\quad \alpha(0)=0,\quad 0<\nu<1, \tag{11}$$

$$\frac{\partial\theta_2}{\partial t}=a_2^{2}\left(\frac{\partial^2\theta_2}{\partial z^2}+\frac{\nu}{r}\frac{\partial\theta_2}{\partial r}\right),\qquad \alpha(t)<r<\infty \tag{12}$$

with the initial conditions

$$\theta_1(0,0)=\theta_m \tag{13}$$

$$\theta_2(r,0)=\varphi(r),\qquad \varphi(0)=\theta_m \tag{14}$$

the boundary conditions

$$\theta_1(0,t)=f(t),\qquad f(0)=\theta_m \tag{15}$$

$$\theta_1\left(\alpha(t),t\right)=\theta_2\left(\alpha(t),t\right)=\theta_m \tag{16}$$

$$\theta_2\left(\infty,t\right)=0 \tag{17}$$

and the Stefan condition

$$-\lambda_1\frac{\partial\theta_1\left(\alpha(t),t\right)}{\partial r}=-\lambda_2\frac{\partial\theta_2\left(\alpha(t),t\right)}{\partial r}+L\gamma\frac{d\alpha}{dt} \tag{18}$$

***2.* The method of solution**

Suggesting that the initial and boundary functions can be expanded in Maclaurin series

$$f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n \qquad \varphi(r)=\sum_{n=0}^{\infty}\frac{\varphi^{(2n)}(0)}{(2n)!}r^{2n} \tag{19}$$

we represent the solution in the form

$$\theta_1(r,t)=\sum_{n=0}^{\infty}A_n\left(4a_1^2t\right)^nL_n^{(\mu-1)}\left(-\frac{r^2}{4a_1^2t}\right)+\sum_{n=0}^{\infty}B_n\left(4a_1^2t\right)^n\left(\frac{r^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{r^2}{4a_1^2t}\right) \tag{20}$$

$$\theta_2(r,t)=\sum_{n=0}^{\infty}C_n\left(4a_2^2t\right)^nL_n^{(\mu-1)}\left(-\frac{r^2}{4a_2^2t}\right)+\sum_{n=0}^{\infty}D_n\left(4a_2^2t\right)^n\left(\frac{r^2}{4a_2^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{r^2}{4a_2^2t}\right) \tag{21}$$

where $\frac{1}{2}<\mu=\frac{\nu+1}{2}<1$.

Satisfying the boundary condition (15) and using the formula (8) for $z=\frac{r}{2a_1\sqrt{t}}$, $\beta=2n$ we obtain

$$\sum_{n=0}^{\infty}A_n\left(4a_1^2t\right)^nL_n^{(\mu-1)}\left(0\right)=f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$$

and

$$A_n=\frac{f^{(n)}(0)}{n!\left(4a_1^2\right)^n\binom{n+\mu-1}{n}} \tag{22}$$

Using the initial condition (14) and the formula (8) for $z=\frac{r}{2a_1\sqrt{t}}$, $\beta=2n$ for the first term with $C_n$ and $\beta=2(n+\mu-1)$ for the second term with $D_n$ we get

$$\varphi(r)=\sum_{n=0}^{\infty}\frac{\varphi^{(2n)}(0)}{(2n)!}r^{2n}=\lim_{t\to 0}\theta_2(r,t)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}C_nr^{2n}+\frac{1}{\Gamma(n+\mu)}D_nr^{2n}\right],$$

Thus

$$\frac{(-1)^n}{n!}C_n+\frac{1}{\Gamma(n+\mu)}D_n=\frac{\varphi^{(2n)}(0)}{(2n)!} \tag{23}$$

Now we should use the conditions (16) and (18) to get additional three equations for the definition of all coefficients and the free boundary. Thus $\alpha(t)$ can be written in the form

$$\alpha(\tau)=\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}$$

where $\tau=\sqrt{t}$.

Now we rewrite the conditions (16) and (18) in terms of $\tau$ and compare the powers in the left and the right sides of equations using $k$-th differentiation and putting then $\tau=0$. We obtain

$$\left.\frac{\partial^k\theta_1\left(\alpha(\tau),\tau\right)}{\partial\tau^k}\right|_{\tau=0}=\left.\frac{\partial^k\theta_2\left(\alpha(\tau),\tau\right)}{\partial\tau^k}\right|_{\tau=0}=0,\quad k=0,1,2,... \tag{24}$$

$$-\lambda_1\frac{\partial^k\theta_{1r}\left(\alpha(\tau),\tau\right)}{\partial\tau^k}=-\lambda_2\frac{\partial^k\theta_{2r}\left(\alpha(\tau),\tau\right)}{\partial\tau^k}+L\gamma k!\alpha_k,\quad k=0,1,2,... \tag{25}$$

At first, we use Leibniz formula for $k$-th derivative for (24) equation and we obtain for the first term of $\theta_i(r,t),\quad i=1,2$.

$$\left.\frac{\partial^k\left[2^{2n}a_i^{2n}\tau^{2n}L_n^{(\mu-1)}(-\delta(\tau))\right]}{\partial\tau^k}\right|_{\tau=0}=2^{2n}a_i^{2n}\frac{k!}{(k-2n)!}\left.\frac{\partial^{k-2n}[L_n^{(\mu-1)}(-\delta(\tau))]}{\partial\tau^{k-2n}}\right|_{\tau=0}$$

and for second term we have

$$\left.\frac{\partial^k\left[2^{2n}a_i^{2n}\tau^{2n}(\delta(\tau))^{1-\mu}\Phi[1-\mu-n,2-\mu,-\delta(\tau)]\right]}{\partial\tau^k}\right|_{\tau=0}=$$

$$=2^{2n}a_i^{2n}\frac{k!}{(k-2n)!}\left.\frac{\partial^{k-2n}\left[(\delta(\tau))^{1-\mu}\Phi[1-\mu-n,2-\mu,-\delta(\tau)]\right]}{\partial\tau^k}\right|_{\tau=0}$$

where $\delta(\tau)=\frac{1}{4a_i^2}\left(\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}\right)^2,\quad i=1,2..$

In particular, when $k=0$ and $\tau=0$ we have

$$A_0=\theta_m,\quad B_0=0,\quad C_0=\theta_m,\quad D_0=0.$$

For this purpose, we use the Faa di Bruno formula (Arbogast formula) for a derivative of a composite function. For the first term of temperature equation we have

$$\left.\frac{\partial^{k-2n}[L_n^{(\mu-1)}(-\delta(\tau))]}{\partial\tau^{k-2n}}\right|_{\tau=0}=\sum_{m=0}^{k-2n}(-1)^m[L_{n-m}^{(\mu-1-m)}(-\delta_1)]\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}$$

$b_2,b_3,...$ satisfy the following equations

$$b_2+b_3+...+b_{k-2n-m+2}=m$$
$$2b_2+3b_3+...+(k-2n-m+2)b_{k-2n-m+2}=k-2n$$

where $\delta_1 = \frac{\alpha_1^2}{4a_i^2}, \delta_2 = \frac{\alpha_2^2}{4a_i^2}, ..., \delta_{k-2n-m+2} = \frac{\alpha_{k-2n-m+2}^2}{4a_i^2}, \quad i = 1,2$

for the second term we have analogously.

Then from condition (16) we get the following recurrent formulas for determine coefficients $A_n, B_n, C_n$ and $D_n$

$$\sum_{n=0}^{k} A_n (2a_1)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} (-1)^m [L_{n-m}^{(\mu-1-m)}(-\delta_1)] \sum_{b_i} \frac{(k-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2! b_3! ... b_{k-2n-m+2}!} +$$
$$+ \sum_{n=0}^{k} B_n (2a_1)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} \frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)} \delta_1^{1-\mu-m} \sum_{b_i} \frac{(k-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2! b_3! ... b_{k-2n-m+2}!} \cdot \tag{26}$$
$$\cdot \sum_{l=0}^{k-2n-m} [\Phi(1-\mu-n, 2-\mu, -\delta_1)]^{(l)} \sum_{b_i} \frac{(k-2n-m)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2! b_3! ... b_{k-2n-m-l+2}!} = 0$$

$$\sum_{n=0}^{k} C_n (2a_2)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} (-1)^m [L_{n-m}^{(\mu-1-m)}(-\delta_1)] \sum_{b_i} \frac{(k-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2! b_3! ... b_{k-2n-m+2}!} +$$
$$+ \sum_{n=0}^{k} D_n (2a_2)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} \frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)} \delta_1^{1-\mu-m} \sum_{b_i} \frac{(k-2n)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2! b_3! ... b_{k-2n-m+2}!} \cdot \tag{27}$$
$$\cdot \sum_{l=0}^{k-2n-m} [\Phi(1-\mu-n, 2-\mu, -\delta_1)]^{(l)} \sum_{b_i} \frac{(k-2n-m)! \delta_2^{b_2} \delta_3^{b_3} ... \delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2! b_3! ... b_{k-2n-m-l+2}!} = 0$$

As coefficient $A_n$ is known from (22), then by making substitution to (26) we can find coefficient $B_n$. From system of equations (24) and (27) we can determine the coefficients $C_n$ and $D_n$

$$B_n = -\frac{f^{(n)}(0)\xi_1}{n!(2a_1)^{2n} \binom{n+\mu-1}{n} \xi_2} \tag{28}$$

$$C_n = \frac{\varphi^{(2n)}(0) n!}{(2n)!(-1)^n} - D_n \frac{n!}{(-1)^n \Gamma(n+\mu)}, \tag{29}$$

$$D_n = \frac{\varphi^{(2n)}(0) n! \xi_3}{(2n)!(-1)^{n+1} \left( \xi_4 - \frac{n!}{(-1)^n \Gamma(n+\mu)} \right)} \tag{30}$$

where $n = 1, 2, ...$ and

$$\xi_{1(3)} = (2a_{1(2)})^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} [L_n^{(\mu-1)}(-\delta_1)]^{(m)} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!},$$

$$\xi_{2(4)} = (2a_{1(2)})^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} \frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)} \delta_1^{1-\mu-m} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!} \cdot$$
$$\cdot \sum_{l=0}^{k-2n-m} [\Phi(1-\mu-n, 2-\mu, -\delta_1)]^{(l)} \sum_{b_i} \frac{(k-2n-m)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}.$$

From Stefan's condition (18) and (24) we have the recurrent formula for free boundary

$$\alpha_k = \frac{\lambda_2}{L\gamma k!}\Bigg[\sum_{n=0}^{k} C_n (2a_2)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} [L_{nr}^{(\mu-1)}(-\delta_1)]^{(m)} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!} +$$
$$+\sum_{n=0}^{k} D_n 2^{2n-1} a_2^{2n-2} \frac{k!(1-\mu)}{(k-2n+2)!} \sum_{m=0}^{k-2n+2} \binom{k-2n+2}{m} \frac{(-1)^m \Gamma(\mu+1)}{\Gamma(1+\mu-m)} \delta_1^{-\mu-m} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!} \cdot$$
$$\cdot \sum_{l=0}^{k-2n-m+2} \binom{k-2n-m+2}{l} \beta_1^{(l)} \sum_{b_i} \frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}$$
$$\cdot \sum_{i=0}^{k-2n-m-l+2} [\Phi_r(1-\mu-n, 2-\mu, -\delta_1)]^{(i)} \sum_{b_i} \frac{(k-2n-m-l+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l-i+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l-i+2}!}\Bigg] -$$
$$-\frac{\lambda_1}{L\gamma k!}\Bigg[\sum_{n=0}^{k} A_n (2a_1)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} [L_{nr}^{(\mu-1)}(-\delta_1)]^{(m)} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!} +$$
$$+\sum_{n=0}^{k} B_n 2^{2n-1} a_2^{2n-2} \frac{k!(1-\mu)}{(k-2n+2)!} \sum_{m=1}^{k-2n+2} \binom{k-2n+2}{m} \frac{(-1)^m \Gamma(\mu+1)}{\Gamma(1+\mu-m)} \delta_1^{-\mu-m} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!} \cdot$$
$$\cdot \sum_{l=0}^{k-2n-m+2} \binom{k-2n-m+2}{l} \beta_1^{(l)} \sum_{b_i} \frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!} \cdot$$

$$\cdot \sum_{l=0}^{k-2n-m+2} \binom{k-2n-m+2}{l} \beta_1^{(l)} \sum_{b_i} \frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}$$
$$\cdot \sum_{i=0}^{k-2n-m-l+2} [\Phi_r(1-\mu-n, 2-\mu, -\delta_1)]^{(i)} \sum_{b_i} \frac{(k-2n-m-l+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l-i+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l-i+2}!}\Bigg] \quad (31)$$

We can find coefficient $A_n, B_n, C_n$ and $D_n$ from (22), (28)-(30) and free boundary we can determine from (31).

### 3. Convergence of series

Convergence of series (20)-(21) can be proved as following. Let $\alpha(t_0) = \eta_0$ for any $t = t_0$. Then series (20) can be written as

$$\theta_1(r,t_0) = \sum_{n=0}^{\infty} A_n \left(4a_1^2 t_0\right)^n L_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2 t_0}\right) + \sum_{n=0}^{\infty} B_n \left(4a_1^2 t_0\right)^n \left(\frac{\eta_0^2}{4a_1^2 t_0}\right)^{1-\mu} \Phi\left(1-\mu-n, 2-\mu, -\frac{\eta_0^2}{4a_1^2 t_0}\right) \quad (32)$$

The series (20) and (21) must be convergence because $\theta_1(r,t)=\theta_2(r,t)=\theta_m$. Then there exists some constant $E_1$ independent of $n$ and for the first term of (32) we have

$$|A_n|< E_1/(4a_1^2t_0)^n L_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2t_0}\right) \tag{33}$$

Since $A_n$ bounded, then multiply both sides of (33) by $(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)$ we obtain

$$\sum_{n=0}^{\infty} A_n(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)< E_1\sum_{n=0}^{\infty}\frac{(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)}{(4a_1^2t_0)^n L_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2t_0}\right)}< E_1\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n \tag{34}$$

Similarly, for the second term of (33) we have some constant $E_2$ which satisfy

$$|B_n|< E_2/\left(4a_1^2t_0\right)^n\left(\frac{\eta_0^2}{4a_1^2t_0}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\eta_0^2}{4a_1^2t_0}\right) \tag{35}$$

Analogously, if we multiple both sides of (35) by $(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)$

we get

$$\sum_{n=0}^{\infty} B_n(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)$$
$$< E_2\sum_{n=0}^{\infty}\frac{(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)}{\left(4a_1^2t_0\right)^n\left(\frac{\eta_0^2}{4a_1^2t_0}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\eta_0^2}{4a_1^2t_0}\right)}< E_2\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n \tag{36}$$

These geometric series and $\theta_1(r,t)$ convergence for all $r<\mu_0$ and the same $\theta_2(r,t)$ convergence for all $r>\mu_0$ and $t<t_0$. Convergence for $\alpha(t)$ can be determined analogously from (31).

## 4. Conclusion

A mathematical model of describing heat distribution for generalized heat in electrical contacts on liquid and solid zone is constructed by two-phase Stefan problem. Temperature for liquid zone $\theta_1(r,t)$ and for solid zone $\theta_2(r,t)$ which given in the form summation two special functions as Laguerre polynomial and confluent hyper-geometric function are determined and their coefficients $A_n, B_n, C_n$ and $D_n$ are founded from equations (22) and (28)-(30) and free boundary on melting isotherm is described in recurrent formula (31). The convergence of series is proved.

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## ЖАЛПЫЛАНҒАН ЖЫЛУ ТЕҢДЕУІ ҮШІН ЕКІ ФАЗАЛЫ СТЕФАН ЕСЕБІ

**Аннотация.** Жалпыланған жылу теңдеуі температуралық градиенттің радиалды компонентін оның осьтік компонентімен салыстырғанда елемеуге болатын ауыспалы қимасы бар денелердегі жылу беруді модельдеу үшін маңызды. Мұндай модельдерді электр байланыстарындағы жылу және масса алмасу теориясында қолдануға болады. Атап айтқанда, ашық электр түйіспелерінде пайда болатын сұйық металл көпіріндегі температуралық өрісті жалпыланған жылу теңдеуі үшін жоғарыда қарастырылған Стефан есебімен модельдеуге болады. Сұйық көпірдің температуралық өрісі жалпыланған жылу теңдеуімен, ал қатты жанасу аймағының температурасы сфералық түрде модельденген кездегі нақты шешім радиалды термиялық полиномдар және қателіктердің интегралдық функциялары түрінде ұсынылуы мүмкін. Осы қатарлардың коэффициенттерінің қайталану формулалары ҚР ҰҒА-ның «Известия» физика-математикалық сериясында жарияланған еңбектерде келтірілген.

Бұл мақалада фазалардың біреуі бастапқы уақытта бір нүктеге төмендеген жағдайда жалпыланған жылу теңдеуі үшін екі фазалы Стефан есебі қарастырылады. Бұл есепті интегралдық теңдеулерге келтірудің стандартты әдісі арқылы шешуге айтарлықтай қиындықтар туғызады, өйткені бұл жағдайда теңдеулер жеке сипатқа ие болады. Бұл жұмыста бастапқы және шекаралық шарттарда пайда болатын функциялар аналитикалық болып, оларды Тейлор қатарына кеңейту мүмкін болған жағдайда біз басқа әдісті қолданамыз. Бұл жағдайда мәселенің шешімі белгісіз коэффициенттері бар арнайы функциялардағы қатарлар түрінде ұсынылуы мүмкін (Лагуерр полиномиясы және дегенеративті гипергеометриялық функция). Бұл арнайы функциялар Розенблум және Д.В. Виддер енгізген жылу полиномдарымен тығыз байланысты.

Алынған қатар априорлы түрде жылу теңдеуін қанағаттандырады, сондықтан бастапқы және шекаралық шарттарды, сондай-ақ еркін шекара үшін Стефан шартын қанағаттандыратын коэффициенттерді табу керек. Бұл тәсіл өте пайдалы болып көрінеді, өйткені шекаралық жағдайлар тек кейбір қателіктермен ғана қанағаттандырылса да, жылу теңдеуінің максималды принципі бойынша шешім қателігі шекара жағдайындағы қателіктен аспауы керек. Бұл ерітіндіні қабылдағанға дейін оны жақындатудың бағасын алуға мүмкіндік береді. Фа-ди-Бруно формуласын қолданып, бастапқы және шекаралық шарттардағы функцияларды қатарға көбейтіп, коэффициенттерді бір уақытта ретке келтіре отырып, біз белгісіз коэффициенттерді іздеу үшін қайталанатын теңдеулер жүйесін аламыз. Еркін шекараның коэффициенттері үшін теңдеулердің ұқсас жүйесін Стефан шартын қолдана отырып табуға болады. Алынған қатарлардың конвергенциясы еркін шекарадағы тұрақты температура жағдайын қолдана отырып дәлелденді. Бұл жағдай серияның геометриялық прогрессиямен масштабталуына және қатардың жақындасуы үшін қажетті бағаларды алуға мүмкіндік береді.

**Түйін сөздер:** Стефан есебі, арнайы функциялар, Лагерра полиномы, Фаа-ди Бруно формуласы.

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## ДВУХФАЗНАЯ ЗАДАЧА СТЕФАНА ДЛЯ ОБОБЩЕННОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

**Аннотация.** Обобщенное уравнение теплопроводности имеет важное значение для моделирования теплообмена в телах с переменным поперечным сечением, когда радиальной составляющей градиента температуры можно пренебречь по сравнению с ее осевой составляющей. Такие модели могут быть использованы в теории тепло- и массопереноса в электрических контактах. В частности, температурное поле в жидкометаллическом мостике, возникающем в размыкающихся электрических контактах, может быть моделировано рассмотренной выше задачей Стефана для обобщенного уравнения теплопроводности. Точное решение для случая, когда температурное поле жидкого мостика моделируется обобщенным уравнением теплопроводности, в то время как температура твердой контактной зоны моделируется в сферическом варианте, может быть представлено в виде рядов по радиальным тепловым полиномам и интегральным функциям ошибок. Рекуррентные формулы для коэффициентов этих рядов даны в работах, опубликованных ранее в «Известиях НАН РК, серия физико-математическая».

В этой статье рассматривается двухфазная задача Стефана для обобщенного уравнения теплопровод-ности для случая, когда одна из фаз в начальный момент времени вырождается в точку. Это создает серьезные трудности для решения задачи стандартным методом ее сведения к интегральным уравнениям, поскольку уравнения в этом случае становятся сингулярными. В данной работе используется другой метод для случая, когда функции, фигурирующие в начальных и граничных условиях, являются аналитическими и могут быть разложены в ряды Тейлора. В этом случае решение задачи можно представить в виде рядов по специальным функциям (многочлены Лагерра и вырожденная гипергеометрическая функция) с неопределен-ными коэффициентами. Эти специальные функции имеют тесную связь с тепловыми полиномами, введенными П.С. Розенблюмом и Д.В. Уиддером.

Построенные ряды априори удовлетворяют уравнению теплопроводности, и нужно найти их коэффициенты, удовлетворяющие начальному и граничному условию, а также условию Стефана для свободной границы. Такой подход представляется весьма полезным, потому что даже если граничные условия выполняются лишь приближенно с некоторой ошибкой, то ошибка решения согласно принципу максимума для уравнения теплопроводности должна быть не больше, чем ошибка в граничных условиях. Это дает возможность получить оценку приближения решения до его получения. Используя формулу Фаа-ди Бруно и разлагая функции в начальных и граничных условиях на ряды и приравнивая коэффициенты в одном и том же порядке времени, можно получить систему рекуррентных уравнений для поиска неизвестных коэффициентов. Аналогичная система уравнений для коэффициентов свободной границы может быть найдена с использованием условия Стефана. Сходимость полученных рядов доказывается, используя условие постоянства температуры на свободной границе. Это обстоятельство позволяет мажорировать ряды геометрической прогрессией и получить оценки, необходимые для сходимости ряда.

**Ключевые слова:** задача Стефана, специальные функции, полином Лагерра, формула Фаа-ди Бруно.

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# ON COMPLETENESS OF ROOT VECTORS OF THE CAUCHY PROBLEM OF THE FIRST ORDER EQUATION WITH DEVIATING ARGUMENT

**Abstract.** The main objective of this paper is to study the conditions under which the system of finite-dimensional invariant (root) subspaces of the operator A turns out to be complete in H or in the range of the operator.

In the case of a general completely continuous operator, completeness may not take place. The simplest example of this kind is the integration operator

$$Af = \int_a^x f(t)dt, a \le x \le b ,$$

which acts in the Hilbert space of square-integrable functions on the interval (a, b). In what follows, we will denote this space by $L^2(\mathrm{a,b})$.

In the present work, by the method of M.V. Keldysh, the completeness of the system of root vectors of the operator corresponding to the Cauchy problem of a first-order equation with a variable coefficient and a deviating argument is proved. The Volterra operator of the ordinary differential equation corresponding to the Cauchy problem is well known, so the result is in sharp contrast with the known facts. It can be expected that the results obtained will find application in theoretical physics, the theory of signal transmission, especially in fiber optic communications. Since the coefficient of the equation is not assumed to be real, the corresponding operator is not self-adjoint; therefore, questions of basicity were not considered. The result obtained is formulated in terms of the coefficient of the equation, and is close to the necessary, which is confirmed by the constructed example. This condition is a consequence of the method used; perhaps with other methods they can take a different look.

Note that the square of the operator A generates a sheaf of operators in the space $L^2(0,1)$; therefore, the results of the paper are of interest also for the theory of sheaf.

**Keywords.** Spectrum, deviating argument, root subspace, completeness, Keldysh theorem, compactness, Hilbert-Schmidt theorem, Green function, resolvent, sheaf of Keldysh operators.

**1.** One of the central concepts of the spectral theory of linear operators is completeness of the system of its root vectors. In this paper, we consider a linear non-self-adjoint operator acting in a separable Hilbert space $H$ and having a discrete spectrum. The latter means that all points of the spectrum of the operator $A$ (with the possible exception of one) are isolated and the corresponding subspaces are finite-dimensional. A finite-dimensional invariant subspace of the operator $A$ related to some point $\lambda_s$ of the spectrum $\sigma(A)$ is usually called the root subspace. We will denote it by $R_s$. The root subspace $R_s$ can be characterized as a set of elements $f$ satisfying for some integer $m \ge 1$ the following equation

$$(A - \lambda_s E)^m f = 0. \tag{1}$$

Discrete spectrum, as well known, is possessed by completely continuous operators, as well as unbounded (for example, differential) operators having completely continuous inverse operators. In fact, we consider only such operators.

The main objective of this paper is to study conditions under which the system of finite-dimensional invariant (root) subspaces of the operator $A$ turns out to be complete in $H$ or in the range of the operator.

We explain that a system of finite-dimensional invariant subspaces of an operator is commonly called complete in a Hilbert space $H$ if any element $h \in H$ can be approximated with a predetermined accuracy by the norm by a finite linear combination of elements, each of which belongs to one of the invariant subspaces. It is well known that if some completely continuous operator is self-adjoint, then the system of its finite-dimensional invariant subspaces is complete in the range of the operator (in this case, the root subspaces turn out to be eigenvalues, in the formula (1) $m = 1$).

In the case of a general completely continuous operator, completeness may not take place. The simplest example of this kind is the integration operator

$$Af = \int_a^x \mathrm{f(t)}\, dt, a \le x \le b, \tag{2}$$

which maps in a Hilbert space of functions having a Lebesgue integrable square on the interval $(a, b)$. Furthermore, we denote this space by $L_2(a, b)$. It is easy to verify that the operator (2), being completely continuous, has only a single point in the spectrum – zero, and does not have any eigenvectors. Consequently, it does not have finite-dimensional invariant subspaces.

Although the question about completeness of the system of root subspaces for operators with a discrete spectrum has long been considered in a large number of works, a decisive step on this path was made only in 1951 by M. Keldysh in the fundamental paper [1], where he proved the general theorem, in which completeness in a large number of boundary value problems for various series of equations with ordinary and partial derivatives was established. In his research, M.V. Keldysh relied on the results of his previous authors [1] - [7]. After this work, a number of papers [2] - [20], devoted to this theme, appeared. After 20 years, M.V. Keldysh continued his research [21]. This paper contains a detailed exposition of the first part of the work “On eigenvalues and eigenfunctions of some classes of non-self-adjoint equations” published by the author in [1]. According to the author, content of the paper was reported in 1951 at the meeting of the Moscow Mathematical Society. Then its manuscript was made available by the author to a number of mathematicians.

Note that the second part of this work did not appear, apparently, it was lost in archives of his students. Works in this direction are published today. In development of this direction in the theory of non-self-adjoint operators, the present paper is also written.

Compact operator $A$ in a linear topological space $E$ is called complete, if the system $\sum_0(A)$ of root vectors, corresponding to eigenvalues, which are nonzero, is complete in $JmA$. In the case $\ker A = 0$, it means that for the operator $A^{-1}: JmA \to E$ system of root vectors is complete in the domain (and complete in $E$, if $D(A^{-1}) = JmA$ is possible). In usual concrete situations original operator is $A^{-1}$, for example, $A^{-1} = L$ is a differential operator, $A -$ is an integral operator, generated by the Green’s function of the operator $L$. Any self-adjoint and compact operator $A$ in a Hilbert space is complete. It is natural to expect that for small perturbations of such operators the completeness is preserved. This expectation is justified if eigenvalues of the unperturbed operator quickly tend to zero. As noted above, the first general result in this direction belongs to I.V. Keldysh [1].

**Keldysh Theorem [1].** Suppose that operator $A$ in a Hilbert space $H$ has the form $A = (I + R)S$, where $S, R$ - are compact operators, moreover $S -$ is self-adjoint and its eigenvalues $\lambda_n \neq 0$ $(n = 0,1,2, \dots)$, taking into account the multiplicities, satisfy the condition

$$\sum_{n=0}^{+\infty} \left|\lambda_n\right|^p \prec \infty$$

at some $p > 0$. Then the operator $A$ - is complete.

The aim of this paper is to study the operator of the Cauchy problem for an equation with a deviating argument for completeness.

**2. Research Methods.**

Let $H = L^2(0,1)$ be a Hilbert space, and $q(x)$ be a continuous function. We consider the operator

$$Ay = S\frac{d}{dx}y + Qy, D(A) = \{y \in C^1(0,1) \cap [0,1], y(0) = 0\} \tag{2}$$

Where $S$, $Q$ are also operators, given by the formulas:

$$Sy(x) = y(1-x), \quad Qy = q(x)\cdot y(x) \tag{3}$$

The question is whether the operator $A^{-1}$ will be complete in $H$. To answer this question, we study the operator $A^{-1}$ according to the scheme of the Keldysh theorem.

Let

$$A_0 = SL \tag{4}$$

Where $L = \frac{d}{dx}$, $D(L) = \{y \in C^1(0,1) \cap C[0,1], y(0) = 0\}$.

Then we get:

$$A = A_0 + Q = A_0\left[I + A_0^{-1}Q\right] = \left[I + QA_0^{-1}\right]\cdot A_0,$$

$$A^{-1} = \left[I + A_0^{-1}Q\right]^{-1}\cdot A_0^{-1}, \tag{5}$$

$$\left[I + A_0^{-1}Q\right]^{-1} = \sum_{m=0}^{+\infty}(-1)^m\left(A_0^{-1}Q\right)^m = I + \sum_{m=1}^{\infty}(-1)^m\left(A_0^{-1}Q\right)^m. \tag{6}$$

These equalities are true if $A_0^{-1}$ exists and the inequality $\|A_0^{-1}Q\| < 1$ holds, therefore we study these operators in detail.

We consider the spectral problem

$$A_0 y = \lambda y \tag{7}$$

for them in expanded form

$$\begin{cases} -y'(1-x) = \lambda y(x) \\ y(0) = 0 \end{cases} \tag{8)-(9}$$

the following lemma holds.

**Lemma 2.1.** Spectral problem (8) +(9) has an infinite set of eigenvalues

$$\lambda_n = (-1)^n\left(n\pi + \frac{\pi}{2}\right), \quad (n = 0,1,2,\ldots) \tag{10}$$

and the corresponding eigenfunctions

$$u_n(x) = \sqrt{2}\cdot\sin\left(n\pi + \frac{\pi}{2}\right)x \tag{11}$$

which form an orthonormal basis in the space $L^2(0,1)$.

**Lemma 2.2.**

(a) Operator $A_0^{-1}$ is self-adjoint and almost continuous;

(b) For any $p > 1$ we have

$$\sum_{n=0}^{+\infty}\frac{1}{|\lambda_n|^p} \prec +\infty,$$

where $\lambda_0^{-1} -$ are eigenvalues of the operator $A_0^{-1}$.

(c) $\|A_0^{-1}\| \le \frac{2}{\pi}$.

**Proof.**

(a) We show that if an operator $A_0$ is symmetric, then the operator $A_0^{-1}$ is also symmetric; since it is defined throughout the space $H$, then it is self-adjoint.

(b) Due to Lemma 1, we have

$$\sum_{n=0}^{\infty}\frac{1}{|\lambda_n|^p}=\sum_{n=0}^{+\infty}\frac{1}{\left(n\pi+\frac{\pi}{2}\right)^p}=\sum_{n=0}^{\infty}\frac{1}{\pi^p\left(n+\frac{1}{2}\right)^p}\le\left(\frac{2}{\pi}\right)^p+\sum_{n=1}^{\infty}\frac{1}{\pi^p n^p}\prec+\infty.$$

(c) Norm of the integration operator is known [3], it is equal, therefore

$$\left\|A_0^{-1}\right\|=\left\|L^{-1}S^{-1}\right\|=\left\|L^{-1}S\right\|\le\left\|L^{-1}\right\|\cdot\|S\|=\left\|L^{-1}\right\|=\frac{2}{\pi}.$$

Now we estimate norm of the operator $A_0^{-1}Q$.

$$\left\|A_0^{-1}Q\right\|\le\left\|A_0^{-1}\right\|\cdot\|Q\|\le\frac{2}{\pi}\cdot\|Q\|,$$

$$\|Qy\|^2=\int_0^1|q(t)y(t)|^2\,dt=\int_0^1|q|^2\cdot|y(t)|^2\,dt\le\left[\max_{0\le x\le1}|q|\right]^2\cdot\|y\|^2,$$

$$\|Qy\|\le\max_{0\le x\le1}|q|\cdot\|y\|,\Rightarrow\|Q\|\le\max_{0\le x\le1}|q|.$$

Thus,

$$\left\|A_0^{-1}Q\right\|\le\frac{2}{\pi}\cdot\max_{0\le x\le1}|q|. \tag{12}$$

**Lemma 2.3.** If

$$\max_{0\le x\le1}|q|\prec\frac{\pi}{2}, \tag{13}$$

then the operator

$$R=\sum_{m=1}^{+\infty}(-1)^m\left(A_0^{-1}Q\right)^m \tag{14}$$

is almost continuous.

**Proof.** From the conditions (12)+(13) it follows that the series (14) converges uniformly; since each term of this series is almost continuous, then the sum is also almost continuous operator. The proved Lemmas imply Theorem 3.1.

**3. Research Results.**

**Theorem 3.1.** If $\max_{0\le x\le1}|q|\prec\frac{\pi}{2}$, then the operator $A$

$$A=S\frac{d}{dx}+Q, D(A)=\{y\in C^1[0,1]\cap[0,1], y(0)=0\},$$

where $Sy(x)=y(1-x), Qy=q(x)-y(x)$ is complete in the space $H=L^2(0,1)$.

Our operator $A$ is invertible, therefore its range is whole space $H=L^2(0,1)$, consequently, Theorem 3.2 also holds.

**Theorem 3.2.** If $q(x)$ is a complex continuous function, satisfying the condition

$$\max_{0\le x\le1}|q(x)|\prec\frac{\pi}{2},$$

then the system of root vector of the operator

$$A=S\frac{d}{dx}+Q, D(A)=\{y\in C^1(0,1)\cap[0,1], y(0)=0\},$$

where $Sy(x)=y(1-x), Qy=q(x)\cdot y(x)$ is complete in the space $H=L^2(0,1)$.

If $q(x)$ is a real function, then the operator $Q$:

$$Qy=q(x)\cdot y(x)$$

will be self-adjoint in $H = L^2(0,1)$. Then the operator

$$A = A_0 + Q$$

Is also self-adjoint. From

$$A^{-1} = \left[I + A_0^{-1}Q\right]^{-1} \cdot A_0^{-1}$$

it follows that the operator $A^{-1}$ is almost continuous, moreover if $A^{-1}f = 0$, then $f = AA^{-1}f = 0$. Therefore $kerA^{-1} = 0$. We have proved the following Theorem 3.3.

**Theorem 3.3.** If $q(x)$ is a real continuous function, satisfying the condition

$$\max_{0\le x\le 1}|q(x)| \prec \frac{\pi}{2},$$

then the system of orthonormal eigenvectors of the operator $A$:

$$Ay = y'(1-x) + q(x)y(x), \quad D(A) = \{y \in C^1(0,1) \cap [0,1], y(0) = 0\}$$

forms an orthonormal basis in the space $H = L^2(0,1)$.

**Proof** follows from the Hilbert-Schmidt theorem [36], and the above discussions.

**4. Discussion.**

**Remark 4.1.** Condition

$$\max_{0\le x\le 1}|q(x)| \prec \frac{\pi}{2}$$

provided invertibility of the operator $A$, as the following example shows, it may not be invertible. Then the application of the Keldysh theorem becomes difficult.

**Example 4.1.** If

$$q(x) = -\frac{\pi}{2}, x \in [0,1]$$

then the equation

$$y'(1-x) + q(x)y(x) = 0$$

has a nontrivial solution. In fact, solution to this equation is the function

$$y(x) = \sin\frac{\pi x}{2},$$

which can be verified by direct calculation:

$$y'(x) = \frac{\pi}{2}\cos\frac{\pi x}{2}, \quad y'(1-x) = \frac{\pi}{2}\cos\frac{\pi}{2}(1-x) = \frac{\pi}{2}\left[\cos\frac{\pi}{2}\cdot\cos\frac{\pi x}{2} + \sin\frac{\pi}{2}\cdot\sin\frac{\pi x}{2}\right] = \frac{\pi}{2}\cdot\sin\frac{\pi x}{2},$$

$$y'(1-x) + q(x)y(x) = \frac{\pi}{2}\cdot\sin\frac{\pi x}{2} - \frac{\pi}{2}\cdot\sin\frac{\pi x}{2} = 0.$$

**Theorem 4.1.** If symmetric operator $A$ has a complete system of eigenvectors, then closure of this operator $\bar{A}$ is self-adjoint in $H$, in other words, the operator $A$ is self-adjoint in essential.

**5. Conclusion.** If $q(x)$ is a real continuous function, satisfying the condition

$$\max_{0\le x\le 1}|q(x)| \prec \frac{\pi}{2}$$

then

(***a***) system of orthonormal eigenvectors of the operator $A$:

$$Ay = y'(1-x) + q(x)y(x), \quad D(A) = \{y \in C^1(0,1) \cap [0,1], y(0) = 0\}$$

forms an orthonormal basis in the space $H = L^2(0,1)$.

(***b***) operator $A$ is self-adjoint in essential.

The latter property is very important, see [21].

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# АРГУМЕНТІ АУЫТҚЫҒАН, БІРІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУДІҢ КОШИЛІК ЕСЕБІНІҢ ТҮПКІ ВЕКТОРЛАРЫНЫҢ ТОЛЫМДЫЛЫҒЫ ТУРАЛЫ

**Аннотация.** Сыңарлы операторлардың спектрәлді теориясы ретінде, олардың спектрәлдік қасиеттерін зерттеуге арналған кең ауқымды сұрақтар тізімі танылады, мысалы, спекрдің орналасу тәртібі мен асимтотикасы, түпкі векторлар тізімінің толымдылығы, түпкі векторлар системасының базистігі, түпкі векторлардан жасақталған қатарлардың жинақтылығын В. Б. Лидскийдің әдісі арқылы зерттеу. Кәделі мәселелерді зерттегенде, және дифференциалдық теңдеулер теориясында осыған ұқсас есептер кездеседі, бірақта, бұл сәтте оператор үшін емес оператор мәнді функциялар үшін. Операторлардың спектрәлдік теориясының бұл түрін, әрине, оператор-функциялардың (о.-ф.) спектрәлді теориясы деп атаған жөн. Оператор-функциялардың (о.-ф.) спектрәлдік қасиеттерін зерттеу есептерінің қажеттілігіне математиктердің назары өткен ғасырдың басында ауғанымен, о.-ф.-лардың абстракті теориясының негізгі нәтиежелерін тек 1951 жылы М. В. Келдыш алды, бұл еңбекте түпкі векторлар системасының n-ретті толымдылығы ұғымы енгізілді,және λ-ге полиномәлді тәуелді оератор-функциялардың түпкі векторлар системасының n-ретті толымдылығы дәлелденді, кейінірек мұндай операторлар М. В. Келдыштың операторлар шоғыры делінді. Бұл теорема оператор-дифференциәлдік тенделердің Кошилік есептерін Фүре әдісімен шешуге жол ашты. Оператор-дифференциәлдік теңдеулердің басқа есептерін зерттегенде, М. В. Келдыштің n-ретті толымдылық ұғымының жеткіліксіздігі байқалды, сондықтан, еселік толымдылықты зерттеу есептері пайда болды.

Түпкі векторлар системасының толымдылығы сызықтық операторлардың спектрәлдік теориясының негізгі ұғымдарының бірі. Бұл еңбекте гилберттің сепарабелді H кеңістігінде спектрі сирек сызықтық сыңарлы оператор қарастырылды. Бұл дегеніміз спектрдің бір нүктесінен басқа нүктелері оқшауланған, сондай ақ оларға сәйкес ішкеңістіктері сансалалы дегенді білдіреді. A операторының σ(A), спектрінің $\lambda_s$ нүктесіне сәйкес сансалалы инвариантты ішкеңістігін түпкі кеңістік деп атау қабылданған. Біз оны $R_s$ арқылы белгілейміз. Түпкі кеңістікті мына, $(A - \lambda_s E)^m f = 0$, $m \geq 1$ (1) теңдеудің шешімдер жиыны деп сипаттауға болады. Егер де m=1 болса, онда f векторы меншікті вектор болады, басқа жағдайда еншілес вектор деп аталады.

Әсіре үзіксіз операторлармен, керісі әсіре үзіксіз (мысалы, дифференциалдық) операторлардың спектрі сирек екені белгілі. Біздің операторымыз да осылардың қатарына жатады және біз тек сондай операторларды ғана қарастырамыз.

Бұл еңбектің негізгі есебі түпкі кеңістіктер системасының толымдылығын қамтамасыз ететін шарттарды табу және зерттеу, ең болмағанда, олар оператордың мәндерінің жиынында толық болуы шарт. Егер гилберттің H кеңістігінің әрбір h∊H элементін, әлгі түпкеңістіктерде жатқан элементтердің сызықтық комбинациясы арқылы, қалаған дәлдікпен, жуықтауға болса, онда әлгі сансалалы кеңістітер системасын осы H кеңістігінде толық деп санаймыз. Егер-де белгілі бір әсіре үзіксіз оператор жалқы болса, яғни оның сыңары оның өзі болса, онда оның сансалалы инвариантты ішкеңістіктер системасы, оның мендерінің жиынында толымды екені белгілі (бұл сәтте оның түпкі кеңістіктері меншікті болады, яғни (1) формулада m =1 болады).Сыңарлы операторлар үшін жағдай мүлдем басқаша мүмкін.

Толымдылық қасиеті әсіре үзіксіз операторлар үшін жалпылай емес. Оператордың әсіре үзіксіздігі оның толымды болуына жеткіліксіз. Жалпы жағдайда, толымдылық үшін оператордың әсіре үзіксіздігі жеткіліксіз. Мұның айқын мысалы ретінде, келесі,

$$Af = \int_a^x f(t)\,dt,\ a \leq x \leq b\ , \tag{2}$$

интегралдау операторын айтуға болады, бұл оператор мөділдерінің квадраттары Лебег бойынша $(a,\ b)$ интервалында интегралданатын функциялардың гилберттік кеңістігінде әрекет етеді. Бұл кеңістікті біз әрі қарай, былай $L_2(a,\ b)$ белгілейміз. Бұл (2) оператордың әсіре үзіксіз болғанына қарамастан, бірде бір меншікті мәні жоқ, оның спектрі тек нөл нүктесінен тұратынын тексеру қыйын шаруа емес. Демек оператордың сансалалы инвариантты кеңістіктері жоқ.

Бұл еңбекте М.В.Келдыштың әдісімен аргументі ауытқыған коэффициенті айнымалы бірінші ретті дифференциалдық теңдеудің Коши есебінің түпкі векторлар системасының толымдылығы көрсетілді. Бірінші ретті кәдімгі дифференциалдық теңдеудің Кошилік есебінің волтерлі екені көпке мәлім, сондықтан, бұл жай белгілі жайлардан ерекшеленеді. Алынған нәтиежелерді теориялық физика мен деректерді тарату теориясында, әсіресе оптикалық таралым жүйелерінде қолданыс табады деп күтуге болады.

Теңдеудің коэффициенті нақты болмағандықтан, есепке сәйкес оператор сыңарлы, сондықтан базистік туралы мәселелер назардан тыс қалды. Алынған нәтиежелер теңдеудің коэффициенті арқылы өрнектелді, және ол қажетті шартқа жақын, онысы нақты мысал арқылы дәйектелген..Бұл шарт қолданылған әдістің салдары,басқа әдіс қолданған сәтте оның түрі өзгеруі де мүмкін.

Қарастырылған оператордың квадраты $L^2(0,1)$ кеңістігінде операторлар шоғырын туындатады, сондықтан алынған нәтиежелер операторлар шоғырының теориясындада қызығушылық туғызуы мумкін.

**Түйін сөздер**. Спектр, ауытқыған аргумент, түпкі кеңістік, толымдылық, Келдыштың теоремасы,компэктілік, Гилберт-Шмидтің теоремасы, Гриннің функциясы, резөлвента.

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## О ПОЛНОТЕ КОРНЕВЫХ ВЕКТОРОВ ЗАДАЧИ КОШИ УРАВНЕНИЯ ПЕРВОГО ПОРЯДКА С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

**Аннотация.** Под спектральной теорией несамосопряженных операторов принято понимать широкий круг вопросов, связанных с изучением спектральных характеристик несамосопряженных операторов, например, исследование асимптотики и локализации спектра, полноты корневых векторов, базисов, составленных из корневых векторов, изучение возможности суммирования корневых векторов методом, предложенным В.Б. Лидским. Но во многих задачах, встречающихся в дифференциальных уравнениях и прикладных вопросах, возникает необходимость изучить аналогичные вопросы, но не для оператора, а для некоторой функции, принимающей значения во множестве операторов. Такое обобщение спектральной теории операторов естественно назвать спектральной теорией оператор-функций (о.-ф.). Хотя на необходимость исследования спектральных свойств о.-ф. внимание математиков было обращено еще в начале нашего века, тем не менее, первые основополагающие результаты в абстрактной теории о.-ф. были получены М. В. Келдышем. В работе, опубликованной в 1951 г., где введено важное понятие n-кратной полноты корневых векторов и доказана фундаментальная теорема об n-кратной полноте корневых векторов для полиномиально зависящих от $\lambda$ о.-ф., получивших впоследствии название пучков операторов М. В. Келдыша. Эта теорема обосновывает принципиальную возможность применения метода Фурье при решении задачи Коши для широкого класса операторно-дифференциальных уравнений. Исследование же других задач для операторно-дифференциальных уравнений диктует изучение кратной полноты корневых векторов, отличное от n-кратной полноты, рассмотренной М. В. Келдышем.

Одним из центральных понятий спектральной теории линейных операторов является полнота системы его корневых векторов. В настоящей работе рассматривается линейный несамосопряженный оператор, действующий в сепарабельном гильбертовом пространстве H, и обладающий дискретным спектром. Последнее означает, что все точки спектра оператора A (за исключением, быть может, одной) являются изолированными и соответствующие им подпространства конечномерны. Конечномерное инвариантное подпространство оператора A, относящееся к некоторой точке $\lambda_s$ спектра σ(A), принято называть корневым подпространством. Мы будем его обозначать через $R_s$. Корневое подпространство $R_s$ может быть охарактеризовано как совокупность элементов f , удовлетворяющих при некотором целом m≥ 1 уравнению $(A - \lambda_s E)^m f = 0$, m≥ 1. (1)

Дискретным спектром, как известно, обладают вполне непрерывные операторы, а также неограниченные (например, дифференциальные) операторы, имеющие вполне непрерывные обратные. По существу, только такие операторы мы и рассматриваем.

Основной задачей настоящей работы является исследование условий, при которых система конечномерных инвариантных (корневых) подпространств оператора A оказывается полной в H или в области значений оператора. Известно, что если некоторый вполне непрерывный оператор является самосопряженным, то система его конечномерных инвариантных подпространств полна в области значений оператора (при этом корневые подпространства оказываются собственными, в формуле (1) m =1).

В случае же общего вполне непрерывного оператора полнота может и не иметь места.

Простейшим примером такого рода служит оператор интегрирования

$$Af = \int_a^x f(t)dt, a \le x \le b , \tag{2}$$

который действует в гильбертовом пространстве функций, обладающих интегрируемым по Лебегу квадратом на интервале $(a, b)$. Это пространство мы будем в дальнейшем обозначать через $L_2(a, b)$. Нетрудно проверить, что оператор (2), будучи вполне непрерывным, обладает лишь единственной точкой спектра – нулем и не имеет ни одного собственного вектора. Следовательно, конечномерные инвариантные подпространства у него вообще отсутствуют.

В настоящей работе, методом М.В.Келдыша доказана полнота системы корневых векторов оператора, соответствующего задаче Коши уравнения первого порядка, с переменным коэффициентом, и отклоняю-щимся аргументом. Вольтерровость оператора соответствующего задаче Коши обыкновенного дифферен-циального уравнения общеизвестно, поэтому результат резко контрастирует с известными фактами. Можно ожидать, что полученные результаты найдут приложение в теоретической физике, теории передачи сигналов, особенно в

оптико-волоконной связи. Поскольку, коэффициент уравнения не предполагается вещественной, то соответствующий оператор не самосопряжен, поэтому вопросы базисности не рассматри-вались. Полученный результат сформулирован в терминах коэффициента уравнения, и близко к необходи-мому, что подтверждается построенным примером. Это условие является следствием использованного метода, возможно при других методах они могут принимать другой вид.

Отметим, что квадрат оператора A порождает пучок операторов в пространстве $L^2(0,1)$, поэтому результаты работы представляет интерес и для теории пучков.

**Ключевые слова.** Спектр, отклоняющийся аргумент, корневое подпространство, полнота, теорема Келдыша, компактность, теорема Гилберта-Шмидта, функция Грина, резольвента.

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# ABOUT ONE INVARIANT MULTIVALUED MAPPING IN THE TASK OF THERMAL CONDUCTIVITY WITH TIME DELAY

**Abstract**. This paper considers a matter on the strong and weak invariance of the constant multi-valued mapping towards thermal conductivity equation with boundary control in the presence of time delay. Sufficient conditions for the strong and weak invariance of this multi-valued mapping were obtained for the heat transfer control task with a delayed argument with boundary and initial conditions. The concept of "invariant sets" is applied to a system with distributed parameters, the physical meaning of which is to "keep" the object in the desired state as long as possible by controlling it. At the same time, here object retention is understood not in the geometric sense, but in the sense of holding the average value relative to the volume of the object. The necessary conditions for keeping the object in the desired state are proposed. To solve the equations, we first expand the definition of the elliptic operator and the operator itself to a self-adjoint operator, and then consider the existence of a solution that belongs to the energy space of this operator. This uses the fact that the extended operator has generalized eigenvalues and generalized eigenfunctions that make up the complete system both in the energy space of the operator and in each space. In this case, a generalized solution is understood as a solution, because it is represented as a Fourier series whose coefficients satisfy infinite ordinary differential equations. Just in this system of differential equations there is a control parameter. An essential point in considering this task is that the controls are located on the border of the area. In this case, the control area is a convex compact polyhedron, and the restriction area and the terminal set are half-spaces. This, in certain conditions, allows the possibility of applying the obtained results in solving practical tasks.

**Keywords**: invariant set, control, multi-valued mapping, control of systems with distributed parameters, time delay.

## 1. Introduction

Note that there are theoretical and practical issues in the controllable systems with distributed parameters, incapable of solution with the help of known methods. Typical examples of such kind of tasks are retention of temperature in the acceptance limits in a specified volume, avoidance of undesirable conditions, etc.

Results in the matter about invariance of specified sets concerning systems with lumped parameters were obtained earlier in the works of A. Feuer, M. Heymann, V.N. Ushakov, Kh.G. Gusseinov, N.S. Rettiyev, A.Z. Fazylov and other authors [1-7]. As against the work [7], this paper considers problems with boundary control. The works [7] consider interesting applied problems on the control by convectors for heat distributions in a volume. As against the work, this paper considers tasks with boundary control.

The theory of differential equations with a deviating argument has been developed in various directions; natural formulations of tasks have been found [8–12].

As you know, many real controlled objects can be considered as objects with distributed parameters, in which control parameters can be located both on the right side of the equation and in boundary conditions.

In the work [13-15] considers tasks with distributed parameters.

Invariant sets with respect to a system with distributed parameters, the physical meaning of which is to keep the object in the desired state as long as possible by controlling it. Moreover, here the retention of the object is understood not in the geometric sense, but in the sense of holding the average value relative to the volume of the object. The works [16-19] considers questions of the invariance of given sets with respect to systems with distributed parameters. In particular, the paper [17] is devoted to the questions of the strong and weak invariance of a constant multivalued mapping with delay, and the paper [18] is devoted to a controlled process, which is described by an equation of parabolic type, with the control parameter participating in the additive form on the right side of the equation. This work [19] introduces concept of the invariance of a multi-valued mapping with respect to a system with distributed parameters. In [20], a numerically approximate method for solving of a control problem for integro-differential equations of parabolic type was considered.

This work considers a matter on the strong and weak invariance of the constant multivalued mapping towards thermal conductivity equation with boundary control in the presence of time delay.

A bounded region $\Omega\left(\subset R^n\right)$ will be called a piecewise-smooth boundary, if a boundary $\Gamma=\overline{\Omega}\setminus\Omega$ of the region $\Omega$ can be presented in the form $\Gamma=\sum_{j=1}^{N}\bar{A}j$, where $\Gamma_j\subset\Gamma$ - the set open towards topology, induced on $\Gamma$ by the topology $R^n$. Each $\Gamma_j$ - a connected surface of class $C^1$, i.e. for each point $x_0\in\Gamma_j$ it is possible to indicate ball $U_\varepsilon(x_0)$ by so small radius $\varepsilon>0$, that the set $\Gamma_j\cap U_\varepsilon(x_0)$ is defined by equation of the type $x_k=f_k(x_1,...,x_{k-1},x_{k+1},...,x_n)$, where $f_k(\cdot)\in C^1$, $k,1\le k\le n$ is a some number.

Let $\Omega$ be the bounded region in $R^n$ with the piecewise-smooth boundary. Through $A$ let us denote the next differential operator [21]:

$$A\varphi=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}), \tag{1}$$

where functions $a_{ij}(x)\in L^\infty(\Omega)$ satisfy conditions: $a_{ij}(x)=a_{ji}(x)$ and

$$\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\ge\gamma\sum_{i=1}^{n}\xi_i^2,\ x\in\Omega \tag{2}$$

for any $(\xi_1,\xi_2,...,\xi_n)\in R^n$. The inequality (2) is called a condition of uniform ellipticity of operator $A(1)$. As the domain of operator $A$ space $\dot{C}^2(\Omega)$ is taken – the set of twice continuously differentiable functions.

Let $P$ be operator, which is defined by the inequality

$$P\varphi=\sum_{i,j=1}^{n}a_{ij(x)}\frac{\partial\varphi}{\partial x_i}\cos(l,x)+k(x)\varphi,\quad x\in\partial\Omega,$$

where $l$ - the unit vector of outer normal to $\partial\Omega$, $k(x)$ - given positive, continuous function in $\partial\Omega$.

It is known that the elliptic operator A with the boundary condition $P\varphi=0$, $x\in\partial\Omega$, has a discrete spectrum, i.e. eigenvalues $\lambda_k$ such that $0<\lambda_1\le\lambda_2\le...,\lambda_k\to+\infty$, and according Eigen functions $\varphi_k, x\in\Omega$, compose the complete orthonormal system in $L_2(\Omega)$ [21, 22]. As well as in [23] it is possible to introduce a scalar product and appropriate norm in the space $\dot{C}^2(\Omega)$. Let's denote completion of this space through $H_r=H_r(\Omega)$, where $r\ge0$ - parameter.

**2. Methods.**

Consider the next task on the control by heat exchange with time delay [24]

$$\frac{\partial z(t)}{\partial t}+Az(t)=z(t-h),\quad 0<t\le T, \tag{3}$$

with boundary

$$Pz(t) = u(t), \quad 0 \le t \le T, \quad x \in \partial\Omega, \tag{4}$$

and initial

$$z(t) = z_0(t), \quad -h \le t \le 0 \tag{5}$$

conditions, where $z_0(\cdot) \in X$, $X = \{z(\cdot): z(t) \in H_r, -h \le t \le 0\}$. The controls are measurable functions $u(\cdot) \in H_r$, i. e. satisfying the condition $\sum_{k=1}^{\infty} \lambda_k^r \left( \int_{\partial\Omega} u(\tau)\varphi_k dx \right)^2 < \infty$. Here $z(\cdot), u(\cdot)$ are abstract functions, i.e. at each $t > 0$ they are unique elements of the space $H_r$, $h$ - positive fixed constant, $T$ - positive number.

A solution of the problem (3)–(5) in $H_r$ is defined by the Fourier method. If $f_k(\cdot)$ denotes Fourier coefficients of the function $f(\cdot)$ towards the system $\{\varphi_k\}$, then solution of the problem (3)–(5) on the segment $[0, h]$ has the next view [24]

$$z(t) = \sum_{k=1}^{\infty} \left( z_k^0 e^{-\lambda kt} + \int_0^t \left[ z_{0k}(\tau - h) + \int_{\partial\Omega} u(\tau)\varphi_k dx \right] e^{-\lambda_k (t-\tau)} d\tau \right) \varphi_k, \tag{6}$$

where $z_k^0 = z_{0k}(0)$, $k = 1, 2 \ldots$. Using this solution, the solution is built on the segment $[h, 2h]$:

$$z(t+h) = \sum_{k=1}^{\infty} \left( z_k(h) e^{-\lambda_k t} + \int_0^t \left[ z_k(\tau) + \int_{\partial\Omega} u(\tau + h)\varphi_k dx \right] e^{-\lambda_k (t-\tau)} d\tau \right) \varphi_k,$$

where $z_k(h)$ is defined from (6) at $t = h$. In the same manner, continuations of the solution are built for the next segments.

Further, through $U$ we denote the totality of controls, which is specified below by a some positive number $\rho$.

**Definition 1.** A multivalued mapping $D: [-h, T] \to 2^R$, where $R = (-\infty, \infty)$, will be called strongly invariant on the segment $[-h, T]$ towards the problem (3)–(5), if for any $\langle z_0(t) \rangle \in D(t)$, $-h \le t \le 0$, and $u(\cdot) \in U$ the inclusion $\langle z(t) \rangle \in D(t)$ holds for all $0 < t \le T$, where $\langle \ \rangle$ - corresponding norm, $z(\cdot)$ - corresponding solution of the problem (3)–(5) [21-23].

**Definition 2.** A multivalued mapping $D: [-h, T] \to 2^R$, where $R = (-\infty, \infty)$, will be called weakly invariant on the segment $[-h, T]$ towards the problem (3)–(5), if for any $\langle z_0(t) \rangle \in D(t)$, $-h \le t \le 0$ there is control $u(\cdot) \in U$ such that $\langle z(t) \rangle \in D(t)$ for all $0 < t \le T$, where $\langle \ \rangle$ - corresponding norm, $z(\cdot)$ - corresponding solution of the problem (3)–(5).

**Statement of the problem**

This work studies the weak and strong invariance of the constant multivalued mapping of the next view

$$D(t) = [0, b], \quad -h \le t \le T,$$

where $b$ - positive constant.

Our further objective is to find the connection between parameters $T, b, \rho$ and $\lambda_i$ so that to provide the weak or strong invariance of the multivalued mapping $D(t)$, $t \in [-h, T]$ towards the problem (3)–(5).

**3. Main results.**

A) Let $\langle z(t) \rangle = \| z(t) \|_{H_r}$, $0 \le t \le T$ and

$$U=\left\{u(\cdot):\sqrt{\sum_{k=1}^{\infty}\lambda_k^r\left(\int_{\partial\Omega}u(t)\varphi_k\,dx\right)^2}\le\rho,\quad t\in[0,T]\right\}.$$

Here $\|z(t)\|_{H_r}^2=\int_{\Omega}|z(t)|^2dx=\sum_{k=1}^{\infty}\lambda_k^r z_k^2(t),\quad 0\le t\le T$.

**Assertion.** For any function $u(\cdot)\in U$ the next inequality holds:

$$\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t\int_{\partial\Omega}e^{-\lambda_k(t-\tau)}u(\tau)\varphi_k\,dx\,d\tau\right)^2\le\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2\rho^2.$$

The proof of the assertion follows from the next correlations:

$$\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t\int_{\partial\Omega}e^{-\lambda_k(t-\tau)}u(\tau)\varphi_k\,dx\,d\tau\right)^2=\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\frac{\lambda_k}{2}(t-\tau)}e^{-\frac{\lambda_k}{2}(t-\tau)}\int_{\partial\Omega}u(\tau)\varphi_k\,dx\,d\tau\right)^2\le$$

$$\le\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\lambda_k(t-\tau)}d\tau\cdot\int_0^t e^{-\lambda_k(t-\tau)}\left(\int_{\partial\Omega}u(\tau)\varphi_k\,dx\right)^2d\tau\right)\le\left(\int_0^t e^{-\lambda_1(t-\tau)}d\tau\right)^2\rho^2=\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2\rho^2.$$

**Theorem 1.** If the follow condition is satisfied

$$\rho\le(\lambda_1-1)b, \tag{7}$$

then the multivalued mapping $D(t)=[0,b]$, $t\in[-h,T]$ is strongly invariant on the segment $[-h,T]$ towards the problem (3)-(5), for any $T>0$.

**Proof.** Let $z_0(t)$ with $c\|z_0(t)\|_{H_r}\le b$, $-h\le t\le 0$, and $u(t)$ with $\|u(t)\|_{H_r}\le\rho$, $0\le t\le h$, be arbitrary functions. Substituting these functions in the equation (3), we have solution of the problem (3)–(5) on the segment $[0,h]$ in the next view (6)

$$z(t)=\sum_{k=1}^{\infty}\left(z_k^0e^{-\lambda_k t}+\int_0^t\left[z_{0k}(\tau-h)+\int_{\partial\Omega}u(\tau)\varphi_k\,dx\right]e^{-\lambda_k(t-\tau)}d\tau\right)\varphi_k\,.$$

If we introduce denotation $f_k(\tau)=z_{0k}(\tau-h)+\int_{\partial\Omega}u(\tau)\varphi_k\,dx,\ k=1,2,\ldots$, then for the function $f(\cdot)$, with the Fourier coefficients $f_k(\cdot)$, we have

$$\|f(\tau)\|_{H_r}=\|z_0(\tau-h)+u(\tau)\|_{H_r}\le\|z_0(\tau-h)\|_{H_r}+\|u(\tau)\|_{H_r}\le b+\rho. \tag{8}$$

Hence,

$$\|z(t)\|_{H_r}^2=\sum_{k=1}^{\infty}\lambda_k^r\left(e^{-\lambda_k t}z_k^0+\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau\right)^2=\sum_{k=1}^{\infty}\lambda_k^r\left(e^{-2\lambda_k t}\left|z_k^0\right|^2+\right.$$

$$\left.2e^{-\lambda_k t}z_k^0\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau+\left(\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)\right)^2\right).$$

From here, using the Cauchy– Bunyakovskii inequality, assertion and inequality (8), we have

$$\sum_{k=1}^{\infty}\lambda_k^r\left(e^{-2\lambda_k t}\left|z_k^0\right|^2+2e^{-\lambda_k t}z_k^0\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau+\left(\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)\right)^2\right)\le$$

$$e^{-2\lambda_1 t}\sum_{k=1}^{\infty}\lambda_k^r\left|z_k^0\right|^2+2e^{-\lambda_1 t}\sum_{k=1}^{\infty}\lambda_k^{\frac{r}{2}}z_k^0\lambda_k^{\frac{r}{2}}\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau+$$

$$\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau\right)^2\le e^{-2\lambda_1 t}b^2+2e^{-\lambda_1 t}\sqrt{\sum_{k=1}^{\infty}\lambda_k^r\left|z_{0k}^0\right|^2}\times$$

$$\sqrt{\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\lambda_k(t-\tau)}f_k(\tau)d\tau\right)^2}+\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)(b+\rho)^2\le e^{-2\lambda_1 t}b^2+$$

$$2e^{-\lambda_1 t}b\frac{1-e^{-\lambda_1 t}}{\lambda_1}(b+\rho)+\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2(b+\rho)^2=\left(e^{-\lambda_1 t}b+\frac{1-e^{-\lambda_1 t}}{\lambda_1}(b+\rho)\right)^2.$$

Let
$$\chi(t)=e^{-\lambda_1 t}b+\frac{1-e^{-\lambda_1 t}}{\lambda_1}(b+\rho),\ 0\le t\le h.$$

Then we have $\chi(0)=b$, $\chi'(t)=-\lambda_1 e^{-\lambda_1 t}b+e^{-\lambda_1 t}(b+\rho)=e^{-\lambda_1 t}(-\lambda_1 b+b+\rho)$. It follows that when fulfilling condition of the theorem 1, $\chi'(t)\le 0$, i.e. the function $\chi(t)$ is nonincreasing. Thus, observing $\chi(t)>0$, we have $\|z(t)\|_{H_r}\le b$, $0\le t\le h$.

Now, by similar reasoning for the time interval $[h,2h]$ we have $\|z(t)\|_{H_r}\le b$. It is seen from here that continuing this process it is possible to reach for any number $T>0$:

$$\|z(t)\|_{H_r}\le b,\quad 0\le t\le T.$$

By this, the theorem 1 is proved.

**Theorem 2.** If $\lambda_1 \ge 1$, then the multivalued mapping $D(t)$ = [0,*b*] is weakly invariant on the segment [–*h*, *T*]towards the problem (3)-(5), where T is any positive number.

**Proof.** Let $\lambda_1\ge 1$. Let's show that the set $W=[0,b]$ is weakly invariant towards the problem (3)-(5). Assume that $z_0(\cdot)$ is arbitrary function from $X$, let $u(\cdot)=0$. Then from presentation of the solution (6) of the problem (3)-(5) we have

$$\|z(t)\|_{H_r}^2=\sum_{k=1}^{\infty}\lambda_k^r\left(e^{-\lambda_k t}z_k^0+\int_0^t e^{-\lambda_k(t-\tau)}z_k(t-\tau)d\tau\right)^2\le e^{-2\lambda_1 t}\sum_{k=1}^{\infty}\lambda_k^r\left|z_k^0\right|^2+$$

$$2e^{-\lambda_1 t}\sum_{k=1}^{\infty}\lambda_k^r\left|z_k^0\right|\int_0^t e^{-\lambda_k(t-\tau)}\left|z_{0k}(t-\tau)\right|d\tau+\sum_{k=1}^{\infty}\lambda_k^r\times \tag{9}$$

$$\left(\int_0^t e^{-\lambda_k(t-\tau)}z_{0k}^0(t-\tau)d\tau\right)^2\le e^{-2\lambda_1 t}b^2+2e^{-\lambda_1 t}b\frac{1-e^{-\lambda_1 t}}{\lambda_1}b+\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2b^2=$$

$$\left(e^{-\lambda_1 t}+\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2b^2.$$

Here, the next inequality and assertion are used:

$$\sum_{k=1}^{\infty}\lambda_k^r\left|z_{0k}^0\right|\left|\int_0^t e^{-\lambda_k(t-\tau)}\left|z_{0k}(t-\tau)\right|d\tau\right| \le \sqrt{\sum_{k=1}^{\infty}\lambda_k^r\left|z_{0k}^0\right|^2}\sqrt{\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\lambda_k(t-\tau)}\left|z_{0k}(t-\tau)\right|d\tau\right)^2}$$

If we introduce notation $\chi(t)=e^{-\lambda_1 t}+\dfrac{1-e^{-\lambda_1 t}}{\lambda_1}$, $0\le t\le h$, then we have $\chi(0)=1,\ \chi'(t)=(1-\lambda_1)e^{-\lambda_1 t}\le 0$.

Hence, $0<\chi(t)\le 1$. From here and from (9) we obtain $\|z(t)\|_{H_r}\le b$, $0\le t\le h$. Sequentially using the same reasoning, if necessary, to the time intervals $[jh,(j+1)h]$, $j=1,2,\ldots$ we come to the fact that $\|z(t)\|_{H_r}\le b$, at $-h\le t\le T$. The theorem 2 is proved.

Let $\langle z(t)\rangle=\|z(t)\|_{H_r}$, $0\le t\le T$, and

$$U=\left\{u(\cdot):\sum_{k=1}^{\infty}\lambda_k^r\int_0^t e^{-\lambda_k(t-\tau)}\left(\int_{\partial\Omega}u(\tau+ih)\varphi_k dx\right)^2 d\tau\le\rho^2\int_0^t e^{\lambda_1(t-\tau)}d\tau\right\},$$

where $0\le t\le h$ and $i=-1,0,1,\ldots$.

**Theorem 3.** If $\rho\le(\lambda_1-1)b$, then the multivalued mapping $D(t)=[0,b]$ is strongly invariant on the segment $[-h,T]$ towards the problem (3)-(5), where T is any positive number.

**Proof.** Let $z_0(\cdot)$ be any element of the set $X$, satisfying the condition $\|z_0(\cdot)\|_{H_r}\in D(t)$, i.e. $\|z_0(\cdot)\|_{H_r}\le b$, $-h\le t\le 0$, and $u(\cdot)$ be any permitted control, i.e. $u(\cdot)\in U$. For $0\le t\le h$ we have

$$\|z(t)\|_{H_r}^2=\sum_{k=1}^{\infty}\lambda_k^r z_k^2(t)=\sum_{k=1}^{\infty}\lambda_k^r\left(z_{0k}^0 e^{-\lambda_k t}+\int_0^t e^{-\lambda_k(t-\tau)}z_{0k}(t-\tau)d\tau+\int_0^t e^{-\lambda_k(t-\tau)}\int_{\partial\Omega}u(\tau)\varphi_k dx d\tau\right)^2 \quad (10)$$

Using the Cauchy– Bunyakovskii inequality and definition of control domain $U$, it is possible to show that

$$\sum_{k=1}^{\infty}\lambda_k^r\left(\int_0^t e^{-\lambda_k(t-\tau)}z_{0k}(t-\tau)d\tau+\int_0^t e^{-\lambda_k(t-\tau)}\int_{\partial\Omega}u(\tau)\varphi_k dx d\tau\right)^2\le\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right)^2(b+\rho)^2$$

After simple calculations from (10) we obtain

$$\|z(t)\|_{H_r}\le be^{-\lambda_1 t}+\frac{\rho+b}{\lambda_1}\left(1-e^{-\lambda_1 t}\right). \quad (11)$$

Let's introduce the following function

$$\xi(t)=be^{-\lambda_1 t}+\frac{\rho+b}{\lambda_1}\left(1-e^{-\lambda_1 t}\right).$$

Note that $\xi(0)=b$. As $\xi'(t)=(-\lambda_1 b+\rho+b)e^{-\lambda_1 t}$, therefore under the condition $\rho\le(\lambda_1-1)b$, we have $\xi'(t)\le 0$. Thus, from (5) we obtain that for all $t\in[0,h]$ the inequality $\|z(t)\|_{H_r}\le b$ holds.

Accepting $z(h)$ as a new initial position of the considered system for the time interval, $[h,2h]$ we have

$$\|z(t+h)\|_{H_r}^2=\sum_{k=1}^{\infty}\lambda_k^r z_k^2(t)=\sum_{k=1}^{\infty}\lambda_k^r\left(z_k^0(h)e^{-\lambda_k t}+\int_0^t e^{-\lambda_k(t-\tau)}z_k(\tau)d\tau+\int_0^t e^{-\lambda_k(t-\tau)}\int_{\partial\Omega}u(\tau+h)dx d\tau\right)^2,$$

where $0\le t\le h$.

Similarly, to the previous step it is possible to obtain the next evaluation

$$\|z(t+h)\|_{H_r} \le b,\ 0 \le t \le h .$$

Repeating the same reasoning we obtain that $\|z(t)\|_{H_r} \le b$, $-h \le t \le T$ .

The theorem 3 is proved.

**4. Conclusions**. Many scientists have studied the conditions of strong and weak invariance of sets with respect to differential inclusion. This work continues these studies for systems with distributed parameters. The question of the strong and weak invariance of a multivalued mapping with respect to the heat equation with boundary control is considered. The results obtained are useful for specialists in control theory and differential games described by equations with distributed parameters.

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## КІДІРУЛЕР ОРЫН АЛАТЫН ЖЫЛУӨТКІЗГІШТІК ЕСЕБІНДЕ ИНВАРИАНТТЫ КӨПМӘНДІ БЕЙНЕЛЕУ ТУРАЛЫ

**Аннотация.** Қазіргі таңдағы күрделі, өту ағыны жылдам және энергосыйымдылықты үрдістерді басқару теориясының заманауи жетістіктері мен ғылыми-техникалық ілгерілеусіз іске асыру мүмкін емес. Көптеген нақты басқару нысандарын таралған параметрлі жүйелер ретінде қарастыруға болындығы белгілі, бұл ретте басқару параметрлері теңдеудің оң жағында немесе шекаралық шарт түрінде орналасуы мүмкін. Соңғы уақытта өндірістік және технологиялық үрдістерді автоматтандыруға барынша маңызды мән беріліп отыр, ал таралған параметрлі теңдеулер жүйесі – бұл үрдістердің математикалық модельдері болып табылады.

Басқару есептерін зерттеудің жаңа әдістерін құруда төмендегі сапалық сипаттағы сұрақтарға жауап беру қажет болады:

а) берілген *G* жиыны күшсіз (немесе күшті) инвариантты ма, яғни қаралып жатқан басқарылатын жүйенің *G* жиындағы кез келген бастапқы нүктесі үшін, еш болмағанда бір траектория табыла ма? Бұл траектория берілген нүктеден шығып *G* жиында толық жата ма?

б) қаралып жатқан басқарылатын жүйеге күшсіз инвариантты *G* жиынның бос емес ішкі жиыны бар ма?

Сонымен қатар мынадай есеп туындайды: *G* жиынның тіршілік ядросын құру есебі, немесе қауіпсіз аймақ құру, яғни қаралып жатқан басқару жиынына қатысты күшсіз инвариантты *G* жиынның ең үлкен ішкі жиынын табу. Айта кетейік, осы мәселе бойынша көптеген зерттеулер негізінен жинақталған параметрлі басқару жүйелеріне арналған. Ұынылып отырған жұмыста үдерістің үлгісін сипаттауда, нақты жағдайларда ақпараттың кешігіп келуі арнайы қосынды арқылы есепке алынады. Әдетте басқару параметрлері еркін болмайды, сондықтан оларға әртүрлі геометриялық және интегралдық шектеулер қойылады. Осы шектеулерден басқа басқару функциясы анықталған функционалдық кеңістіктен алынады.

Бұл жұмыста кідірулер орын алатын шекаралық басқарумен берілген жылуөткізгіштік теңдеуіне сәйкес тұрақты көпмәнді бейнелеулердің әлді және әлсіз инварианттылығы туралы мәселе қаралған. Кідірулер орын алатын аргументті шекаралық және бастапқы шарттарда жылу алмасуды басқару есебі үшін осы көпмәнді бейнелеулердің әлді және әлсіз инвариантты болуына жеткілікті шарттар алынған. Таралған көрсеткіштері бар жүйеге қатысты «инвариантты жиын» ұғымынан пайдаланылған. Оның физикалық мағынасы объектіні басқару арқылы ниет еткен жай-күйде мүмкіндігінше ұзақ «ұстап тұру». Бұл ретте, объектіні ұстап тұру геометриялық тұрғыдан емес, объектінің көлеміне қарай орташа мәнді ұстап тұру деген ұғыммен түсіндіріледі. Объектіні ниет еткен жай-күйде ұстап тұру үшін кейбір жағдайларда жеткілікті шарттар ұсынылады. Теңдеулерді шешу үшін алдымен эллиптикалық оператордың мен оператордың өзіне-өзі түйіндеске дейін анықтау аймағы кеңейтіледі, содан кейін осы оператордың энергетикалық кеңістігіне жататын шешімнің болуы қарастырылады. Бұл ретте, кеңейтілген оператордың толық жүйені және оператордың энергетикалық кеңістігінде және әрбір кеңістікте құрайтын жалпыланған меншікті сандары мен жалпыланған меншікті функциялары бар факт пайдаланылады. Бұл жағдайда шешім жалпылама деп қарастырылады, яғни шешім коэффициенттері шексіз жай дифференциалдық теңдеулерді қанағаттандыратын Фурье қатары түрінде беріледі. Тап осы дифференциалдық теңдеулер жүйесінде басқару параметрлері кездеседі. Сондықтан, шешімді алу үшін Фурье әдісі, ал бастапқы есепті басқаруда Фурье коэффициенттері қолданылады.

Бұл мәселені қарастырудағы маңызды сәт, басқару элементтері аймақтың шекарасында орналасқан. Бұл ретте басқару аймағы дөңес ықшам көп қырлы, ал шектеу аймағы мен терминалдық көп - жартылай кеңістіктер болып табылады. Бұл белгілі бір жағдайларда алынған нәтижелерді практикалық есептерді шешуде қолдануға мүмкіндік береді.

**Түйін сөздер:** инвариантты жиындар, басқару, көпмәнді бейнелеу, таралған параметрлі жүйелерді басқару, кідіру.

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## ОБ ОДНОМ ИНВАРИАНТНОМ МНОГОЗНАЧНОМ ОТОБРАЖЕНИИ В ЗАДАЧЕ ТЕПЛОПРОВОДНОСТИ С ЗАПАЗДЫВАНИЕМ

**Аннотация.** Современные сложные, быстро протекающие и энергоемкие процессы невозможно реализовать без дополнения их современными достижениями теории управления и научно-технического прогресса. Как известно, многие реальные управляемые объекты можно рассматривать как объекты с распределенными параметрами, в которых управляющие параметры могут находиться как в правой части уравнения, так и в граничных условиях. В последнее время всевозрастающее значение придается автоматизации производственных и технологических процессов, математическими моделями которых являются системы уравнений с распределенными параметрами. При разработке новых методов исследования задач управления следует ответить на следующие вопросы качественного характера:

а) является ли множество $G$ сильно (или слабо) инвариантным, т.е. для любой начальной точки из $G$ существует ли хотя бы одна траектория (все траектории) рассматриваемой управляемой системы, выходящая из данной точки, определенная на бесконечном интервале времени и целиком лежащая в $G$?

б) существует ли хотя бы одно непустое подмножество множества $G$, слабо инвариантное относительно данной управляемой системы?

Кроме того, возникают задачи: задача построения ядра живучести $G$, или то же самое безопасной зоны, т.е. максимального подмножества множества $G$, слабо (или сильно) инвариантного относительно рассматриваемой управляемой системы. Отметим, что почти все исследования по этой проблеме посвящены управляемым системам со сосредоточенными параметрами. В предлагаемой работе в описании модели процесса с помощью специального слагаемого учитывается то, что в реальных ситуациях все информация поступает с запаздыванием. Так как управляющие параметры не бывают произвольными, поэтому на них налагаются различные ограничения в виде геометрических и интегральных. Кроме этих ограничений, управляющие функции берутся из определенного функционального пространства.

В данной работе рассмотрен вопрос о сильной и слабой инвариантности постоянного многозначного отображения относительно уравнения теплопроводности с граничным управлением при наличии запаздывания. Для задачи управления теплообмена с запаздывающим аргументом с граничными и начальными условиями получены достаточные условия для сильной и слабой инвариантности данного многозначного отображения. Применено понятие – «инвариантные множества» относительно системы с распределенными параметрами, физический смысл которых заключается в том, что бы по возможности дольше «удержать» объект в желаемом состоянии с помощью управления им. При этом, здесь под удержанием объекта понимается не в геометрическом смысле, а в смысле удержания усредненного значения относительно объема объекта. Предложены необходимые условия для удержания объекта в желаемых состояниях. Для решения уравнений сначала расширяется область определения эллиптического оператора и самого оператора до самосопряженного, и затем рассматривается существования решения, принадлежащего энергетическому пространству данного оператора. При этом используется тот факт, что расширенный оператор имеет обобщенные собственные числа и обобщенные собственные функции, составляющие полную систему и в энергетическом пространстве оператора, и в каждом пространстве. В данном случае в качестве решения понимается обобщенное, потому что оно представляется в виде ряда Фурье, коэффициенты которого удовлетворяют бесконечным обыкновенным дифференциальным уравнениям. Как раз в этой системе дифференциальных уравнений присутствует управляющий параметр. Так что для получения решения применяется метод Фурье, а управление исходной задачи осуществляется через коэффициенты Фурье.

Существенным моментом рассмотрения данной задачи является то, что управления находятся на границе области. При этом область управления является выпуклым компактным многогранником, а область ограничения и терминальное множество - полупространствами. Это, в определенных условиях, позволяет

использовать возможность применения полученных результатов при решении практических задач.

**Ключевые слова:** инвариантное множество, управление, многозначное отображение, управление системами с распределенными параметрами, запаздывание.

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# M-FUNCTION NUMBERS: CYCLES AND OTHER EXPLORATIONS. PART 2

**Abstract.** This paper establishes the cyclic properties of the M-Function, which we define as a function, [M(n)], that takes a positive integer, adds to it the sum of its digits and the number produced by reversing its digits, and then divides the entire sum by three. Our definition of the M-Function is influenced by D. R. Kaprekar's work on a remarkable class of positive integers, called self- numbers, and his procedure, [K(n)], of adding to any positive integer the sum of its digits [1]. We analyze the distribution of numbers that make the defined M-Function behave like a cyclic function, and observe that many such "cycles" form arithmetic sequences. We examine the distribution of numbers that produce integer ratios between the outputs of Kaprekar's and the M-Function functions, [K(n)/M(n)]. We also prove that the set of numbers with equal outputs to both Kaprekar's and M-Function functions, [K(n)=M(n)], is infinite.

**Key words:** M-Function, D.R.Kaprekar, self-numbers.

**5. Integer ratios between two numbers from $n$, $K(n)$, $M(n)$.** Kaprekar defined a *digitaddition* as $K(n) = n + S(n) > n$. Hence, the difference in numbers $K(n)$ and $n$ is equal to $K(n) - n = S(n)$. $S(n)$ is a lot less than $n$ when $n$ is large, hence the equation $t * n = K(n)$ when $t \geq 2, t \in N$, does not have any solutions.

Because $K(n) > n$, the investigation of the equation $t * K(n) = n$, when $t \geq 2$ will not produce any solutions: $t * K(n) > K(n) > n$.

Therefore, we will study integer ratios between numbers $n$ and $M(n)$ in paragraph 5.1, and between $K(n)$ and $M(n)$ in paragraph 5.2.

**5.1. Integer ratios between numbers $n$ and $M(n)$.** As defined earlier, if $n = M(n)$, the number $n$ is stationary. So, we need to consider two cases:

$t * n = M(n)$, when $t \geq 2$, $t \in N$, and

$$n = t * M(n), \quad \text{when } t \geq 2, \ t \in N.$$

**A)** Consider $t * n = M(n)$, when $t \geq 2, t \in N$.

**Proposition 5.** If $t \geq 4$, the equation $t * n = M(n)$ will not have solutions.

**Proof.** Let $t \geq 4$, and $t * n = M(n)$.

Then:

$$t * n = \frac{1}{3}(n + S(n) + \bar{n}).$$

Multiply by 3, and we get $3t * n = n + S(n) + \bar{n}$.

Re-arranging, we get $(3t - 1) * n = S(n) + \bar{n}$. (Equation 5)

Note that $3t - 1 \geq 11$ when $t \geq 4$ .

Multiplying both sides by n, we get: $(3t-1)n \geq 11n > S(n) + \bar{n}$.

(Note, $10n > \bar{n}$ and $n \geq S(n)$, thus $11n > S(n) + \bar{n}$.)

Hence, when $t \geq 4$ , Equation 5 doesn't have any solutions, and proposition 5 has been proven.

Let's consider the two remaining situations.

**i)** $t = 2 \rightarrow \quad 2 * n = M(n)$. Substituting in the definition of M(n) and re-arranging the equation, we can simplify the equation in the following steps:

$$2 * n = \frac{1}{3}(n + S(n) + \bar{n}).$$

$$5 * n = S(n) + \qquad \text{(Equation 6)}$$

Using a C++ program, we solve Equation 6 and get the following results for numbers to $10^{10}$:

| | | | | |
|---|---|---|---|---|
| 18 | 1206 | 120006 | 12000006 | 1200000006 |
| 126 | 12006 | 1200006 | 120000006 | |

**Proposition 6.** When $d(n) > 10$ , all numbers with the type $a_d = 12\underbrace{0\ldots0}_{d-3}6$ are solutions of Equation 6. In other words, $2 * a_d = M(a_d)$ for all $d(n) > 10$.

**The proof** is similar to the proof of Proposition 1.

**ii)** $t = 3 \rightarrow \quad 3 * n = M(n)$. Substituting in the definition of M(n) and re-arranging the equation, we can simplify the equation in the following steps:

$$3n = \frac{1}{3}(n + S(n) + \bar{n}).$$

$$8n = S(n) + n \qquad \text{(Equation 7)}$$

Using a C++ program, no solutions were found for Equation 7 for numbers up to $10^{10}$.

**B)** Consider $n = t * M(n)$, when $t \geq 2$, $t \in N$.

**Proposition 7.** If $t \geq 3$ such that $t \in N$, the equation $n = t * M(n)$ will not have solutions.

**Proof.** Since $t \geq 3$, we know the following:

$$t * M(n) = t * \frac{1}{3}(n + S(n) + \bar{n}) \geq n + S(n) + \bar{n}.$$

(Note, $n + S(n) + \bar{n} > n$.)

Hence, when $t \geq 3$, the equation $n = t * M(n)$, doesn't have any solutions.

It remains to consider only 1 case:

$t = 2$, which means we consider the equation

$$n = 2 * M(n).$$

By definition of M(n), it is equivalent to: $n = \frac{2}{3}(n + S(n) + \bar{n})$.

$$n = 2 * S(n) + 2 * \bar{n} \qquad \text{(Equation 8)}$$

Using a C++ program, we found numbers that satisfy Equation 8 for numbers up to $10^{10}$:

| | | | | | |
|---|---|---|---|---|---|
| 72 | 9054 | 120060 | 4920642 | 90000054 | 1200000060 |
| 180 | 12060 | 492642 | 9000054 | 120000060 | 4920000642 |
| 954 | 49842 | 900054 | 12000060 | 492000642 | 9000000054 |
| 1260 | 90054 | 1200060 | 49200642 | 900000054 | |

**Proposition 8.** When $d(n) > 10$, the following numbers with types $b_d = 12\underbrace{0\ldots0}_{d-4}60$, $c_d = 492\underbrace{0\ldots0}_{d-6}642$, $e_d = 9\underbrace{0\ldots0}_{d-3}54$ are solutions of Equation 8.

In other words, $b_d = 2 * M(b_d)$, $c_d = 2 * M(c_d)$, $e_d = 2 * M(e_d)$.

**The proof** is similar to the proof of Proposition 1.

**Conjecture 4.** When $d(n) > 10$, Equations 6 and 8 don't have any other solutions, except the solutions specified at propositions 6 and 8, respectively. Equation 7 doesn't have solutions in the set N.

**5.2. Integer ratios between numbers $\boldsymbol{K(n)}$ and $\boldsymbol{M(n)}$.** Earlier, we investigated the equation $K(n) = M(n)$. Hence, in this section we investigate 2 specific cases:

$$t * K(n) = M(n), \qquad \text{when } t \geq 2, \;\; t \in N,$$
$$K(n) = t * M(n), \qquad \text{when } t \geq 2, \;\; t \in N.$$

**A)** Let's consider $t * K(n) = M(n)$, when $t \geq 2$ and $t \in N$.

**Proposition 9.** Equation $t * K(n) = M(n)$ doesn't have any solutions when $t \geq 4, t \in N$.

**Proof.** Let's simplify this equation:

$$t * K(n) = M(n)$$
$$t * \big(n + S(n)\big) = \frac{1}{3}(n + S(n) + \bar{n})$$

Multiply by 3:

$$3t * n + 3t * S(n) = n + S(n) + \bar{n}$$
$$(3t - 1) * n + (3t - 1) * S(n) = \bar{n}$$

Because we are considering $t \geq 4$, $3t - 1 \geq 11$.

Hence $$(3t - 1) * n + (3t - 1) * S(n) \geq 11 * (n + S(n)) > \bar{n}$$

If number $n$ has $d$ digits, number $11 * \big(n + S(n)\big)$ will have at least $d$+1 digits.

Thus, when $t \geq 4$, the equation $t * K(n) = M(n)$ doesn't have any solutions.

It remains to consider two other subcases: when $t = 2$ and $t = 3$.

**i**) $t = 2 \rightarrow 2 * K(n) = M(n)$. We can simplify the equation further:

$$2\big(n + S(n)\big) = \frac{1}{3}(n + S(n) + \bar{n})$$
$$5n = -5 * S(n) + \bar{n} \qquad \text{(Equation 9)}$$

**ii**) $t = 3 \rightarrow 3 * K(n) = M(n)$. We can also simplify this equation:

$$3\big(n + S(n)\big) = \tfrac{1}{3}(n + S(n) + \bar{n}).$$
$$8n = -8 * S(n) + \bar{n} \qquad \text{(Equation 10)}$$

Using a C++ program, no solutions were found for Equations 9 and 10 for numbers up to $10^{10}$.

**Conjecture 5.** Equations 9 and 10 don't have any solutions in N, which means that n ∈ N doesn't exist for the following 2 equations:

$$2 * K(n) = M(n), \qquad 3 * K(n) = M(n).$$

**B)** Let's consider $K(n) = t * M(n)$, when $t \geq 2$, and $t \epsilon N$.

**Proposition 10.** When $t \geq 3$, the equation $K(n) = t * M(n)$ doesn't have solutions in the set *N*.

**Proof.** We can simplify the following equation by substituting in the definitions of K(n) and M(n), and performing algebraic manipulation:

$$K(n) = t * M(n)$$
$$n + S(n) = \tfrac{t}{3}(n + S(n) + \bar{n}).$$

Since we are considering the case

$t \geq 3$, we can set up the following inequalities:

$$\frac{t}{3}(n + S(n) + \bar{n}) \geq n + S(n) + \bar{n} > n + S(n).$$

Hence, when $t \geq 3$, $t * M(n) > K(n)$ and the equation $K(n) = t * M(n)$ doesn't have solutions ($t \in N$).

It remains to consider just 1 case.

**i)** $t = 2 \rightarrow K(n) = 2 * M(n)$. We can simplify the equation:

$$n + S(n) = \frac{2}{3}(n + S(n) + \bar{n}).$$

$$n = -S(n) + 2 * \bar{n} \qquad \text{(Equation 11)}$$

Using a C++ program, we get the following solutions to Equation 11 for numbers up to $10^{10}$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 201 | 8004 | 200001 | 4999952 | 60000003 | 1079999350 |
| 2 | 402 | 8734 | 340071 | 5400072 | 74000073 | 1400000070 |
| 3 | 603 | 9474 | 400002 | 6000003 | 80000004 | 2000000001 |
| 4 | 804 | 14070 | 540072 | 6999943 | 94000074 | 3079999351 |
| 5 | 1470 | 20001 | 600003 | 7400073 | 140000070 | 3400000071 |
| 6 | 2001 | 34071 | 740073 | 8000004 | 200000001 | 4000000002 |
| 7 | 2731 | 40002 | 800004 | 8999944 | 340000071 | 5079999352 |
| 8 | 3471 | 54072 | 940074 | 9400074 | 400000002 | 5400000072 |
| 9 | 4002 | 60003 | 1400070 | 14000070 | 540000072 | 6000000003 |
| 21 | 4732 | 74073 | 2000001 | 20000001 | 600000003 | 7079999353 |
| 42 | 5472 | 80004 | 2999941 | 34000071 | 740000073 | 7400000073 |
| 63 | 6003 | 94074 | 3400071 | 40000002 | 800000004 | 8000000004 |
| 84 | 6733 | 140070 | 4000002 | 54000072 | 940000074 | 9079999354 |
| | 7473 | | | | | 9400000074 |

**Proposition 11.** When $d(n) > 10$, the following types of numbers are solutions to Equation 11:

$a_d = 14\underbrace{0 \ldots 0}_{d-4}70$, $b_d = 2\underbrace{0 \ldots 0}_{d-2}1$, $c_d = 34\underbrace{0 \ldots 0}_{d-4}71$, $e_d = 4\underbrace{0 \ldots 0}_{d-2}2$, $f_d = 54\underbrace{0 \ldots 0}_{d-4}72$, $g_d = 6\underbrace{0 \ldots 0}_{d-2}3$, $h_d = 74\underbrace{0 \ldots 0}_{d-4}73$, $l_d = 8\underbrace{0 \ldots 0}_{d-2}4$, $q_d = 94\underbrace{0 \ldots 0}_{d-4}74$

Which, equivalently, means that all of these numbers satisfy the equation

$$K(n) = 2 * M(n).$$

**The proof** is similar to the proof of Proposition 1.

**Note**. The equation $K(n) = 2 * M(n)$ satisfies all 9 types of numbers described in proposition 11, starting with $d(n) \geq 4$, (i.e. starting with four-digit numbers). However, there are also other four-digit, seven-digit and ten-digit solutions to Equation 11:

| | | | | | |
|---|---|---|---|---|---|
| 2731 | 6733 | 2999941 | 6999943 | 1079999350 | 5079999352 |
| 4732 | 8734 | 4999942 | 8999944 | 3079999351 | 7079999353 |

We can't observe a clear relationship between these 3 groups of numbers.

Recall the following fact: cycles with length 10 can only contain four-digit, seven-digit or ten-digit numbers. According to all of our discovered facts, our investigation of function $M(n)$ for four-digit, seven-digit and ten-digit numbers take special place.

In the case of Equation 11, we can't observe a general pattern of solutions for numbers up to $10^{10}$. Hence, we cannot formulate a conjecture, but we can instead formulate a problem.

**Problem 1.** It is possible to find all solutions to Equation 11 for all $n \in N$, when $d(n) > 10$.

**6. The distribution and properties of sets of m-generated numbers.** Let $N$ be the set of positive integer numbers. Let us choose any $n \epsilon N$ and denote the sum of its digits by $S(n)$. The number $M(n) = \frac{1}{3}(n + S(n) + \bar{n})$ is called a *m-generated number* and the inputted number *n* is its *generator*. Positive integer that has no generator is called a *m-self number*. Let's denote $E$ as the set of all m-self numbers, and $G$ as the set of all $m-$generated numbers. Clearly,

$$N = G \cup E$$

In contrast to Kaprekar function's set of generated numbers, $m-$generated numbers are much rare than $m-$self numbers. The distribution of the set of $m-$generated numbers in the set of natural numbers $N$ brings strong interest to us.

Using a C++ program, we can find $m-$ generated numbers for numbers up to $10^8$, and compile the following table.

Table 2 - The distribution of set of $m-$ generated numbers for numbers to $10^8$

| $d(n)$ | Interval | Quantity of generated numbers | % percentage out of all numbers |
|---|---|---|---|
| 1 | [1, 9] | 9 | 100% |
| 2 | [10, 99] | 29 | 32.22% |
| 3 | [100, 999] | 188 | 20.89% |
| 4 | [1000, 9999] | 594 | 6.60% |
| 5 | [10000, 99999] | 3668 | 4.076% |
| 6 | [100000, 999999] | 11352 | 1.261% |
| 7 | [1000000, 9999999] | 69819 | 0.776% |
| 8 | [10 000 000, 99 999 9999] | 215985 | 0.240% |

Let's calculate $\sqrt[7]{215985/9} \approx 4.224$. This means that, if we increase the order to 1, the quantity of $m-$generated numbers in the next order $d(n)$ will be increased, on average, by a factor of 4.224. However, the percentage of $m-$generated numbers from all natural numbers decreases from 100% among one-digit numbers to 0,24% among eight-digit numbers. The percentage of $m-$generated numbers, on average, decreases by a factor of 2,37 when the order $d(n)$ decreased by one, because $\sqrt[7]{100/0{,}24} \approx 2{,}37$.

The decreasing in the proportion of $m-$generated numbers out of all natural numbers while the order of numbers is increasing suggests that the average number of $m-$generators for one $m-$generated number is increasing. $m-$ generated numbers with more $m-$ generators appear when the order of numbers is increasing.

**Conjecture 6.** For any $k \epsilon N$, there exist $m-$ generated numbers, and the quantity of $m-$ generators greater than or equal to $k$.

Let's consider the distribution of m-generated numbers by intervals of hundreds, thousands, ten-thousands, hundred-thousands, millions, and ten-millions.

Table 3

| Intervals | Quantity |
|---|---|
| [100, 199] | 30 |
| [200, 299] | 28 |
| [300, 399] | 31 |
| [400, 499] | 32 |
| [500, 599] | 31 |
| [600, 699] | 20 |
| [700, 799] | 6 |
| [800, 899] | 6 |
| [900, 999] | 4 |

Table 4

| Intervals | Quantity |
|---|---|
| [1000, 1999] | 57 |
| [2000, 2999] | 57 |
| [3000, 3999] | 72 |
| [4000, 4999] | 84 |
| [5000, 5999] | 85 |
| [6000, 6999] | 65 |
| [7000, 7999] | 60 |
| [8000, 8999] | 60 |
| [9000, 9999] | 54 |

Table 5

| Intervals | Quantity |
|---|---|
| [10000, 19999] | 570 |
| [20000, 29999] | 570 |
| [30000, 39999] | 598 |
| [40000, 49999] | 625 |
| [50000, 59999] | 624 |
| [60000, 69999] | 350 |
| [70000, 79999] | 114 |
| [80000, 89999] | 114 |
| [90000, 99999] | 103 |

Table 3

| Intervals | Quan-tity |
|---|---|
| [100000, 199999] | 1083 |
| [200000, 299999] | 1083 |
| [300000, 399999] | 1364 |
| [400000, 499999] | 1646 |
| [500000, 599999] | 1646 |
| [600000, 699999] | 1206 |
| [700000, 799999] | 1140 |
| [800000, 899999] | 1140 |
| [900000, 999999] | 1044 |

Table 4

| Intervals | Quan-tity |
|---|---|
| [1000000, 1999999] | 10830 |
| [2000000, 2999999] | 10830 |
| [3000000, 3999999] | 11366 |
| [4000000, 4999999] | 11906 |
| [5000000, 5999999] | 11900 |
| [6000000, 6999999] | 6671 |
| [7000000, 7999999] | 2166 |
| [8000000, 8999999] | 2165 |
| [0000000, 9999999] | 1985 |

Table 5

| Intervals | Quan-tity |
|---|---|
| [10000000, 19999999] | 20577 |
| [20000000, 29999999] | 20577 |
| [30000000, 39999999] | 25980 |
| [40000000, 49999999] | 31383 |
| [50000000, 59999999] | 31379 |
| [60000000, 69999999] | 22915 |
| [70000000, 79999999] | 21660 |
| [80000000, 89999999] | 21659 |
| [90000000, 99999999] | 19855 |

From the investigation of the distribution, we can observe that by the order $d(n)$ where $1 \leq d(n) \leq 8$ , the quantity of $m-$ generated numbers increases.

**7. "Neighboring" numbers**. If we consider a variety of $m-$generated numbers, we can observe 2 or more consecutive numbers belonging to the defined set $G$. So, if numbers $a_1 + i \in G$, such that $i = 0,1,\dots,k-1$, we will call them ($\{a_1, a_1+1,\dots,a_1+k-1\}$) "neighboring" with length $k$, where $k \geq 2$.

Using a C++ program, we find "neighboring" numbers up to $10^9$ and study them. We completed a table showing the distribution of "neighboring" numbers up to $10^9$:

Table 9 - The table of "neighboring" numbers' distribution up to $10^9$

| $d(n)$ | Intervals | The length of "neighboring" numbers | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | [1,9] | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | [10,99] | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | [100,999] | 87 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | [1000,9999] | 127 | 17 | 2 | 0 | 0 | 0 | 0 | 0 |
| 5 | [10000,99999] | 26 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | [100000,999999] | 28 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | [1000000,9999999] | 77 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | [10000000,99999999] | 21918 | 27 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | [100000000,999999999] | 138510 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

On this list (considering lengths 2-9), there exist "neighboring" numbers only with length $k = \{2,3,4,9\}$. Among the "neighboring" numbers with length 9, there is just one: $\{1,2,3,4,5,6,7,8,9\}$. Among "neighboring" numbers with length 4, there are a total of 2: $\{3671,3672,3673,3674\}$ and $\{4340,4341,4342,4343\}$. According to the table, "neighboring" numbers of length 2 are more frequent than such numbers of length 3.

Let's consider the following table of the greatest and the smallest "neighboring" numbers in terms of each order $d(n)$ of $m$ –generated numbers.

Table 10 - The table of smallest and largest "neighboring" numbers up to $10^9$

| $d(n)$ | The smallest "neighboring" numbers | The greatest "neighboring" numbers |
|---|---|---|
| 2 | {40,41} | {96,97} |
| 3 | {102,103} | {963,964} |
| 4 | {1000,1001,1002} | {6566,6567,6568} |
| 5 | {10002,10003} | {65535,65536} |
| 6 | {358903, 358904} | {655204,655205} |
| 7 | {35852237, 3585238} | {6551873,6551874} |
| 8 | {35521871,35521872} | {99966311, 99966312} |
| 9 | {100033705, 100033706} | {429966350, 429966351} |

According to all the gathered facts about “neighboring” numbers, we can propose the following conjectures.

**Conjecture 7.** The “neighboring” numbers with length $k \geq 4$ don’t exist among the set of $m-$generated numbers with order $d(n) > 9$.

**Conjecture 8.** There is an infinite amount of “neighboring” numbers with length $k = 2$ among the set of $m-$generated numbers.

**Conjecture 9.** There is an infinite amount of “neighboring” numbers with length $k = 3$ among the set of $m-$generated numbers.

**Conclusion.** Following Kaprekar, three years ago we came up with a simple procedure of generating positive integers

$$n \to M(n) = \frac{1}{3}(n + S(n) + \bar{n}).$$

Although the case $n < K(n)$ occurs for all positive integers $n$, in the M-Function, it is possible to have 3 different cases: $n < M(n)$, $n = M(n)$ and $n > M(n)$. Through investigation, new facts and notions were discovered. Based on the results of investigation, 9 conjectures and 1 problem were formulated.

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**М-ФУНКЦИЯ САНДАРЫ: ЦИКЛДЕР ЖӘНЕ БАСҚА ЗЕРТТЕУЛЕР**

**Аннотация.** Индиялық математик Д.Р. Капрекар өзі ашқан“ Капрекар Константасы” - 6174 санымен аса танымал.

Капрекардың тағы бір ашқан жаңалығы – өзіндік туындаған сандар класы белгілі америкалық ғылым насихаттаушысы Мартин Гарднердің «Уақыт бойынша саяхат» [1] атты кітабында баяндалған. Кез-келген натурал $n$ санын аламыз және ол санға цифрларының қосындысы $S(n)-$ді қосамыз. Шыққан сан $K(n) = n + S(n)$ туындаған сан, ал алғашқы сап $n$ – оның генераторы деп аталады. Мысалы, егер 53 санын алсақ, онда туындаған сан $53 + 3 + 5 = 61$ саны болады.

Туындаған санның генераторларының саны бірден артық болуы мүмкін. Екі генераторы бар ең кіші сан 101, ал оның генераторлары 91 и 100 сандары. Өзіндік туындаған сандар - генераторлары жоқ сандар. «The American Mathematical Monthly» [2] журналында жарияланған мақалада өзіндік туындаған сандардың шексіз көп екендігі және өзіндік туындаған сандар туындаған сандарға қарағанда өте сирек кездесетіндігі дәлелденген.

Капрекар ашқан осы жаңалықтар көптеген математиктерді қызықтырды. Әртүрлі елдерде «Капрекар Константасының» , өзіндік туындаған және туындаған сандары жиындарының жаңа қасиеттерін жан-жақты зерттеген көптеген мақалалар, математикалық ғылыми жобалар және программалық өнімдер жарық көрді.

Мен Капрекарға сүйене отырып натурал сандарды алудың жаңа әдісін таптым: $n \to M(n) = \frac{1}{3}(n + S(n) + \bar{n})$, мұнда $\bar{n}$ - сол цифрлармен, бірақ кері бағытта жазылған сан. $M(n)$ саны әрқашан бүтін сан болады, себебі $n, S(n), \bar{n}$ сандарының 3- ке бөлгендегі қалдықтары әрқашан тең болады. Егер Капрекар жағдайында $n < K(n)$ теңсіздігі кез –келген натурал $n$ сандарында орындалса, менің құрған функциямда әртүрлі қатынастар болады, яғни барлық 3 жағдай да орын алады : $n < M(n), n = M(n)$ и $n > M(n)$.

Менің тапқан жаңа натурал сан алу функциясы $n \to M(n)$ әрі қарапайым, табиғи және ол Капрекар $n \to K(n)$ функциясының аналогы болып табылады.

Мақаланың 1-ші бөлімінде $m$-циклдарының таралуы және олардың қасиеттері зерттеледі. ( Егер $M^l(n) = n$ теңдігі орындалатындай ең кіші натурал сан $l$ болса, онда $n \to M(n) \to M^2(n) \to \ldots \to M^{l-1}(n) \to M^l(n) \to n$ сандары $m$-циклді құрайды. Ал $K(n)$ жағдайында циклдар туындамайды, себебі. $n < K(n) < K^2(n)$ ....). Сонымен қатар, $K(n) = M(n)$ функцияларының теңдігі сұрағы және $n, K(n)$ и $M(n)$ сандарының қандай да бір ретпен арифметикалық прогрессия құрайтын сұрақтары зерттелген.

Мақаланың 2-ші бөлімінде $n, K(n)$ и $M(n)$ сандарының арасындағы еселік қатынастар қарастырылған. Яғни, қандай натурал $t$ сандарында, $t \geq 2$, $K(n) = tM(n), tK(n) = M(n), n = t\,M(n), n * t = M(n)$ теңдіктері орындалатындығы зерттелген. ( Айта кетейік, $n$ және $K(n)$ сандарының арасында еселік қатынастар болуы мүмкін емес ). Сонымен қатар $m-$туындаған сандар жиының таралуы және қасиеттері зерттелген.

($m$ – туындаған сандар $m -$ өзіндік туындаған сандарға қарағанда өте сирек кездеседі. Сондықтан $m$ – туындаған сандар жиынын зерттеу $m -$өзіндік туындаған сандар жиынына қарағанда маңыздырақ). Осы бөлімде "көрші", яғни қатарлас тұрған $m -$туындаған сандар жиыны зерттелген.

Зерттеу барысында 1 мәселе және 9 гипотезалар тұжырымдалған.

**Түйін сөздер**: М-функция, Д.Р.Капрекар, өзіндік туындаған сандар.

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## ЧИСЛА М-ФУНКЦИИ: ЦИКЛЫ И ДРУГИЕ ИССЛЕДОВАНИЯ

**Аннотация.** Индийский математик Д.Р. Капрекар особенно известен своим открытием "Константу Капрекара"- числом 6174.

Другое выдающиеся открытие Капрекара, описанная известным американским популяризатором науки Мартином Гарднером в своей книге "Путешествие во времени"[1], это класс самопорожденных чисел. Выберем любое натуральное число $n$ и прибавим к нему сумму его цифр $S(n)$. Полученное число $K(n) = n + S(n)$ называется *порожденным*, а исходное число $n$ – его *генератором*. Например, если выберем число 53, порожденное им число равно $53 + 3 + 5 = 61$. Порожденное число может иметь более одного генератора. Наименьшее число с двумя генераторами равно 101, и его генераторами являются числа 91 и 100. Самопорожденное число – это число, у которого нет генератора. В статье журнала «The American Mathematical Monthly»[2] доказывалось, что существует бесконечно много самопорожденных чисел, но встречаются они гораздо реже, чем порожденные числа. Эти открытия Капрекара заинтересовали многих математиков и в разных странах мира появились много научных статьей, научных проектов по математике, программных продуктов, в которых исследовались различные новые свойства "Константы Капрекара" и множеств самопорожденных чисел и порожденных чисел.

Следуя Капрекару, я нашел новый способ получения натуральных чисел: $n \to M(n) = \frac{1}{3}(n + S(n) + \bar{n})$, где $\bar{n}$ - число, записанное теми же цифрами, но в обратном порядке. Число $M(n)$ будет всегда целым, так как числа $n, S(n), \bar{n}$ дают одинаковые остатки при делении на 3. Если в случае Капрекара неравенство $n < K(n)$ выполняется при всех натурального $n$, то в моем случае положение разнообразнее, т.е. возможны все 3 случая: $n < M(n)$, $n = M(n)$ и $n > M(n)$.

Моя функция получения новых натуральных чисел $n \to M(n)$ простая, естественная и она является аналогом функции Капрекара $K(n)$. В 1-й части данной статьи исследованы распределение $m$-циклов и их свойства. (Если $l$ – наименьшее натуральное такое, что $M^l(n) = n$, то числа $n \to M(n) \to M^2(n) \to \ldots \to M^{l-1}(n) \to M^l(n) \to n$ образуют $m$-цикл. В случае функции $K(n)$ циклы невозможны, т.к. $n < K(n) < K^2(n)$ ....). Также изучены вопрос равенства чисел $K(n) = M(n)$ и вопрос образования арифметической прогрессии в некотором порядке числами $n, K(n)$ и $M(n)$.

Во 2-й части статьи изучены кратные отношения между числами $n, K(n)$ и $M(n)$. Т.е. исследованы вопросы: при каких $t$ натуральном, $t \geq 2$, возможны равенства $K(n) = tM(n), tK(n) = M(n), n = t\, M(n), n * t = M(n)$. (Отметим, что кратные отношения между числами $n$ и $K(n)$ невозможны). Также исследованы распределение и свойства множества $m$ – порожденных чисел ($m -$порожденныех чисел встречаются гораздо реже, чем $m -$ самопорожденные, поэтому изучение множества $m$ – порожденных намного важнее, чем изучение класса $m$ – самопорожденных чисел). В этой части исследовано множество "соседних", т.е. последовательных $m$ – порожденных чисел.

В процессе исследования сформулированы 1 проблема и 9 гипотез.

**Ключевые слова**: М-функция, Д.Р.Капрекар, самопорожденные числа.

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# IMPROVING THE QUALITY AND RELIABILITY OF SIGNAL TRANSMISSION AND RECEPTION IN MULTISERVICE NETWORKS»

**Abstract.** This article describes the entire complete cycle of signal transmission and reception from the terminal device on the transmitting side to the terminal device on the receiving side, due to the fact that the reliability and quality of information transmission and reception in public access networks depend on many different parameters. The article describes the process of converting a signal from an analog form to a digital one. For this purpose, the so-called encoding of the graphic signal is used. To get a rich palette of colors, different intensities were set for the base colors. During the experiment, an IP video camera was used and digital traffic was prepared to transmit it over multiservice or open networks. The Novus IP video camera uses a web interface. During the initial installation, in order to access the camera interface, you must assign an IP address, subnet mask, and Ethernet adapter gateway to the PC. To ensure greater protection of information, it is necessary to apply encryption, which will increase the reliability of the process of transmitting and receiving information. Today, cryptography is one of the most used ways to ensure the confidentiality and authenticity of information. There are symmetric and asymmetric cryptosystems. In the symmetric encryption process, the same key is used for both encryption and decryption. Asymmetric systems use public and private keys that are mathematically related to each other. The DHCP-installer program was used and the distribution of IP addresses was obtained by agreeing to its terms. First, you need to configure the DHCP server. The DHCP Protocol is commonly used in most cases used in TCP/IP networks. In addition to the IP address, DHCP can also tell the client additional parameters that are necessary for normal network operation and these are called DHCP options. Today, there is a modern and high-speed Wi-Fi router that supports the wireless communication standard 802.11 a/b/g/n/ac. The router, in turn, transmits private information through any built-in interfaces. After high-speed transmission of traffic, the receiving side must perform the reverse encryption procedure-decryption, in order to get the original signal. As a result of the experiment on the equipment, we come to the conclusion that the studied parameters such as reliability, quality and secrecy of the transmitted information depend on the technical characteristics of the real equipment that was used in the experiment.

**Key words:** signal, IP-video camera, format, encryption, decryption, broadcast, receiving, quality, reliability, traffic.

Today, the reliability and quality of data transmission and reception in public access networks depend on many different parameters such as:

1) The operation of routers.
2) The quality of converting the source signal from one format to another (for example, digitization in IP cameras, etc.).
3) The use of various types of hiding information itself (for example, encryption).
4) Network service quality.

It means that the relevance of this task is undeniable. This article will cover the entire complete cycle of signal transmission and reception, starting from the terminal device on the transmitting side to the terminal device on the receiving side.

At the beginning, the process of converting a signal from an analog form to a digital one will be briefly considered. For this purpose, we will use the so-called encoding of the graphic signal. The screen resolution and color depth determines image quality. The number of colors (K) displayed on the display screen depends on the number of bits (N) allocated in video memory for each pixel:

$$K=2^N . \quad (1)$$

Where N is the value of the bit depth.

To achieve a intense palette of colors, the base colors can be set to different intensities. For instance, if the color depth is 24 bits, 8 bits (RGB) are allocated for each color, i.e. K = 28 = 256 intensity levels are possible for each color. One bit of video memory is occupied by information about one pixel on a black-and-white screen (without halftones). Figure 1 depicts an example of this transformation [1, 3].

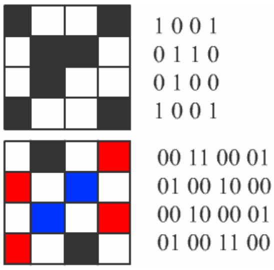


Figure 1 - Example of converting pixels (image points) to a bit sequence

For the experiment, an IP video camera "NOVUS NVIP-TC2400D / MPX1. 3-II" was used, which was installed on a laboratory stand designed for the study and research of analogue CCTV systems, on the left end wall of the laboratory table № 1 (figure 2) [2].

Figure 2 - Appearance of the laboratory table №1

The Novus IP video camera uses a web interface. During the initial installation, in order to access the camera interface, IP address, subnet mask, and Ethernet adapter gateway to the PC have to be assigned. In the address bar of the Internet browser, select the IP address:IP-адрес: 192.168.0.83.

Subnet mask: 255.255.255.0

sluice: 192.168.0.200

After all the pre-settings described above on endpoints such as IP video cameras, digital traffic was prepared for transmission over multiservice or open networks. To better protect information, you can now use encryption, which will increase the reliability of the process of transmitting and receiving information.

Today, cryptography is one of the most used ways to ensure the confidentiality and authenticity of information. There are symmetric and asymmetric cryptosystems.

In the symmetric encryption process, the same key is used for both encryption and decryption.

Asymmetric systems use public and private keys that are mathematically related to each other. Information is encrypted using a public key that is shared, and decrypted using a private key that is known only to the recipient of the message.

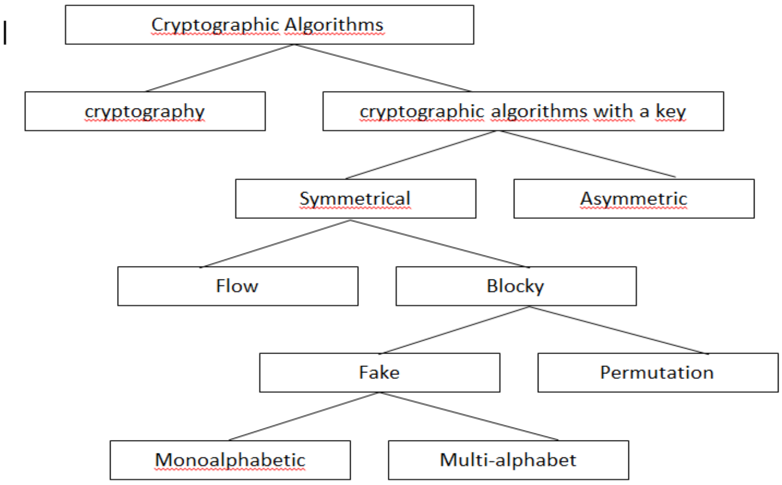


Figure 3 – General classification of cryptographic algorithms

After selecting and applying the encryption algorithm, the traffic is ready to transfer the already encrypted information over the communication channels. To do this, let's briefly consider how packet traffic is transmitted and received over communication networks [4].

First, it is needed to configure the Dhcp server. The Dhcp Protocol is commonly used in most cases used in TCP/IP networks.

In addition to the IP address, Dhcp can also tell the client additional parameters that are necessary for normal network operation and these are called Dhcp options.

There are the most frequently used options:

- The IP address of the default router;
- Subnet mask;
- DNS server address and
- DNS domain name.

Running the dhcp-installer program and agreeing to its terms, we get the distribution of IP addresses, which is shown in figure 4.

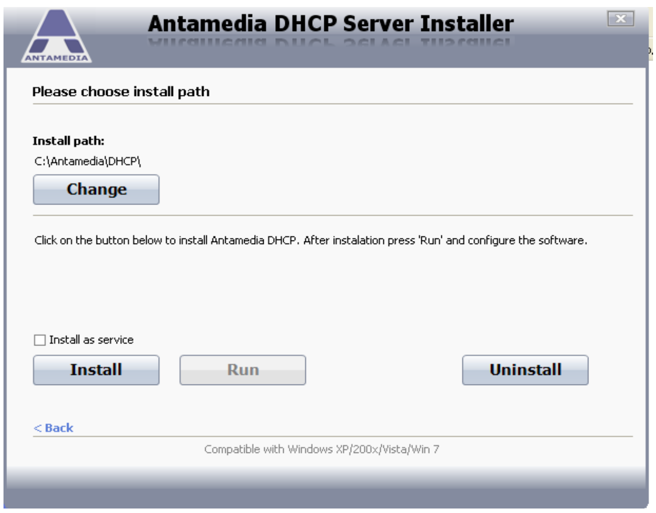


Figure 4 - Antamedia DHCP Server Installer window

Now the original signal is completely ready for transmission to the router, which in turn will transmit the closed information via any interfaces built into it, such as fiber, twisted pair or Wi-Fi, to the communication channels (figure 5) [5].

Figure 5 -ASUS RT-AC52U B1 router

Today, there are modern and high-speed Wi-Fi router which supports wireless standards 802.11 a/b/g/n/ac, and can operate in 2 frequency bands such as 2.4 GHz and 5 GHz. Therefore, there is a possibility of traffic with high speed is 733 Mbit/s, which will significantly affect the figure as processing time and packet delay, in addition to Wi-Fi interface it enables connection of copper source and optical fiber [5].

After the high-speed traffic on the receiver side, it was necessary to carry out the opposite procedure of encryption – decryption, in order to obtain the original signal. In this case, the IP video image.

**Conclusion.** After conducting an experiment on the equipment, it can be seen that the studied parameters such as reliability, quality and secrecy of the transmitted information depend on the technical characteristics of the real equipment that was used in the experiment and the use of an encryption system that generally improves the quality of service in multi-service public access networks.

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## МУЛЬТИСЕРВИСТІК ЖЕЛІДЕ ТАРАТУ ЖӘНЕ ҚАБЫЛДАУ КЕЗІНДЕ СИГНАЛДЫҢ САПАСЫ МЕН СЕНІМДІЛІГІН АРТТЫРУ

**Аннотация.** Бұл мақалада тарату мен қабылдаудың барлық аяқталған циклі қарастырылады, таратушы жақтан терминалды құрылғыдан қабылдау жағындағы терминалды құрылғыға, өйткені ақпараттың сенімділігі мен сапасы көпшілікке арналған желілерде әртүрлі параметрлерге байланысты болады. Мақалада сигналды аналогтық форматтан сандық түрге түрлендіру процесі сипатталған. Ол үшін графикалық сигналды кодтау деп аталатын әдіс қолданылады. Түстердің бай палитрасын алу үшін негізгі түстерге әр түрлі қарқындылық берілді. Тәжірибе барысында IP бейнекамерасы қолданылды және сандық трафик мультисервистік немесе ашық желілер арқылы берілуге дайындалған. Novus IP бейнекамерасы веб-интерфейсті қолданады. Алғашқы орнатудан кейін камераның интерфейсіне кіру үшін компьютерге IP мекенжайын, ішкі желі маскасын және Ethernet адаптерінің шлюзін тағайындау керек. Ақпараттың үлкен қорғалуын қамтамасыз ету үшін ақпаратты беру және қабылдау процесінде сенімділікті арттыратын шифрлауды қолдану қажет. Бүгінгі таңда криптография ақпараттың құпиялылығы мен шынайылығын қамтамасыз етудің ең көп қолданылатын әдістерінің бірі болып табылады. Симметриялық және асимметриялық криптожүйелер бар. Симметриялық шифрлау процесінде бірдей кілт шифрлау үшін де, шифрлау үшін де қолданылады. Асимметриялық жүйелерде бір-бірімен математикалық байланысқан ашық және жеке кілттер қолданылады. DHCP-installer бағдарламасы қолданылды және оның шарттарына сәйкес IP адрестердің таралуы алынды. Алдымен сіз DHCP серверін конфигурациялауыңыз керек. DHCP көбінесе TCP / IP желілерінде қолданылады. IP мекенжайынан басқа, DHCP клиентке желіде қалыпты жұмыс істеу үшін қажет болатын қосымша параметрлерді айта алады және оларды Dhcp опциялары деп атайды. Бүгінгі таңда 802.11a / b / g / n / ac сымсыз байланыс стандартын қолдайтын заманауи және жоғары жылдамдықты Wi-Fi маршрутизаторы бар. Маршрутизатор өз кезегінде жабық ақпаратты оған салынған кез-келген интерфейс арқылы жібереді. Трафикті жоғары жылдамдықпен бергеннен кейін, қабылдаушы жағында бастапқы сигналды алу үшін кері шифрлау процедурасын орындау керек - шифрлау. Жабдықта жүргізілген эксперимент нәтижесінде алынған мәліметтердің сенімділігі, сапасы және құпиясы сияқты зерттелген параметрлер экспериментте қолданылған нақты жабдықтың техникалық сипаттамаларына байланысты деп тұжырымдаймыз.

**Түйін сөз:** сигнал, IP-бейнекамера, формат, шифрлау, шифрлау, тарату, қабылдау, сапа, сенімділік, трафик.

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## УЛУЧШЕНИЕ КАЧЕСТВА И НАДЕЖНОСТИ СИГНАЛА ПРИ ПЕРЕДАЧЕ И ПРИЕМЕ В МУЛЬТИСЕРВИСНЫХ СЕТЯХ

**Аннотация.** В данной статье рассмотрен весь законченный цикл по передаче и приему сигнала, начиная от оконечного устройства с передающей стороны до оконечного устройства на приемной стороне, в связи с тем, что надежность и качество при передаче и приеме информации в сетях общего доступа зависят от множества различных параметров. Описан процесс преобразования сигнала с аналоговой формы в цифровую. Для этого использовано так называемое кодирование графического сигнала. Для получения богатой палитры цветов базовым цветам были заданы различные интенсивности. При проведении эксперимента использовалась IP-видеокамера и был подготовлен цифровой трафик для передачи его по мультисервисным или открытым сетям. Для использования IP-видеокамеры Novus используется веб-интерфейс. При первоначальной установке, для того чтобы получить доступ к интерфейсу камеры, необходимо назначить IP-адрес, маску подсети и шлюз Ethernet-адаптера на ПК. Для большего обеспечения защиты информации необходимо применить шифрование, что повысит надежность при самом процессе передачи-приема информации. На сегодняшний день криптография является одним из наиболее используемых способов обеспечения конфиденциальности и подлинности информации. Существуют симметричные и асиметричные криптосистемы. В процессе симметричного шифрования и для шифрования, и для дешифрования используется один и тот же ключ. В асиметричных системах используются открытый и закрытый ключи, связанные друг с другом математически. Использовалась программа DHCP-installer и, соглашаясь с ее условиями, была получена раздача IP – адресов. Вначале надо настроить DHCP-сервер. Протокол DHCP является часто используемым в большинстве случаев используемым в сетях TCP/IP. Кроме IP-адреса, DHCP также может сообщить клиенту дополнительные параметры, которые необходимы для нормальной работы в сети и они называются опциями DHCP. На сегодняшний день имеется современный и скоростной Wi-Fi роутер, который поддерживает стандарт беспроводной связи 802.11a/b/g/n/ac. Маршрутизатор в свою очередь передает закрытую информацию через любые встроенные в нем интерфейсы. После высокоскоростной передачи трафика, на приемной стороне необходимо провести обратную процедуру шифрованию – дешифрование, для того что бы получить исходный сигнал. В результате проведения эксперимента на оборудовании, приходим к заключению, что исследуемые параметры, такие как надежность, качество и скрытость передаваемой информации зависят от технических характеристик реального оборудования, которое было использовано в эксперименте.

**Ключевые слова:** сигнал, IP-видеокамера, формат, шифрование, дешифрование, трансляция, прием, качество, надежность, трафик.

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# EXPERIMENTAL RESEARCHES OF JUMP-SHAPED CHANGES OF CURRENT AND VOLTAGE IN TRANSIENT SWITCHING CIRCUITS OF AC GENERATION

**Abstract.** The article considers the transient process in switching electrical circuits, which are available in electric current generators in the form of a surge of supercurrents and voltage when removing, from isolated plates, brushes and which are lost in unused form. These bursts in collected form account for about-30-35% of the energy generated by the generator. The work is devoted to analysis of the available theoretical works on transients and experimental study of current surges and voltage in switching circuits, similar in electric motors and elk-regenerators. The results are given of the experimental study in the switching circuits of the electric generator in the work.

**Keywords:** current intensity, voltage, oscilloscope, oscillogram, delay line, switching, transient processes, current and voltage hopping, electro motoring force-EMF.

**Introduction** Here, it is necessary to recall the evolution of the development of an electromagnetic field based on Faraday 's experiments "on the relationship of a changing magnetic field to the appearance of a changing induction current," and Ersted 's experiments "that, a changing electric field is accompanied by a changing magnetic field." Another consequence of the field theorem should be mentioned. Let there be a turn through which the current arising from the Volt battery flows. Suddenly, the conductor 's connection to the current source breaks. Now, of course, there’s no current! But at the moment of this brief break, there is a complex process that can again be predicted by field theory. There was a magnetic field around the conductor before the current broke. It ceased to exist when the current was interrupted. Consequently, due to the current rupture, the magnetic field disappeared. The number of force lines passing through the surface surrounded by the chain changed very rapidly. But such a rapid change, no matter how fast it occurs, should cause an induction current. What really matters are that it is the change in the magnetic field that excites the induction current, the stronger the field change is. This conclusion is another test of theory. The current break must be accompanied by a strong short-term induction current. The experiment again confirms the prediction of the theory. Whoever ever broke the current noticed that there was a good spark. This spark indicates a huge potential difference caused by a rapid change in the magnetic field. The spark has quite considerable energy, so the magnetic field must have no less energy. To consistently apply the concept of field and its language, we must view the magnetic field as a supply of energy. Only by taking this path will we be able to describe magnetic and electrical phenomena in accordance with the law of energy conservation. - (A. Einstein, L. Infeld-Evolution of Physics, M. ed. ACT, 2018, c-151) All this determines the importance of considering switching analysis methods. Switching refers to any change in circuit parameters, its configuration, connection or disconnection of sources, which leads to transient processes. Switching will be considered instantaneous, but the transition process, as noted above, will take some time. In theory, it takes infinitely long to complete the transition, but in practice it is taken to be finite, depending on the parameters of the chain. Let us assume that switching is performed using the ideal key K (figure 1), whose resistance in the open state is infinitely large, and in the closed state is zero. The direction of closing or opening the key will be indicated by the arrow. We will also consider, unless otherwise stated, that switching takes place at the time of t = 0. The first and second switching laws are distinguished. The first switching law is related to the continuity of the change of the magnetic field of the inductor WL = Li2/2 and states that at the initial moment t = 0 immediately after switching, the current in the inductance has the same value as at the moment t = 0- before switching and from this moment gradually changes (hereinafter f (0-) means the left-hand limit of the function f(t) if t $\rightarrow$ 0-, and under f(0+ ) - right-hand limit f(t) if t $\rightarrow$ 0+ )

$$i_L = i_L(0_+) \quad (1)$$

The second switching law is related to the continuity of the change of the electric field of the capacity WC = Cu2/2: at the initial moment t = 0 immediately after switching, the voltage on the capacity has the same value as at the moment t = 0- before switching and from that moment gradually changes:

$$U_C\,(0_-) = U_C(0_+) \quad (2)$$

In contrast to the current in the iL inductance and the voltage at the uC capacitance, the voltage at the inductance uL and the current in the iC capacitance can be varied by a jump since according to (1. 9) and (1. 12) they are derived from iL and uC and are not directly related to magnetic and electric field energy. The values of the currents in the inductance of the iL (0) and the voltages at the capacitances of the uC (0) form the initial conditions of the task. Depending on the initial energy state of the circuit, two types of transient calculation tasks are distinguished: tasks with zero initial conditions, when immediately after switching (at t = 0) iL (0) = 0; uC (0) = 0 (i.e. WL (0) +WC (0) = 0) and tasks with non-zero initial conditions when iL (0+) 0 and/or uC (0+) 0 (i.e. WL(0 ) + WC(0 ) 0). The zero and non-zero values of the initial conditions for iL and uC are called independent and the initial conditions of the remaining currents and stresses are dependent. Independent initial of conditions is determined by switching laws (1) and (2).

The classical method of calculating transients in electrical circuits is based on the compilation of integral-differential equations for instantaneous values of currents and voltages. These equations are derived from Kirchhof laws, contour current methods, nodal stresses, and can contain both independent and dependent variables. For ease of solution, it is generally accepted to draw up differential equations with respect to an independent variable, which may be iL or uC. Solving the obtained differential equations with respect to the selected variable and constitutes the essence of the classical method.

Considering that in some cases the solution of differential equations is simpler than integral-differential equations, the obtained system is reduced to one differential equation of the corresponding order relative to the selected independent variable iL or uC. The order of the differential equation is determined by the number of independent electric and magnetic field energy accumulators. Denote an independent of variable (iL or uC) through x = x (t).

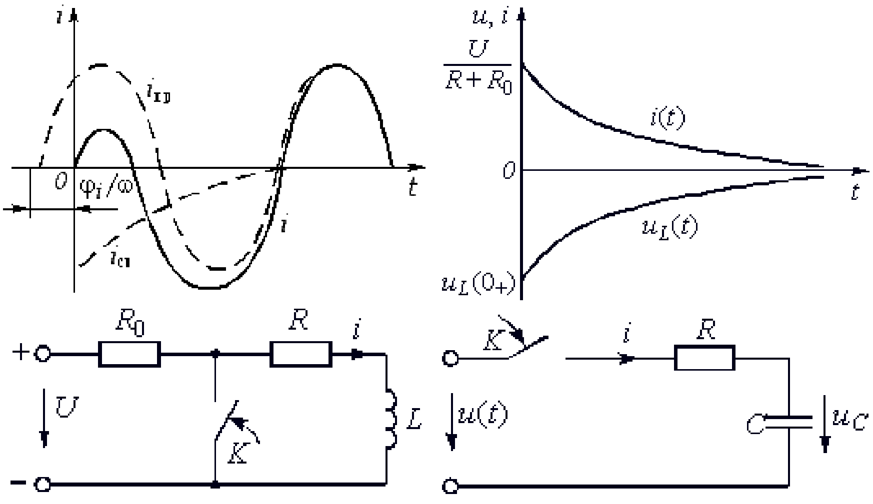


Figure 1

Analysis of the equations shows that in the case $\omega$ where the frequency of the applied voltage is substantially higher $\omega$ 0 where $\varphi$ C $\approx$ 0 han the resonant frequency of loop 0 at C 0, the circuit may be overvoltage, and in the case of $\omega \ll \omega_0$ и $\varphi$ C $\approx$ $\pi$/2 — overcurrents.

Analysis of equation shows that in the case of a circuit connection to source u (t) at the moment when $\varphi$ u = $\varphi$ $\pm$ $\pi$/2 Supercurrents may occur in the latter. If the chain time constant $\tau$ large enough, then the surge of current in the initial period can reach $i_{max}$ $\approx$ 2Im. In contrast, when the circuit is switched on, when $\varphi$ u = $\varphi$, In it immediately comes a steady-state regime. Similar picture is observed with voltage of inductance.

As a calculation example, consider the case of non-zero initial conditions in the RL chain. Magnetic field energy equal to WL = Li2(0– )/2, where i(0– ) = U/(R0 + R). After switching in the RL circuit, the transient process described by:

$$L\frac{di}{dt} + Ri = 0, \tag{3}$$

where iпр = 0. Solving equation (1), we find, taking into account (2) - (4):

$$i = i_{CB} = Ae^{pt} = Ae^{-t/\tau}.$$

Constant A is found from initial condition i (0-) and switching law (1):

$$i(_{-0}) = i(0_+) = \frac{U}{(R + R_0)} = A.$$

Finally, the law of changing the current in the transition mode is described by the equation

$$i = \frac{U}{R + R_0} e^{-t/\tau}. \tag{4}$$

The uL voltage is defined as

$$u_L = L\frac{di}{dt} = -\frac{U}{R + R_0} \mathrm{R} e^{-t/\tau}. \tag{5}$$

Figure 1 shows graphs i and uL. Note that all energy WL stored in the inductance over time is consumed for heat loss in R. Under non-zero initial conditions, L behaves as a source of current.

Switching on the RLC circuit to harmonic voltage

When RLC circuit is switched to harmonic voltage u = Umsin( $\omega$ t + $\varphi$ u ) forced voltage component on capacitance $u_{C\,пр} = U_{mC} \sin(\omega t + \varphi_C)$, (5) where $\varphi$ C = $\varphi$ u + $\varphi$ — $\pi$/2. Here, the phase shift between the current in the circuit and the applied voltage

$\varphi = \mathrm{arctg}(\omega L - 1/\omega C)/R$, (6) And amplitude of forced voltage at the capacitance

$$U_{mC} = \frac{1}{\omega C} \cdot \frac{U_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = \frac{1}{\omega C} \cdot I_{mC}. \tag{7}$$

Considering that the oscillatory circuit in radio engineering devices, as a rule, has high quality, that is, the condition is met R $\ll$ 2 $\rho$, that the free component of us is defined by the equation and the law of voltage change on the capacitance will be

$$u_C = U_{mC} \sin(\omega t + \varphi_C) + Ae^{-\alpha t} \sin(\omega_c t + \theta). \tag{8}$$

Taking the derivative from the expression (8), and taking into account that for the specified contour $\alpha \ll \omega_0 \approx \omega_C$, let's receive the current equation

$$i = C\frac{du_C}{dt} \approx I_{mC} \cos(\omega t + \varphi_C) + \omega_0 C A e^{-\alpha t} \cos(\omega_0 t + \theta). \tag{9}$$

Permanent integration A and $\theta$ We find from initial conditions and switching laws:

$$\left.\begin{aligned} u_C(0_-) = u_C(0_+) = 0 = U_{mC}\sin\varphi_C + A\sin\theta, \\ i(0_-) = i(0_+) = 0 = I_{mC}\cos\varphi_C + \omega_0 CA\cos\theta. \end{aligned}\right\} \quad (10)$$

Where

$$A = U_{mC}\sqrt{\sin^2\varphi_C + \left(\frac{\omega}{\omega_0}\right)^2\cos^2\varphi_C}; \quad (11)$$

$$\theta = \operatorname{arctg}\{\omega_0\operatorname{tg}\varphi/\omega\}. \quad (12)$$

Substituting values A and $\theta$ from the equations (11), (12) in (8) and (9), get the final law of change of voltage on capacitance and current in RLC- contour:

$$u_C = U_{mC}\sin(\omega t + \varphi_C) + U_{mC}\sqrt{\sin^2\varphi_C + (\omega/\omega_0)^2\cos^2\varphi_C}\cdot e^{-\alpha t}\sin(\omega_0 t + \theta); \quad (13)$$

$$i = I_{mC}\cos(\omega t + \varphi_C) + I_{mC}\sqrt{(\omega_0/\omega)^2\sin^2\varphi_C + \cos^2\varphi_C}\cdot e^{-\alpha t}\cos(\omega_0 t + \theta). \quad (14)$$

Analysis of the equations (8. 1), (8. 2) shows that in the case $\omega$ where the frequency of the applied voltage is substantially higher $\omega$ 0 where $\varphi$ C $\approx$ 0 han the resonant frequency of loop 0 at C 0, the circuit may be overvoltage, and in the case of $\omega \ll \omega_0$ и $\varphi$ C $\approx$ $\pi$/2 – overcurrents.

If the frequency of the reference voltage $\omega$ = $\omega$ 0, that is, isochronism phenomena occur in the circuit when the voltage on the capacitance and the current in the circuit change smoothly according to equations:

$$u_C = U_{mC}(1 - e^{-\alpha t})\sin(\omega_0 t + \varphi_C); \quad (15)$$

$$i = I_{mC}(1 - e^{-\alpha t})\cos(\omega_0 t + \varphi_C). \quad (16)$$

At the same time the transient process proceeds without overvoltage and overcurrents (figure 2, a).

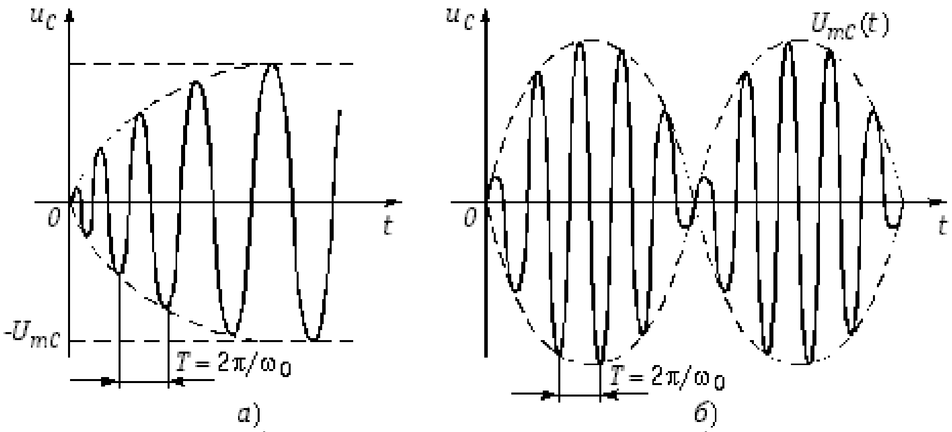


Figure 2

When the frequency of a given voltage $\omega$ and resonant frequency of the circuit $\omega$ 0 close to each other, then beat phenomena occur in the circuit. Let's put that

$$u_C = U_{mC}\left[\sin(\omega t + \varphi_C) + \sin(\omega_0 t + \varphi_C)\right] = U_{mC}(t)\sin(\omega_0 t + \varphi_C), \quad (17)$$

wheree UmC(t) = 2UmCcos $\Omega$t - amplitude of beats with angular frequency $\Omega$ = ( $\omega$ — $\omega$0 )/2. In figure 2, b, is a graph of the change in beat voltages (17).

**Experimental study of surge voltages and currents in transient processes during generation of electric energy**

For experimental investigation of surge-like pulsation of voltages and currents in generators during generation of electric energy, which are as a result of transient processes taking place in current-collecting contacts (brushes), specifically for this purpose, we have assembled an electric installation based on a DC motor. At the same time, standard stator windings, DC motors, were rewound in the number of 9 windings simulcast along stator walls. (figure 3). Electric motor is supplied from accumulator with voltage of 24B, which through contactors is alternately supplied to stator windings symmetrically wound by us 9 (figure 4).

Figure 3 Photo – Symmetrically wound 9 stator windings

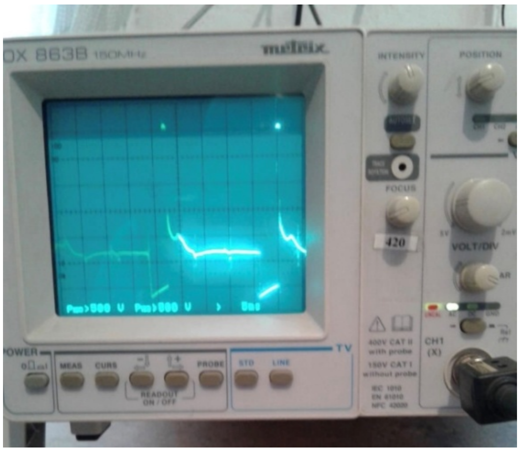


Figure 4 – Ostsilogramm of impulses

When voltage is transferred from one winding to another, transient processes occur in contactor where voltage surge is generated and which is supplied to base of transistor "JR-260," which operates as electronic switch, in collector circuit of which stator winding is connected. Such electronic keys are only 9-pieces, one for each stator winding (figure 4).

Thus, 9-pulses of voltage surges are generated in one revolution of motor in collector circuit of winding as a result of phenomenon of transient processes due to switching of 24B supply voltage from one winding to another. The number of stator windings, we specially increased from 3 to 9 by hand winding 9 new windings located symmetrically on the inner surface of the stator. (figure 1). Such changes allow us to obtain in one revolution the collector of 9-pulses of voltage jump as a result of transient processes when switching the supply voltage from one winding to another. During the study, the parameters of voltage hopping pulses were carefully studied by their oscillograms. With (figure 3, U = 250V, t = 0, 6 ms).

According to the electronic diagram assembled by us (figure 6), used as an electronic switch, the 24B supply voltage is switched from one winding to another. At each switching (break and connection of inding circuit) U = surge voltage pulse appears as a result of transient processes (figure 5).

Then this pulse is supplied to the base of transistor of JR-260 type, as a result of which emmiter-collector transition of transistor and supply voltage is opened 24B supplied to the next winding of stator. Such transistor-based electronic switches JR-260 only 9, one for each winding.

The obtained pulse voltages and currents on transistor collectors we can smooth and accumulate on capacitances via RC-chain diodes for further use of accumulated electric energy. For example: -to charge the power battery or for the direct underside of the stator rim. A similar use of energy obtained from the hopping pulses generated in the respective transients of each coil will increase the efficiency of motors and electric current generators by about 30%.

Typically, these pulses occurring in transients accompanying the operation of motors and generators based on the principle of electromagnetic induction are not used and wasted.

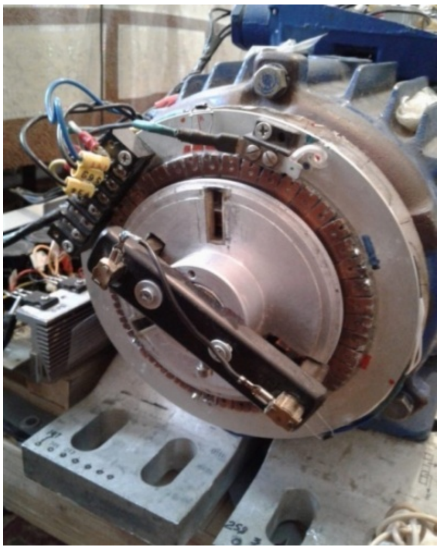

Figure 5 – Photo of manifold with contactors

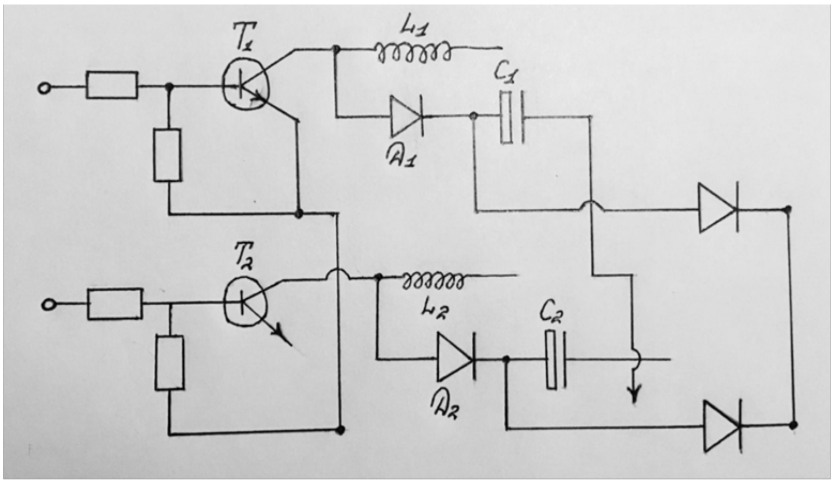


Figure 6 – Diagram of electronic switch for obtaining transient process in stator windings

**Analysis of results of experimental studies of jump pulses in transient processes**

1. Each time the 24B power supply switches from one winding to another, as a result of the power circuit break of the next winding, a surge voltage pulse appears as in standard transients.

2. Voltage pulse as a result of switching of power supply sources from one winding to another, by oscillograms obtained on the collector of the corresponding electronic key is estimated to be about 250B at pulse signal duration of about 0, 6 m/s. (figure 5).

At the same time duration of signal-pulse depends on rotation speed of rotor, windings of which are powered from the same power supply source of 24B, which powers simultaneously windings of stator and circuit of electronic switch.

3. Increase of battery supply voltage increases frequency of rotor turn and amplitude of pulse-signal on oscilloscope.

4. Pulse currents generated on the collectors of electronic key transistors can be smoothed and accumulated at capacitances through diodes and RC-chains (figure 6) followed by using it to charge batteries or use it to directly feed stator and rotor windings.

5. Similar use of surge voltage pulses appearing in transient processes in all electrical systems applying electromagnetic induction phenomenon makes it possible to use pulses and increase their efficiency up to 30% and more.

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## АЙНЫМАЛЫ ТОКТЫ ГЕНЕРАЦИЯЛАУ КЕЗІНДЕ, ӨТПЕЛІ ПРОЦЕСТІ ТУҒЫЗАТЫН,КОММУТАЦИЯ ТІЗБЕГІНДЕГІ ТОК КҮШІНІҢ ЖӘНЕ КЕРНЕУДІҢ СЕКРМЕЛІ ТҮРДЕ ТЕЗ ӨСУІН ЭКСПЕРИМЕНТ ЖҮЗІНДЕ ЗЕРТТЕУ

**Аннотация:** Мақалада электр тогының генераторларындағы асқын токтардың қалқуы және бір-бірімен оқшауланған, щеткалардан тұратын пластиналардан алу кезіндегі кернеу түріндегі электр тогы бар коммутациялық электр тізбектеріндегі ауыспалы процесті және олар пайдаланылмаған күйінде жоғалуы қарастырылады. Жиналған түрдегі бұл қалқалар генератор өндіретін энергияның шамамен – 30-35% құрайды. Жұмыс өтпелі кезеңдегі теориялық жұмыстарды талдауға және электр қозғалтқыштары мен электр генераторларына ұқсас коммутациялық тізбектердегі ток пен кернеу қалқандарын тәжірибелік зерттеуге арналған. Мақалада электрлік генератордың коммутациялық тізбектеріндегі тәжірибелік зерттеу нәтижелері келтірілген.

Тізбектің ауыспалы жұмыс режимі онда магнит және электр өрістерінің энергиясы жинақталатын реактивті элементтердің (индуктивтілік, сыйымдылық) болуымен байланысты. Әр түрлі әсер ету кезінде (тізбекке қосу немесе электр энергиясы көздерін ажырату, тізбек параметрлерін өзгерту) тізбектің энергетикалық жұмыс режимі өзгереді, бұл өзгерістер электр және магнит өрістері энергиясының үздіксіздігіне байланысты (үздіксіздік принципі) бірден жүзеге асырыла алмайды, бұл өтпелі процестердің туындауына алып келеді. Көптеген Байланыс құрылғылары мен жүйелеріндегі өтпелі процестер олардың жұмыс режимінің құрамдас "қалыпты" бөлігі болып табылатынын атап өткен жөн. Сонымен қатар, бірқатар жағдайларда өтпелі процестер аса ағындар мен асқын кернеулердің пайда болуы сияқты жағымсыз құбылыстарға әкелуі мүмкін.

Оның барлығы коммутацияны талдау әдістерін қарастырудың маңыздылығын анықтайды. Коммутация тізбек параметрлерінің, оның конфигурациясының кез келген өзгеруін, өтпелі процестердің туындауына әкелетін көздердің қосылуын немесе өшірілуін атауға болады. Коммутация бір сәттік деп есептейміз, бірақ өтпелі процесс, жоғарыда айтылғандай, белгілі бір уақытта өтеді. Өтпелі процесті аяқтау үшін теориялық тұрғыдан шексіз үлкен уақыт талап етіледі, бірақ іс жүзінде оны тізбек параметрлеріне байланысты шекті түрінде қабылдайды. Коммутация К идеалды кілтінің көмегімен жүзеге асырылады деп есептейміз. Тұйықталған күйде кедергісі шексіз үлкен, ал тұйықталмаған жағдайда нөлге тең. Кілттің тұйықталу немесе ажыратылу бағытын көрсеткімен көрсетеміз. Сонымен қатар, егер басқа айтылмаса, коммутация t = 0 сәтінде жүзеге асырылады деп есептейміз. Коммутацияның бірінші және екінші заңдары бар. Коммутацияның бірінші заңы $W_L = Li^2/2$ индуктивтілік катушкасының магнит өрісінің өзгеруінің үздіксіздігімен байланысты және бастапқы сәтте $T=0_+$ тікелей коммутациядан кейін индуктивтілік тогы бірдей мәнге ие, бұл $T=0_-$ коммутацияға дейін және осы сәттен бастап бірқалыпты өзгереді (мұнда және одан әрі $F(0_-)$ деп T=0 кезіндегі F(t) функциясының сол жақты шегі, ал $F(0_+)$ деп-$t=0_+$ кезіндегі f(t) оң жақты шегі түсініледі)

$$i_L(0_-) = i_L(0_+). \tag{1}$$

Коммутацияның екінші заңы сыйымдылығы $W_C = Cu^2/2$ электр өрісінің үздіксіз өзгеруімен байланысты: бастапқы сәтте $T=0_+$ коммутациядан кейін сыйымдылықтағы кернеу $t=0_-$ коммутацияға дейін және осы сәттен бастап бірқалыпты өзгереді:

$$u_C(0_-) = u_C(0_+). \tag{2}$$

$i_L$ индуктивтілік тогынан және $u_C$ сыйымдылығындағы кернеуден айырмашылығы $u_L$ индуктивтілік кернеуі және $i_C$ сыйымдылығындағы ток секіру түрінде өзгереді.

**Түйін сөздер:** Ток күші, кернеу, осциллограф, осциллограмма, кешігу сызығы, линия задержки, коммутация, ауысу процестері, ток пен кернеудің секірмелі өзгеруі, электр қозғаушы күш – ЭҚК.

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## ЭКСПЕРИМЕНТАЛЬНЫЕ ИССЛЕДОВАНИЯ СКАЧКООБРАЗНЫХ ИЗМЕНЕНИЙ СИЛЫ ТОКА И НАПРЯЖЕНИЙ В ПЕРЕХОДНЫХ КОММУТАЦИОННЫХ ЦЕПЯХ ГЕНЕРАЦИИ ПЕРЕМЕННОГО ТОКА

**Аннотация.** В статье рассматривают переходный процесс в коммутационных электрических цепях, которые имеются в генераторах электрического тока в виде всплеска сверхтоков и напряжения при снятии, с

изолированных между собой, пластин щетками и которые теряются в неиспользованным виде. Эти всплески в собранном виде составляют около 30-35% энергии, вырабатываемого генератором. Работа посвящена анализу имеющихся теоретических работ по переходным процессам и экспериментальному исследованию всплесков тока и напряжений в коммутационных цепях, аналогичного в электродвигателях и электрогенераторах. В работе приведены результаты экспериментального исследования в коммутационных цепях электрогенератора.

Переходной режим работы цепи обусловлен наличием в ней реактивных элементов (индуктивности, емкости), в которых накапливается энергия магнитного и электрического полей. При различного рода воздействиях (подключении к цепи или отключения источников электрической энергии, изменении параметров цепи) изменяется энергетический режим работы цепи, причем эти изменения не могут осуществляться мгновенно в силу непрерывности изменения энергии электрического и магнитного полей (принцип непрерывности), что и приводит к возникновению переходных процессов. Следует подчеркнуть, что переходные процессы во многих устройствах и системах связи являются составной "нормальной" частью режима их работы. В то же время в ряде случаев переходные процессы могут приводить к таким нежелательным явлениям, как возникновение сверхтоков и перенапряжений.

Все это определяет важность рассмотрения методов анализа коммутации. Коммутацией принято называть любое изменение параметров цепи, ее конфигурации, подключение или отключение источников, приводящее к возникновению переходных процессов. Коммутацию будем считать мгновенной, однако переходный процесс, как было отмечено выше, будет протекать определенное время. Теоретически для завершения переходного процесса требуется бесконечно большое время, но на практике его принимают конечным, зависящим от параметров цепи. Будем считать, что коммутация осуществляется с помощью идеального ключа К (рисунок 6. 1), сопротивление которого в разомкнутом состоянии бесконечно велико, а в замкнутом равно нулю. Направление замыкания или размыкания ключа будем показывать стрелкой. Будем также считать, если не оговорено иное, что коммутация осуществляется в момент t = 0. Различают первый и второй законы коммутации. Первый закон коммутации связан с непрерывностью изменения магнитного поля катушки индуктивности WL = Li2/2 и гласит: в начальный момент t = 0+ непосредственно после коммутации ток в индуктивности имеет то же значение, что и в момент t = 0– до коммутации и с этого момента плавно изменяется (здесь и далее под f(0- ) понимается левосторонний предел функции f(t) при t → 0-, а под f(0+ ) - правосторонний предел f(t) при t → 0+ )

$$i_L(0_-) = i_L(0_+). \quad (1)$$

Второй закон коммутации связан с непрерывностью изменения электрического поля емкости WC = Cu2/2: в начальный момент t = 0+ непосредственно после коммутации напряжение на емкости имеет то же значение, что и в момент t = 0 – до коммутации и с этого момента плавно изменяется:

$$u_C(0_-) = u_C(0_+). \quad (2)$$

В отличие от тока в индуктивности iL и напряжения на емкости uC напряжение на индуктивности uL и ток в емкости iC могут изменяться скачком.

**Ключевые слова:** сила тока, напряжение, осциллограф, осциллограмма, линия задержки, коммутация, переходные процессы, скачкообразные изменения тока и напряжения, электродвижущая сила-ЭДС.

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# RESEARCH OF CHARACTERISTICS OF HEAT AND MASS TRANSFER AT THE INTRODUCTION OF TECHNOLOGY OF STEPS FUEL BURNING ON THE BKZ-75 BOILER OF THE SHAKHTINSKAYA TPP

**Abstract**. This article presents computational experiments on the introduction of step-by-step fuel combustion (OFA) technology using the example of the combustion chamber of the BKZ-75 boiler at the Shakhtinskaya TPP. OFA technologies are based on the separation of the supplied oxidizing agent into the combustion space in such a way as to reduce the amount of fuel NOx in the area of the burners by reducing excess air, and the amount of thermal NOx by reducing the temperature of the flame in the area of the OFA injectors. Using computer simulation methods, various modes of supplying additional air to the combustion chamber of the BKZ-75 boiler through OFA injectors were studied: OFA=0% (basic version), OFA=10%, OFA=18%. As a result of computational experiments, the distributions of the concentrations of carbon monoxide CO and nitrogen dioxide NO2 were obtained over the entire volume of the combustion chamber. The most important result of the introduction of staged fuel combustion (OFA=18%) is a decrease in the concentration of nitrogen dioxide NO2 at the outlet of the combustion chamber by 25% and carbon monoxide CO by 36%. The results allow us to conclude that the introduction of Overfire Air (OFA) technology has a positive effect on the heat and mass transfer in the combustion chamber and minimizes emissions of harmful substances.

**Key words.** heat and mass transfer, fuel combustion, numerical simulation, computational experiment, OFA technologies (Overfire Air), carbon oxides, nitrogen dioxides, ecology.

## Introduction

Modern environmental problems that have arisen as a result of anthropogenic overload and irrational use of natural resources have undoubtedly affected the economic and environmental status of the Republic of Kazakhstan. For heat power engineering and other related industries, the task of reducing the cost of obtaining the required products and emissions of harmful substances is paramount.

In this regard, the issue of choice, operation, and, first of all, the creation of new, highly efficient energy and resource-saving and "clean" technologies of energy processes becomes relevant. This requires the implementation of a whole range of measures, the most important of which is the use of modern technologies, as well as world achievements in the field of development to optimize the combustion of solid pulverized coal fuel at thermal power plants (TPP) of Kazakhstan [1-9]. OFA technology ("Over Fire Air") is currently successfully used all over the world, and especially in Europe, since the introduction of such technology at existing thermal power plants requires low investment and contributes to a significant reduction in $NO_x$ emissions.. When used in combination with other measures to control and suppress the formation of $NO_x$, it is possible to reduce their emissions to 85%.

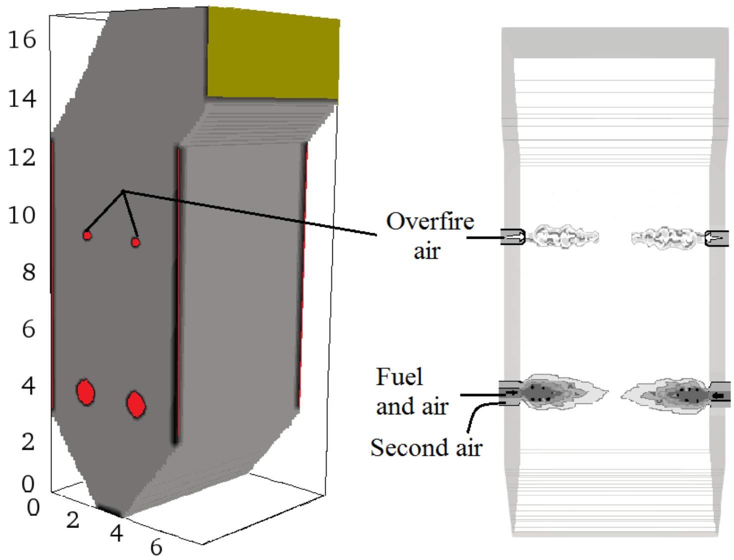
The OFA method, or as it is also called the "step-by-step method of burning fuel", includes the supply of the entire volume of combustion air (primary and secondary) in two stages: 70-90% of the air is supplied to the burners, and the rest is supplied to the combustion device over the burner "sharp blast". By mixing fuel with a controlled air flow in the burner, a relatively low-temperature, oxygen-depleted and fuel-rich combustion zone is created in the lower part of the combustion burner, which helps to reduce the formation of $NO_x$ from the nitrogen contained in the fuel (fuel $NO_x$) [10].

The remaining part of the air is supplied above the main combustion zone to several air channels located on the front and rear walls of the combustion chamber above the upper level of the burners, in order to achieve the most complete combustion of the fuel. The relatively low temperature in the oxygen-enriched afterburning zone leads to reduced formation of $NO_x$ from the air (thermal $NO_x$).

To model heat and mass transfer in the presence of physicochemical processes, the fundamental laws of conservation of such quantities as mass, momentum, energy are used. Since heat and mass transfer in the presence of physico-chemical transformations is an interaction of turbulent movements and chemical processes, we must also take into account the law of conservation of the components of the reacting mixture, turbulence, multiphase environment, heat generation due to radiation from a heated medium and chemical reactions [11-16].

**Object of research**

The combustion chamber of the BKZ-75 boiler of the Shakhtinskaya TPP (Shakhtinsk, Kazakhstan) was selected for numerical experiments to suppress nitrogen and carbon oxides using OFA technologies [17-24]. Figure 1 shows a general view of the combustion chamber of the BKZ-75 boiler (Figure 1a) and the layout of burner devices and injectors for the introduction of OFA technology (Figure 1b). The finite difference grid for numerical modeling has steps along the X, Y, Z axes: 90×32×158, which is 455 040 control volumes. Dust of Karaganda coal is burned in the boiler, with an ash content of 35.1%, a volatile yield of 22%, a moisture content of 10.6% and a heat of combustion of 18.55 MJ/kg. The main structural characteristics of the combustion chamber of the boiler BKZ-75 are presented in table. 1.




a) b)

Figure 1 - General view of the combustion chamber of the boiler BKZ-75 of the Shakhtinskaya TPP (a) and the layout of the burner devices and OFA injectors (b)

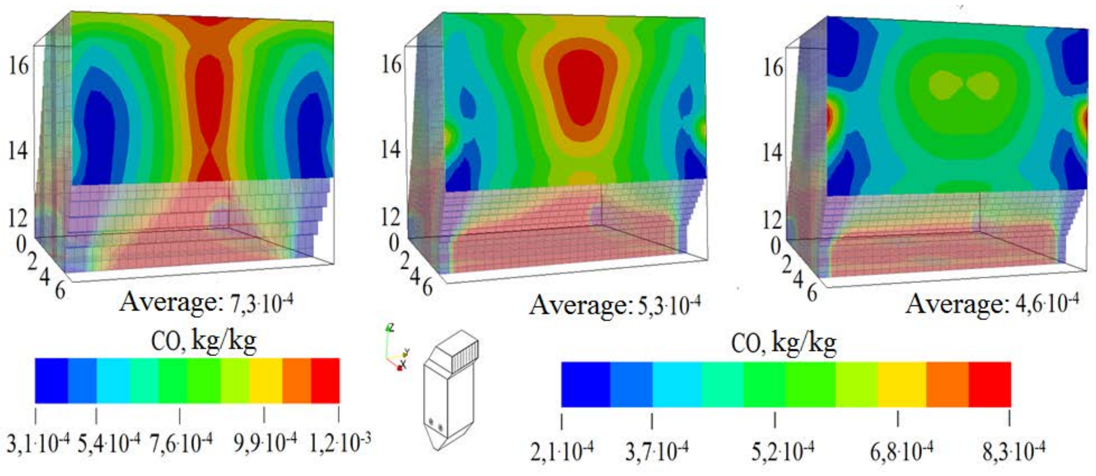
Table 1 - Structural characteristics of the boiler BKZ-75 of the Shakhtinskaya TPP during the organization of staged fuel combustion

| Characteristic | Value |
|---|---|
| Number of OFA injectors | 4 |
| The height of the burner, m | 4 |
| The height of the tier of OFA injectors, m | 9 |
| Diameter of OFA injectors, m | 0.325 |

Various modes of supplying additional air to the combustion chamber of the BKZ-75 boiler through OFA injectors were studied: OFA=0% (basic version), OFA=10% and OFA=18%. As a result of the computational experiments, the distributions of the concentrations of carbon monoxide CO and nitrogen dioxide $NO_2$ were obtained over the entire volume of the combustion chamber; at the exit from it, a comparative analysis was carried out for all the studied modes.

**Results of computational experiments**

Figure 2 shows the 3-D distribution of carbon monoxide concentrations CO at the outlet of the combustion chamber of the BKZ-75 boiler for three options for supplying additional air through OFA injectors: a) OFA=0% (basic version), b) OFA=10%, c ) OFA=18%. Analysis of Figure 2 shows that an increase in the volume of air supplied through OFA injectors reduces the concentration of carbon monoxide CO at the outlet of the combustion chamber from $7.3*10^{-4}$ kg/kg to $4.6*10^{-4}$ kg/kg, which makes up about 36%.




a) b) c)

Figure 2 - The distribution of the concentration of carbon monoxide CO at the outlet of the combustion chamber of the boiler BKZ-75 at various values of air supplied through OFA nozzles:
OFA = 0% (a), OFA = 10% (b), OFA = 18% (c)

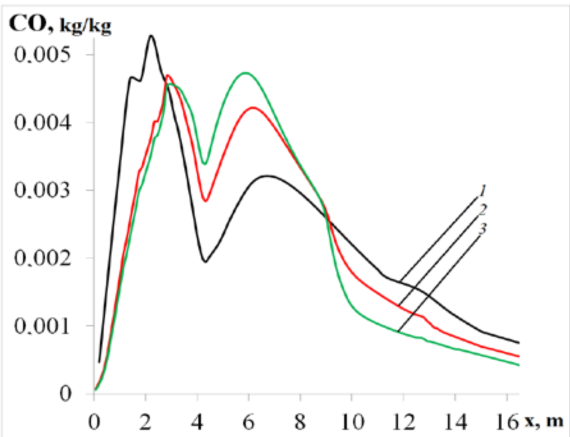



Figure 3 - Distribution of the concentration of carbon monoxide CO over the height of the combustion chamber of the BKZ-75 boiler at various values of air supplied through OFA nozzles:
1 - OFA = 0%, 2 -OFA = 10%, 3 - OFA = 18%

Figure 3 shows the distribution of carbon monoxide concentrations over the height of the combustion chamber of the BKZ-75 boiler for various values of air supplied through OFA nozzles: 1 - OFA= 0%, 2 - OFA=10%, 3 - OFA=18%. It can be noted that carbon monoxide is concentrated mainly in the zone of the main distribution of the fuel flow and oxidizer (air) from the burners, i.e. where there is a large amount of carbon fuel. With an increase in the volume of air supplied through OFA nozzles, further oxidation of carbon monoxide CO to carbon dioxide $CO_2$ occurs, which leads to a decrease in CO in the exhaust gases and at the exit from the combustion space (as shown in figure 2).

Distributions of $NO_2$ concentrations at the outlet of the combustion chamber of the BKZ-75 boiler for three options for supplying additional air through OFA injectors: a) OFA=0% (basic version), b) OFA=10%, c) OFA=18% are shown in figure 4. Analysis of the concentration field of nitrogen dioxide NO2 at the exit from the combustion space indicates a significant effect of stepwise combustion technology on the distribution of the concentration of this component. It can be seen that with an increase in the volume of air supplied through OFA nozzles, there is a significant decrease in the concentration of $NO_2$ at the outlet of the combustion chamber compared to the basic mode: at OFA=0% - 564.4 mg/nm$^3$, at OFA=10% - 509.44 mg/nm$^3$, at OFA=18% - 424.88 mg/nm$^3$. This is primarily due to the relatively low temperature in the oxygen enriched zone of OFA injectors, which leads to a decrease in the formation of $NO_x$ from the air (thermal $NO_x$). The maximum permissible concentration (MPC) for nitrogen oxides $NO_X$, adopted in Kazakhstan, is about 850 mg/nm$^3$.

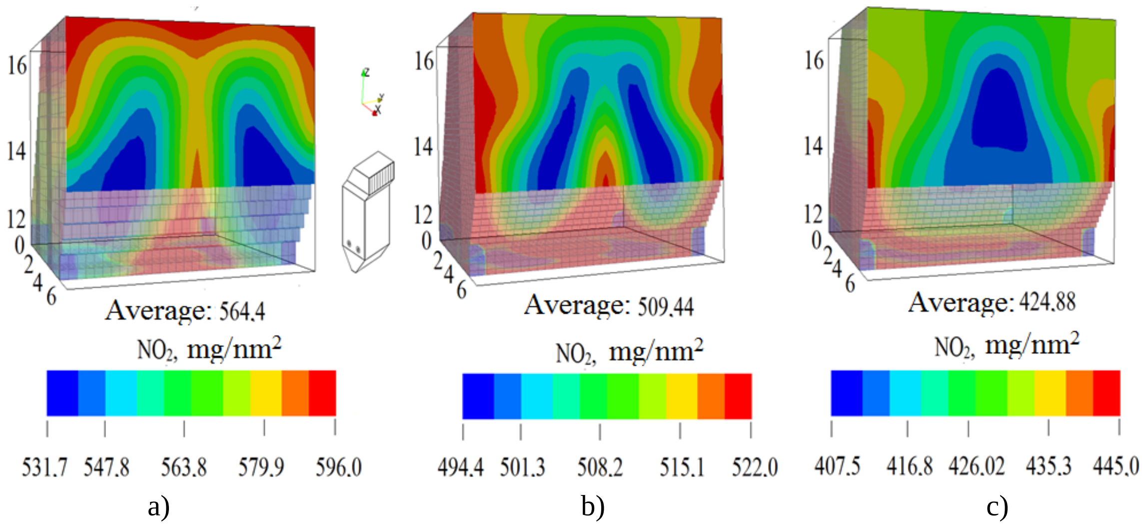



Figure 4 - Distribution of the concentration of nitrogen dioxide $NO_2$ at the outlet of the combustion chamber of the BKZ-75 boiler at various values of air supplied through OFA nozzles: OFA = 0% (a); OFA = 10% (b); OFA = 18% (c)

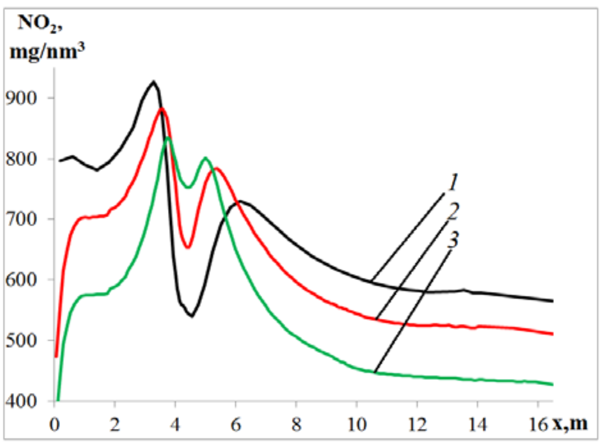



Figure 5 - Distribution of the concentration of nitrogen dioxide $NO_2$ along the height of the combustion chamber of the BKZ-75 boiler at various values of air supplied through OFA nozzles: 1 - OFA = 0%, 2 -OFA = 10%, 3 - OFA = 18%

This pattern of $NO_2$ behavior is confirmed by figure 5, which shows the distribution of $NO_2$ concentration over the height of the combustion chamber of the BKZ-75 boiler for the cases: OFA=0%, OFA=10%, OFA=18%. An analysis of this figure shows that the main gas generation of $NO_X$ occurs in the region of propagation of the mixtures of air from the burners. The nature of the distribution of the curves in this region is ambiguous, which indicates a complex process of the formation of nitrogen dioxide $NO_2$ in this region. Like figure 4, figure 5 talks about the effect of step-by-step combustion technology on the formation and suppression of nitrogen oxides (at OFA=18%, the $NO_2$ concentration at the outlet decreases by almost 25%).

**Conclusion**

The results of a study on the introduction of OFA technology for heat and mass transfer processes occurring in areas of real geometry, which are the combustion chambers of TPPs, when burning energy fuel in them are presented. Numerical experiments were carried out using 3-D computer simulation methods. A comparison was made for different modes of supplying additional air through OFA injectors into the combustion chamber of the BKZ-75 boiler: OFA=0% (basic version), OFA=10%, OFA=18%. It has been shown that the introduction of OFA technology at the BKZ-75 boiler of the Shakhtinskaya TPP can significantly reduce emissions of harmful substances, such as carbon monoxide CO and nitrogen dioxide $NO_2$, which will improve the environmental situation at coal-burning thermal power plants in the republic.

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## ШАХТИНСК ЖЭО БКЗ-75 ҚАЗАНДЫҒЫНДА ОТЫННЫҢ САТЫЛЫ ЖАНУ ТЕХНОЛОГИЯСЫН ЕНГІЗУ КЕЗІНДЕГІ ЖЫЛУ МАССА АЛМАСУ СИПАТТАМАЛАРЫН ЗЕРТТЕУ

**Аннотация.** Табиғи энергия ресурстарының сарқылуы және қоршаған ортаның ластануы жағдайында энергетикалық және экологиялық қауіпсіздік проблемасын шешу жаңа энергетикалық стратегияның маңызды міндеттері болып табылады. Қазақстан өзінің қажеттіліктерін қанағаттандыру үшін ғана емес, басқа өңірлерге экспорттау үшін де жеткілікті энергетикалық ресурстардың үлкен қорына ие. Республиканың энергетикалық ресурстарының теңгерімінде тас және аз дәрежеде қоңыр көмір басым. Бүгінде әлемде көмір ЖЭС-да электр және жылу энергиясының 50%-дан астамы, ал Қазақстанда – 85%-ға жуығы өндіріледі. Қазақстанда зиянды заттарды шығарудың қатаң нормаларын сақтай отырып, энергия өндіруге байланысты процестердің тиімділігін арттыру туралы мәселе өте өткір тұр. Осыған байланысты зиянды Шаң-газ шығарындыларын қалыптастырудың негізгі процестерін бақылауға мүмкіндік беретін энергиялық тиімді технологияларды құру және оларды төмендету жөніндегі ұсыныстарды әзірлеу жылу энергетикасының өзекті міндеті болып табылады. Шаңкөмірлі отынын жағудың энергетикалық қондырғыларын жетілдіру жөніндегі прогрессивті технологиялық процестер саласындағы зерттеулер және отынның әр түрлерін жағудың баламалы әдістерін пайдалану қазіргі уақытта Қазақстан Республикасының барлық энергетикалық кешені үшін неғұрлым өзекті болып табылады.

Азот оксидтерінің шығарындыларын азайтудың әртүрлі әдістері бар, олардың ішінде отындық камерада отынды жағу сатысында азот оксидтерін басу технологиясын енгізу неғұрлым орынды болып табылады. Отынды сатылы жағу - "Overfire Air" (OFA) технологиясы $NO_x$ азот оксиді концентрациясын төмендетудің тиімді әдістерінің бірі болып табылады. Сатылы ауаны оттық кеңістік кезінде OFA технологиялар жасалады

беруде қажетті ауа көлемі үшін көмірді жағу былайша: 70-90% ауа беріледі оттықтары және 10-30% – арқылы OFA-инжекторы орналасқан үстінен жанарғы құрылғылары. Бұл жағдайда отындық құрылғының төменгі бөлігінде оттегімен аздаған төмен температуралы және отынмен байытылған жану аймағы құрылады, бұл отын азотынан NOx түзілуін төмендетуге мүмкіндік береді (отындық NOx). Сонымен қатар, OFA-инжекторлардың оттегімен байытылған аймағындағы төмен температура ауадан NOx түзілуін азайтуға әкеледі (термиялық NOx).

Бұл мақалада Шахтинск ЖЭО БКЗ-75 қазандығының оттық камерасы мысалында "Overfire Air" (OFA) технологиясын енгізу бойынша есептеу эксперименттері ұсынылған. БКЗ-75 қазандығы майданнан және тылдан бір қабатқа екі жанарғы орнатылған төрт шаң бұрышымен жабдықталған. Қазандықта Қарағанды қатардағы көмірдің (КР-200) шаңы жағылады, күлдігі 35,1%, ұшқыштың шығымы 22%, ылғалдылығы 10,6% және жану жылуы 18,55 Мдж/кг. Газдар мен сұйықтықтардың ағымын сипаттайтын математикалық модель масса мен импульсты сақтау теңдеулеріне негізделген. Жылу беру процестері болатын ағындар үшін, сондай-ақ қысылған орталар үшін энергияны сақтау теңдеуін қосымша шешу қажет. Әртүрлі құрамдастарды араластыру процестерімен, жану реакцияларымен және т.б. ағымдарда қоспа компоненттерінің сақталу теңдеуін қосу қажет. Турбуленттік ағыстар үшін теңдеулер жүйесі турбуленттік сипаттамаларға арналған көліктік теңдеулермен толықтырылады. Отын мен ауаның айналмалы ағындары үшін жалпы жағдайда күрделі үшөлшемді есепті шешу талап етіледі.

Компьютерлік модельдеу әдістерімен зерттелді әр түрлі режимдері беру оттық камераны БКЗ-75 қосымша ауа арқылы OFA-инжекторы: OFA=0% (базалық-нұсқа), OFA=10%, OFA=18%. Жүргізілген есептеу эксперименттерінің нәтижесінде оттық камераның барлық көлемі бойынша CO көміртегі оксидтерінің және $NO_2$ азот диоксиді концентрацияларының бөлінуі алынды. OFSA-технологиясын енгізудің ең маңызды нәтижесі оның көмегімен OFA=18%-ға 25%-ға және CO көміртегі оксиді 36%-ға пайдалану кезінде оттық камерасынан шығуда $NO_2$ азот диоксиді концентрациясының төмендеуі болып табылады. Алынған нәтижелер "Overfire Air" (OFA) технологиясын енгізу жылу масса алмасу процесіне оң әсер етеді және зиянды заттардың шығарылуын азайтуға мүмкіндік береді деген қорытынды жасауға мүмкіндік береді.

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## ИССЛЕДОВАНИЕ ХАРАКТЕРИСТИК ТЕПЛОМАССОПЕРЕНОСА ПРИ ВНЕДРЕНИИ ТЕХНОЛОГИИ СТУПЕНЧАТОГО ГОРЕНИЯ ТОПЛИВА НА КОТЛЕ БКЗ-75 ШАХТИНСКОЙ ТЭЦ

**Аннотация.** В условиях истощения природных энергоресурсов и загрязнения окружающей среды решение проблемы энергетической и экологической безопасности являются важнейшими задачами новой энергетической стратегии. Казахстан располагает огромными запасами энергетических ресурсов, достаточными для удовлетворения не только своих потребностей, но и для экспорта в другие регионы. В балансе энергетических ресурсов республики преобладают каменные и, в меньшей степени, бурые угли. В мире на угольных ТЭС вырабатывается более 50% электрической и тепловой энергии, а в Казахстане – почти 85%. В Казахстане очень остро встает вопрос о повышении эффективности процессов, связанных с производством энергии, при соблюдении строгих норм выброса вредных веществ. В этой связи создание энергоэффективных технологий, позволяющих контролировать основные процессы формирования вредных пылегазовых выбросов, и разработка рекомендаций по их снижению является актуальной задачей теплоэнергетики. Исследования в области прогрессивных технологических процессов по совершенствованию энергетических установок сжигания пылеугольного топлива и использованию альтернативных методов сжигания различных видов топлива являются в настоящее время наиболее актуальными для всего энергетического комплекса Республики Казахстан.

Имеются различные методы сокращения выбросов вредных веществ, наиболее целесообразным из которых является внедрение технологии подавления оксидов азота на стадии сжигания топлива в топочной камере. Ступенчатое сжигание топлива – технология «Overfire Air» (OFA) является одним из эффективных методов снижения концентрации оксидов азота $NO_x$. Ступенчатая подача воздуха в топочное пространство при OFA-технологии заключается в подаче необходимого объема воздуха для сжигания угля следующим

образом: 70-90% воздуха подается в горелки и 10-30% – через OFA-инжекторы, которые расположены над горелочными устройствами. В этом случае в нижней части топочного устройства создается низкотемпературная обедненная кислородом и обогащенная топливом зона горения, что позволяет снизить образование NOx из азота топлива (топливные NOx). В то же время низкая температура в обогащенной кислородом зоне OFA-инжекторов приводит к минимизации образования NOx из воздуха (термические NOx).

В данной статье представлены результаты вычислительных экспериментов по внедрению технологии «Overfire Air» (OFA) на примере топочной камеры котла БКЗ-75 Шахтинской ТЭЦ. Котел БКЗ-75 оборудован четырьмя пылеугольными горелками, установленными по две горелки с фронта и с тыла в один ярус. В котле сжигается пыль Карагандинского рядового (КР-200) угля, зольностью 35,1%, выходом летучих 22%, влажностью 10,6% и теплотой сгорания 18,55 MJ/kg. Течение газов и жидкостей описывается математической моделью, основанной на уравнениях сохранения массы и импульса. Для потоков, в которых происходят процессы теплопередачи, а также для сжимаемых сред необходимо дополнительно решать уравнение сохранения энергии. В течениях с процессами смешивания различных составляющих, с реакциями горения и др. необходимо добавить уравнение сохранения компонентов смеси. Для турбулентных течений система уравнений дополняется транспортными уравнениями для турбулентных характеристик. Для вращающихся потоков топлива и воздуха требуется в общем случае решение сложной трехмерной задачи.

Методами 3-D компьютерного моделирования исследованы различные режимы подачи в топочную камеру котла БКЗ-75 дополнительного воздуха через OFA-инжекторы: OFA=0% (базовый вариант), OFA=10%, OFA=18%. В результате проведенных вычислительных экспериментов были получены распределения концентраций оксидов углерода CO и диоксида азота $NO_2$ по всему объему топочной камеры. Наиболее важным результатом внедрения ступенчатого сжигания топлива (OFA=18%) является снижение концентраций диоксида азота $NO_2$ на выходе из топочной камеры на 25% и оксида углерода CO на 36%. Полученные результаты позволяют сделать вывод о том, что внедрение технологии ступенчатого сжигания топлива (OFA) положительно влияет на процесс тепломассообмена в камере сгорания и позволяет минимизировать выбросы вредных веществ на казахстанских ТЭЦ.

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# THE MODELING OF A FLOW IN FLAT AND RADIAL CONTACT UNITS WITH A STILL GRANULAR LAYER. EVALUATIONS OF A CYLINDRICAL REACTOR WITH A RADIAL GAS INPUT AND A FREE OUTPUT (PART 1)

**Abstract.** Heterogeneous catalytic processes conducted in axial or radial type reactors with a still catalytic layer are some of the most important elements of the chemical technology. The attention of scientists and manufacturers to the investigation and application of these contact units deals with the following advantages: a highly developed surface of a phase separation, a possibility to provide a high flow velocity and hence to decrease sizes and a material consumption, a construction simplicity and a reliability of an exploit. Improving an operation of contact units may be achieved by refining present technologies, catalysts, disperse system structures and by creating new ones. Nevertheless, in some cases large scale hydrodynamic heterogeneities in a working zone of the unit cancel out efforts to increase an efficiency of chemical, heat/mass transfer and other processes. The exploration of reasons of the hydrodynamic heterogeneities formation requires an investigation of liquid and gas motion physics features in granular layers. A practice of a chemical reactors exploitation reveals that technical and economical indicators of an industrial process are as a rule lower than the calculated ones, derived on a stage of the process design. Now it can be considered proven that one of the reasons affecting the reactor output is the heterogeneity of a reagents flow in a granular catalyst layer. The article deals with a mathematical modeling of an incompressible liquid flow in flat and radial contact units with the still granular layer and a creation of numerical realization methods for the model

We propose a cycle of articles dealt with a model of a real reactor that consists of three parts: a distributing manifold, a collecting manifold and a working zone, where the still layer of a granular catalyst is loaded. An input and an output are made with a Z-shaped scheme. We consider processes and their equations in each reactor zone in detail.

**Keywords:** chemical reactor, still granular layer, catalyst, Ergun law, stream function, granular layer resistance factor, Green's function, pressure field, velocity field, layer resistance.

The vast amounts of works are dealt with revealing the equations of an incompressible liquid motion in the still granular layer. These equations are constructed by phenomenological and statistical methods [1-4]. In the first case equations are written down phenomenologically and an interpretation of some parts is conducted using the averaging of a microscopic model [1-2]. The statistical method is based on time, ensemble and space ways of averaging correspondent micro-equations, that describe a continuous one-phase medium motion and the motion of several one-phase media with account for boundary conditions on inter-phase surfaces [3- 4]. For deriving the averaging equations the kinetic theory of a disperse media and Vokker-Planck differential equation were applied. As a result of these approaches there were obtained either different modifications of Darcy and Ergun equations or, as in a turbulence theory, non-closed systems of equations that may be closed with account for a structure and physical properties of phases in the mixture [5-7]. This is the main problem in modeling heterogeneous media.

Contact units of a radial type with the still granular material are widely used in technological processes of different industries. A chemical reactor with the still layer of a tableted catalyst that is used in a large-capacity petrochemical industry can be mentioned as an example. One of the reasons that decreases the efficiency of such units is a heterogeneity of a reagents flow in a reactor working zone. It is known that the appearance of heterogeneities in a steam and raw mixture flow is caused by two factors. The first factor is the heterogeneity of the catalyst layer structure, for example, its porosity (or density) that appears during the process of a layer making (in filling the unit) [8-10] and during the further operation as a result of packing by gravity, vibration, breaking catalyst granules and so on. The second one is a bad choice of a ratio between geometrical and hydraulic parameters of a unit during its design.

It is considered that the heterogeneity of the reagents flow in the reactor working zone sufficiently influences process indicators only if a chemical reaction takes place either near the catalyst surface or on it. Indeed, at these conditions the velocity of reacting products directly defines the time of a contact with the catalyst. Main characteristic parameters of the reaction depend on this time. If the reaction takes place inside a porous space of catalyst granules then the contact time is defined by a diffusive reagents velocity and does not depend on a flow velocity near the granule. In the case it is assumed that the flow heterogeneity does not influence the chemical reaction kinetics.

Indeed that is not so. The majority of practically using reactions are accompanied by heat consumption or emission, so they are endothermic or exothermic. Hence if the reaction takes place in an interdiffusive area then some heat should be brought in or out, because the efficiency of the reaction often depends upon a temperature. To hold the specified temperature regime of the catalyst layer a neutral heat carrier, for example an overheated steam, is added to source reactants. It is well known that the flow heterogeneity of such steam-raw mixture causes an inhomogeneous temperature field and therefore leads to an appearance of overcooled or overheated parts in the catalyst layer. In addition to decreasing the output of a target product that results in sintering the catalyst or losing its catalytic properties.

Heterogeneities in the catalyst layer structure may be removed by using special ways of loading [10-11] or by an application a modular catalyst, where it is possible. By now these ways of loading and the technology of the catalyst module production have been already invented and continue to be developed. The flow nonuniformity that is caused by the reactor construction may be investigated and removed on the base of hydroaerodynamic calculations which allow to define the velocity and pressure fields in the unit in dependence on its geometrical and hydraulic parameters [12-13].

**The reactor model and the problem setting**

Let us produce a scheme of a real reactor as a model that consists of three parts (figure 1). The distribution manifold *I* is a cylindrical pipe of $R_1$ radius with a blind end and a perforated side surface. The collecting manifold *II* is made by a cylindrical perforated shell of $R_2$ radius and an outer wall of the unit. The working zone *III* is the still layer of the granular catalyst placed between two perforated coaxial cylindrical surfaces. At PJSC "Nizhnekamskneftekhim" Z-shaped reactors are used for obtaining a styrene, so an input and an output of a gas stream in the model is realized according to Z-shaped scheme.

The pressure drop in radial reactors is not large and it is about some tenth parts of an atmosphere, as a rule. The velocity of the steam-raw mixture is about 1 m/s and Mach number $M \ll 1$, therefore the gas passing the reactor is considered as incompressible.

$$\mathrm{div}\,\boldsymbol{v} = 0 \tag{1}$$

A stream energy change in *I* and *II* domains on account the gas outflow and inflow through perforated surfaces considerably exceeds energy dissipation due to a viscous friction [14]. Hence the flow in *I* and *II* domains is supposed to be potential:

$$\mathrm{rot}\,\boldsymbol{v} = 0 \tag{2}$$

Since gas velocities are about 1 m/s we assume a correctness of Ergun's square law and write it down in an invariant form:

$$grad\boldsymbol{p} = -f|v|v \tag{3}$$

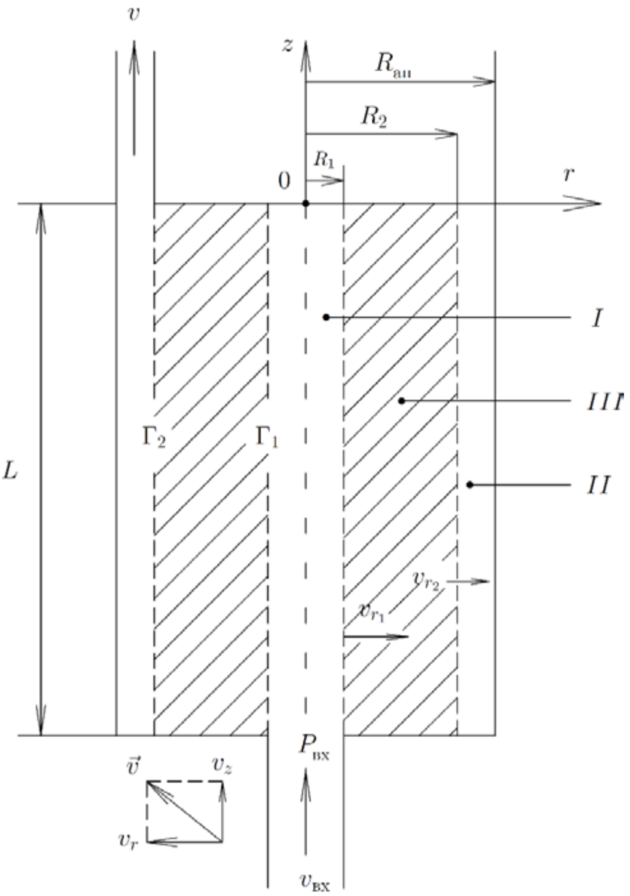


Figure 1 - The unit scheme with the radial gas input

We assume that the flow in units under consideration as axisymmetric, choose the origin and directions of coordinate axis as it is shown in the fig.1 and write down equations (1) – (3) in cylindrical coordinates: the continuity equation

$$\frac{1}{r} \bullet \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \tag{4}$$

the rotor of the velocity vector

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0 \tag{5}$$

Ergun’s law:

$$\frac{\partial p}{\partial r} = -f \; v_r |v|$$

$$\frac{\partial p}{\partial z} = -f \; v_z |v| \tag{6}$$

In *I* and *II* domains the pressure is defined according to Bernoulli's principle:

$$\frac{\rho v^2}{2} + p = const \tag{7}$$

In these equations:
$\upsilon_r$ and $\upsilon_z$ are the radial and axis components of the velocity correspondingly; $\rho$ is a gas density;

$$f = \frac{1{,}75\rho(1-\varepsilon)}{d\varepsilon^3}$$

is a factor of the granular medium resistance; *d* is an effective diameter of a granule; *ε* is a medium porosity.

Let Ψ be a flow function, so we write the solution of the equation (1.4) as

$$v_r = \frac{1}{r} \bullet \frac{\partial \Psi}{\partial z}$$
$$v_z = -\frac{1}{r} \bullet \frac{\partial \Psi}{\partial r} \quad (8)$$

by virtue of the pressure mixed derivatives equality and in case of *f* = const we derive from eq. (6):

$$\frac{\partial^2 \Psi}{\partial z^2}(1+\frac{v_r^2}{v^2}) + \frac{\partial^2 \Psi}{\partial r^2}(1+\frac{v_z^2}{v^2}) - 2\frac{\partial^2 \Psi}{\partial r \partial z} \bullet \frac{v_r v_z}{v^2} + 2v = 0 \quad (9)$$

At $\Gamma_1$ and $\Gamma_2$ boundaries (see fig. 1) between *I – III* and *III – II* domains correspondingly the continuity conditions for normal components of velocities and pressure are obeyed:

$$v_{r_1} = v_{r_3}, \quad v_{r_2} = v_{r_3},$$
$$p^{(I)} = \Delta p_1 + p^{(III)}, \; p^{(II)} = p^{(III)} - \Delta p_2; \quad (10)$$

where $\Delta p_{1,2}$ is a pressure jump on perforated walls of distributing and collecting manifolds, i.e. on $\Gamma_1$ and $\Gamma_2$ boundaries, that equals

$$\Delta p_{1,2} = \sigma_{1,2} v_{r_{1,2}}^2 . \quad (11)$$

According to [15-20] the resistance coefficient σ that corresponds to a midrange flow rate through the side surface $\upsilon_{1,2}$ may be expressed by a free section $\varphi_{1,2}$:

$$\sigma_{1,2} = \frac{\rho}{2}[\phi_{1,2}^{-1}(1-\phi_{1,2} + \sqrt{\frac{1-\phi_{1,2}}{2}})]^2, \quad (12)$$

where 1 and 2 indexes are for $\Gamma_1$ and $\Gamma_2$ boundaries.

If the normal velocity component on $\Gamma_1$ and $\Gamma_2$ boundaries is set then the full determination of the flow parameters can be conducted for all three domains separately and comes down the solution of elliptical equations like (5) and (9) for the flow function in each domain. The determination of the velocity normal component at $\Gamma_1$ and $\Gamma_2$ boundaries are made due to the continuity condition for pressure (10) in passing the boundaries. To accomplish this condition at specified velocity normal components a target function is constructed that equals to the mean-square pressure jump at boundaries:

$$\Phi(v_{r_1} v_{r_2}) = \frac{1}{L}\int_{\Gamma_1}(P^{(I)} - P^{(III)} - \Delta P_1)^2 dr + \frac{1}{L}\int_{\Gamma_2} P^{(III)} \text{-} P^{(II)} - \Delta P_2)^2 dz, \quad (13)$$

where *L* is a length of $\Gamma_1$ and $\Gamma_2$ (the unit height). The procedure of $\upsilon_r$ and $\upsilon_z$ determination comes down to minimization of the target function Φ.

On eq. (7):

$$P^{(I)} = P_{вх} - \frac{\rho}{2}(v_{r_1}^2 + v_{z_1}^2)$$
$$P^{(II)} = P_{вых} - \frac{\rho}{2}(v_{r_2}^2 + v_{z_2}^2) \quad (1.14)$$

where $p_{in}$ and $p_{out}$ are the full pressure at entering *I* domain and at leaving *II* domain, correspondingly, at $\upsilon_{1,2} = 0$. Since the pressure is recovered up to a constant $p_0$, the constant should be entered in the number of the target function parameters and $p_0$ should be determined from the minimum condition, assuming that

$$p^{(III)} = p_0 + \tilde{p}. \quad (15)$$

We select the units of the pressure and velocity measurements so that

$$p_{\text{вх}} - p_{\text{вых}} = 1$$

$$v_B = \sqrt{\frac{2(p_{BX} - p_{BЫX}}{\rho}} = 1 \tag{16}$$

and set the value

$$q_1 = p_{in} - p_0 \tag{17}$$

and rewrite eq. (13) in account for (11), (14) and (15 – 17) as

$$\Phi\left(\nu_{r_1}, \nu_{r_2}\right) = \frac{1}{L}\int_{\Gamma_1} \left[\tilde{p} + \left(1+\tilde{\sigma}_1\right)\nu_{r_1}^2 + \nu_{z_1}^2 - q_1\right]^2 dx + \frac{1}{L}\int_{\Gamma_2} \left[\tilde{p} + \left(1-\tilde{\sigma}_2\right)\nu_{r_2}^2 + \nu_{z_2}^2 - q_1 + 1\right]^2 dz \tag{18}$$

where $\tilde{\sigma}_{1,2} = \dfrac{2\sigma_{1,2}}{\rho}$ is a dimensionless coefficient of a perforation resistance, that in general case may be a function of *z*.

**Results.** So, the problem solution method is a searching for $\upsilon_{r_1}$ and $\upsilon_{r_2}$ that make a minimum of the functional (18). The solution of the direct task in *I* domain (the distribution manifold) is shown in [17] and looks like:

$$\pm\frac{1}{\pi}\int_{z-h}^{z+h} \frac{(z-\bar{z})}{\Delta^2 + (z-\bar{z})^2} \bullet v_{1,2}(\hat{z}) d\ddot{z} = \frac{1}{\pi} \bullet \frac{\partial v_1}{\partial \bar{z}}\Big|_{\bar{z}=z} \bullet \int_{z-h}^{z+h} \frac{(z-\bar{z})^2}{\Delta^2 + (z-\bar{z})^2} d\bar{z} = \pm\frac{2h}{\pi} \bullet \frac{\partial v_{1,2}}{\partial \bar{z}}\Big|_{\bar{z}=z} \tag{19}$$

This equation allows to reach an accuracy $O(h^2)$ in the determination of the velocity tangential component and pressure on boundaries of *I*, *III* and *III*, *II* domains.

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## АҒЫНДЫ ҚОЗҒАЛМАЙТЫН ТҮЙІРШІКТІ ҚАБАТЫ БАР ЖАЗЫҚ ЖӘНЕ РАДИАЛДЫ БАЙЛАНЫС АППАРАТТАРДА МОДЕЛЬДЕУ. ГАЗДЫ РАДИАЛДЫ ЕНГІЗУМЕН ЖӘНЕ ЕРКІН ШЫҒАРУМЕН ЦИЛИНДРЛІК РЕАКТОРДЫ ЕСЕПТЕУ (1-БӨЛІМ)

**Аннотация.** Химиялық технологияның маңызды элементтерінің бірі катализатордың қозғалмайтын қабаты бар аксиальді немесе радиалды түрдегі реакторларда іске асырылатын гетерогенді каталитикалық процестер болып табылады. Ғалымдар мен өндірушілердің назарына осындай байланыс құрылғыларын зерттеу мен қолдануға бірқатар артықшылықтар себеп болған: фазалар бөлімінің жоғары дамыған беті, ағындардың жоғары жылдамдықтарын қамтамасыз ету мүмкіндігі, демек, габариттер мен материал сыйымдылығын азайту, конструкцияның қарапайымдылығы мепайдаланудағы сенімділік. Байланыс аппараттарының жұмысын жақсартуға қолданыстағы технологияларды жетілдіру және жаңа технологияларды, катализаторлар мен дисперсиялық жүйелердің құрылымдарын құру есебінен қол жеткізілуі мүмкін. Алайда, бірқатар жағдайларда аппараттың жұмыс аймағында ірі масштабты гидродинамикалық біртекті еместіктердің болуы химиялық, жылу-масса алмасу және басқа да процестердің тиімділігін арттыру бойынша іс-әрекетті жоққа шығарады. Гидродинамикалық біртекті емес құбылыстардың пайда болу себептерін анықтау түйіршікті қабаттарда сұйықтық пен газдың қозғалыс физикасының ерекшеліктерін зерттеуді талап етеді. Химиялық реакторларды пайдалану тәжірибесі өнеркәсіптік процестің техникалық-экономикалық көрсеткіштері, әдетте, осы процесті жобалау сатысында алынған есептік мәндерден төмен екендігін куәландырады. Қазіргі уақытта реактордың өнімділігіне әсер ететін себептердің бірі түйіршікті катализатордың қабатындағы реагенттер ағынының біртекті еместілігі болып табылатыны дәлелденген деп санауға болады. Жұмыс қозғалмайтын түйіршікті қабаты бар жазық және радиалды контактілі аппараттарда қысылмайтын сұйықтық-тың ағынын математикалық моделдеуге және осы модельді сандық іске асыру әдістерін құруға арналған. Үш бөліктен тұратын нақты реактордың моделі бойынша жұмыс циклі ұсынылды: таратушы коллектор, жинайтын коллектор және түйіршікті катализатордың қозғалмайтын қабаты жүктелетін жұмыс аймағы. Газ ағынын модельге енгізу және шығару Z - бейнелі схема бойынша жүзеге асырылады. Реактордың әрбір аймағындағы процестер мен олардый сипаттайтын теңдеулерді егжей-тегжейлі қарастырайық.

**Түйін сөздер**: химиялық реактор, қозғалмайтын түйіршікті қабат, катализатор, Эрган заңы, ток функциясы, түйіршікті ортаның кедергі факторы, Грин функциясы, қысым өрісі, жылдамдық өрісі, қабат кедергісі.

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## МОДЕЛИРОВАНИЕ ТЕЧЕНИЯ В ПЛОСКИХ И РАДИАЛЬНЫХ КОНТАКТНЫХ АППАРАТАХ С НЕПОДВИЖНЫМ ЗЕРНИСТЫМ СЛОЕМ. РАСЧЕТ ЦИЛИНДРИЧЕСКОГО РЕАКТОРА С РАДИАЛЬНЫМ ВВОДОМ ГАЗА И СВОБОДНЫМ ВЫХОДОМ (ЧАСТЬ 1)

**Аннотация.** Одними из важнейших элементов химической технологии являются гетерогенные каталитические процессы, реализуемые в реакторах аксиального или радиального типа с неподвижным слоем катализатора. Вниманию учёных и производственников к исследованию и применению таких контактных устройств обусловлено рядом преимуществ: высокоразвитой поверхностью раздела фаз, возможностью обеспечения высоких скоростей потоков и, следовательно, уменьшения габаритов и материалоёмкости, простотой конструкции и надёжностью в эксплуатации. Улучшение работы контактных аппаратов может быть достигнуто за счёт усовершенствования существующих и создания новых технологий, катализаторов и структур дисперсных систем. Однако в ряде случаев наличие крупномасштабных гидродинамических неоднородностей в рабочей зоне аппарата сводит на нет усилия по повышению эффективности химических, тепло-массообменных и других процессов. Выяснение причин возникновения гидродинамических неоднородностей требует изучения особенностей физики движения жидкости и газа в зернистых слоях. Опыт эксплуатации химических реакторов свидетельствует о том, что технико-экономические показатели промышленного процесса, как правило, ниже расчётных значений, полученных на стадии проектирования этого процесса. В настоящее время можно считать доказанным, что одной из причин, влияющих на производительность реактора, является неоднородность потока реагентов в слое зернистого катализатора. Работа посвящена математическому моделированию течения несжимаемой жидкости в плоских и радиальных контактных аппаратах с неподвижным зернистым слоем и построению методов численной реализации этой модели. Предложен цикл работ по модели реального реактора, состоящего из трех частей: раздающего коллектора, собирающего коллектор и рабочей зоны, в которую загружается неподвижный слой зернистого катализатора. Ввод и вывод газового потока в модели осуществлен по Z - образной схеме. Рассмотрим подробно процессы и описываемые их уравнения в каждой зоне реактора.

**Ключевые слова:** химический реактор, неподвижный зернистый слой, катализатор, закон Эргана, функция тока, фактор сопротивления зернистой среды, функция Грина, поле давлений, поле скоростей, сопротивление слоя.

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# SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH NON-LOCAL CONDITIONS FOR MULTIDIMENSIONAL HYPERBOLIC EQUATIONS

**Abstract**. In this paper, we study the solvability of new nonlocal boundary value problems for hyperbolic equations in a multidimensional bounded domain. For the problem under study, the existence and uniqueness theorems of regular solutions are proved.

**Keywords**: hyperbolic equation, boundary value problems, regular solutions, existence, uniqueness.

*1. Introduction and statement of the problem.* It is well known that studies of the properties of correcting boundary value problems for differential equations are important both for applications and for mathematical and mechanical problems. The theory of boundary value problems for high-order elliptic equations attracts particular attention of mathematicians.

The aim of the work is to study the solvability in classes of regular solutions of a new boundary value problem for the equation

$$u_{tt} - \Delta u + c(x,t)u = f(x,t), (x,t) \in Q. \quad (1)$$

This equation is an ordinary second-order hyperbolic equation, and the solvability of natural boundary value and initial-boundary value problems for it is well studied [24-26].

The study of the solvability of boundary value problems for quasi-hyperbolic equations (by analogy with quasi-elliptic equations) apparently began with the works of V.N. Vragova [1,2]. A study of the solvability of problems for high orders of equations in Sobolev spaces was carried out in [1-3], a number of similar results were obtained in [4-11].

Investigations of nonlocal problems with integral conditions for linear parabolic equations, for differential equations of odd orders, and for some classes of non-stationary equations, have recently been actively studied in the works of A.I. Kozhanova [7,9,10]. In [1–6], the main attention was paid to situations related to degenerate equations of the form (1).

In [12], a criterion was obtained for the strong solvability of the mixed Cauchy problem for the Laplace equation.

It is known that the Dirichlet problem for polyharmonic equations is uniquely solvable for any right-hand side of the equation. In [13–16], a new representation of the Green function of the Dirichlet problem for the polyharmonic equation in a multidimensional ball is constructed explicitly. In [17,18], a representation of the Green function of the Neumann problem for the Poisson equation in a multidimensional unit ball was obtained.

The problems of finding solvability conditions for boundary value problems for a polyharmonic equation in a ball were investigated in [19]. In [20], for derivatives of the $2l$-th order equation with constant (and only higher) real coefficients, normal derivatives were studied under boundary conditions. For these problems, sufficient conditions for the Fredholm solvability of the problem are obtained and formulas for the index of the problem are given.

In [21-23], the behavior of solutions of the Dirichlet problem for the Poisson equations and the biharmonic equation in an unbounded domain was studied.

In this paper, we obtain a theorem on the existence and uniqueness of a regular solution of a nonlocal in time boundary value problem for hyperbolic equations in a multidimensional bounded domain by the method of a priori estimation and passage to the limit.

Let $\Omega$ - be a bounded region of space $R^n$ with a smooth boundary $\Gamma$, $Q$ – a cylinder $\Omega\times(0,T), 0<t<T<+\infty$, $S=\Gamma\times(0,T)$ - a side boundary $Q$, $f(x,t)$ – given functions defined for $x\in\overline{\Omega}$, $t\in[0,T]$

For equation (1) we study the time-nonlocal boundary value problem

$$u|_S = 0, \tag{2}$$

$$u(x,0) = \alpha u(x,T), \tag{3}$$

$$u_t(x,0) = \beta u_t(x,T), \alpha,\beta \in R. \tag{4}$$

In studies of this kind of nonlocal problems, the parameter continuation method, a priori estimation method, and passage to the limit are usually used. For a hyperbolic equation, the parameter continuation method is not applicable, since the smoothness of the right side of the equation will be lost.

We define a functional space in which the properties of uniqueness and existence of a solution to the boundary value problem (1) - (4) will be studied. Namely, we define space $W_2^{2,2}(Q)$ as the set of functions from space $L_2(Q)$, that have generalized derivatives of spatial variables up to the second order inclusive and with respect to variable $t$ up to the second order inclusive, belonging to the same space. We define the norm in space $W_2^{2,2}(Q)$ .

$$\|\upsilon\|_{W_2^{2.2}(Q)} = \left(\int_Q \left[\upsilon^2 + \sum_{i,j=1}^{n} \upsilon_{x_i x_j}^2 + (\upsilon_{tt})^2\right] dxdt\right)^{1/2}$$

obviously, space $W_2^{2,2}(Q)$ with this norm will be a Banach space.

*2. The regularized nonlocal boundary value problem and the main result.* We consider the following regularized problem in order to apply the continuation method with respect to the parameter

$$u_{tt} - \Delta u + c(x,t)u - \varepsilon\Delta u_t = f(x,t) \in L_2(Q) \tag{5}$$

$$u|_S = 0 \tag{6}$$

$$u(x,0) = \lambda\alpha u(x,T), \tag{7}$$

$$u_t(x,0) = \lambda\beta u_t(x,T). \tag{8}$$

For the regularized problem (5) - (8), we obtain estimates in the corresponding spaces. To this end, we multiply equation (5) by $u_t$ integrate over $Q$, then we obtain.

$$\int_Q u_{tt}u_t dxdt - \int_Q \Delta u u_t dxdt + \int_Q cuu_t\, dxdt - \varepsilon\int_Q \Delta u_t\, u_t dxdt = \int_Q fu_t dxdt\,;$$

$$I_1 = \frac{1}{2}\int_Q \frac{\partial}{\partial t}(u_t^2)dxdt = \frac{1}{2}\int_\Omega [u_t^2(0,T) - u_t^2(x,0)]dx = \frac{(1-\lambda^2\beta^2)}{2}\int_\Omega u_t^2(x,T)dx\,;$$

Here we take into account the boundary condition (7) and $|\beta| < 1$, such that $\frac{1-\lambda^2\beta^2}{2} > 0$ .

$$I_2 = -\int\limits_Q \Delta u u_t = \sum_{i=1}^n \int\limits_Q u_{x_i} u_{x_i t} dxdt = \frac{1}{2}\sum_{i=1}^n \int\limits_Q \frac{\partial}{\partial t}(u_{x_i}^2) dxdt =$$

$$= \frac{1}{2}\sum_{i=1}^n \int\limits_\Omega \left[u_{x_i}^2(x,T) - u_{x_i}^2(x,0)\right] dx = \frac{1-\lambda^2\alpha^2}{2}\sum_{i=1}^n \int\limits_Q u_{x_i}^2(x,T)dx\,;$$

$$I_3 = \int\limits_Q cuu_t dxdt = \frac{1}{2}\int\limits_Q \frac{\partial}{\partial t}(cu^2)dxdt - \frac{1}{2}\int\limits_Q c_t u^2 dx\,dt =$$

$$= \frac{1}{2}\int\limits_\Omega [c(x,T)u^2(x,T) - c(x,0)u^2(x,0)]dx - \frac{1}{2}\int\limits_Q c_t u^2 dx\,dt =$$

$$= \frac{1}{2}\int\limits_\Omega [c(x,T) - \lambda^2\alpha^2 c(x,0)]u^2(x,T)dx - \frac{1}{2}\int\limits_Q c_t u^2 dx\,dt;$$

$$I_4 = -\varepsilon\int\limits_Q \Delta u_t\, u_t dx = -\,\varepsilon\sum_{i=1}^n \int\limits_Q u_{x_i x_i t}\, u_t\, dxdt = \varepsilon\sum_{i=1}^n \int\limits_Q {u_{x_i t}}^2 dxdt\,;$$

$$\frac{1-\lambda^2\beta^2}{2}\int\limits_\Omega u_t^2(x,T)dx + \frac{1-\lambda^2\alpha^2}{2}\sum_{i=1}^n \int\limits_\Omega u_{x_i}^2(x,T)dx +$$

$$+\frac{1}{2}\int\limits_\Omega [c(x,T) - \lambda^2\alpha^2 c(x,0)]\, u^2(x,T)dx + \frac{1}{2}\int\limits_Q \Delta u_t u_t dx\,dt + \varepsilon\sum_{i=1}^n \int\limits_Q u_{x_i t}^2\, dxdt = \int\limits_Q f u_t dxdt;$$

We will require that the conditions are met

$$c(x,T) - \lambda^2\alpha^2 c(x,0) \ge 0 \quad, c_t(x,T) \le 0\,.$$

Under the condition $\vartheta|_S = 0$ the embedding theorem holds, i.e. Friedrichs inequality [20]:

$$\int\limits_\Omega \vartheta(x,T)dx \le m_0 \sum_{i=1}^n \int\limits_\Omega {\vartheta_{x_i}}^2(x,T)dx\,.\ (F)$$

We apply it to the function $\vartheta = u_t$.

$$I_f = -\int\limits_Q f u_t dxdt \le \frac{\delta^2}{2}\int\limits_Q u_t^2 dx\,dt + \frac{1}{2\delta^2}\int\limits_Q f^2 dxdt \le \frac{\delta^2 m_0}{2}\sum_{i=1}^n \int\limits_Q u_{x_i t}^2 dx\,dt + \frac{1}{2\delta^2}\int\limits_Q f^2 dxdt.$$

We choose $\frac{\delta^2 m_0}{2} = \frac{\varepsilon}{2}$.

Then we have

$$\frac{1-\beta^2}{2}\int_\Omega u_t^2(x,T)dx + \frac{1-\alpha^2}{2}\sum_{i=1}^n \int_\Omega {u_{x_i}}^2(x,T)dx + \varepsilon\sum_{i=1}^n \int_Q {u_{x_i t}}^2 dxdt \le \frac{1}{2\varepsilon}\int_Q f^2 dxdt; \qquad (9)$$

At the beginning, we must obtain a uniform estimate for λ for a fixed $\varepsilon$. Equation (5) is multiplied by $u_{tt}$ and integrated over $Q$, then we obtain

$$-\int\limits_Q u_{tt}\Delta u_t dxdt + \int\limits_Q \Delta u\,\Delta u_t dxdt - \int\limits_Q c\,u\,\Delta u_t dxdt + \varepsilon\int\limits_Q {\Delta u_t}^2 dxdt = -\int\limits_Q f\,\Delta u_t dxdt.$$

Then we have

$$\frac{1-\lambda^2\beta^2}{2}\sum_{i=1}^n {u_{x_i t}}^2(x,T)dx + \frac{1-\lambda^2\alpha^2}{2}\int\limits_\Omega \left[\Delta u \quad (x,T)\right]^2 dx + \varepsilon\int\limits_Q (\Delta u_t)^2 dxdt =$$

$$= -\int_Q f\Delta u_t dxdt + \int_Q cu\Delta u_t dxdt\,;$$

Estimating the right-hand side and taking into account $C_0 = \max_Q |c|$ we have

$$\frac{1-\lambda^2\beta^2}{2}\sum_{i=1}^{n}\int_\Omega {u_{x_it}}^2(x,T)dx + \frac{1-\lambda^2\alpha^2}{2}\int_\Omega [\Delta u\,(x,T)]^2 dx + \frac{\varepsilon}{2}\int_Q (\Delta u_t)^2 dxdt \le$$

$$\le \frac{1}{\varepsilon}\int_Q f^2 + \frac{{c_0}^2}{\varepsilon}\int_Q u^2 dxdt = I_f. \qquad (10)$$

Applying the Newton-Leibniz formula:

$$u\,(x,t) = \int_0^t u_\tau(x,\tau)d\tau + u(x,0) \qquad \text{(N-L)}$$

using the boundary condition (7) we obtain

$$u(x,T) = \frac{1}{1-\lambda\alpha}\int_0^T u_t(x,t)dt\,.$$

Using the inequality $|(a+b)^2 \le 2(a^2+b^2)|$ we obtain

$$u^2(x,t) \le 2\left(\int_0^t u_t(x,t)dt\right)^2 + 2\alpha_1\left(\int_0^T u_t(x,t)dt\right)^2 \le$$

$$\le 2\left(\int_0^t u_\tau^2\,d\tau\right)\left(\int_0^t d\tau\right) + 2\alpha_1\left(\int_0^T u_t^2\,dt\right)\left(\int_0^T dt\right) \le (2T + 2\alpha_1 T)\int_0^T u_t^2 dt;$$

Integrate over $Q$:

$$\int_Q u^2\,(x,t)dxdt \le 2(1+\alpha_1)T^2\int_Q u_t^2 dxdt; \qquad (11)$$

Here $\alpha_1 = \left(\frac{\lambda\alpha}{1-\lambda\alpha}\right)^2$.

This is the required inequality. Denote by $k_0 = 2(1+\alpha_1)T^2$ . Further, continuing inequality (11), we have:

$$I_f \le \frac{1}{\varepsilon}\int_Q f^2 dxdt + \frac{C_0^2 k_0}{\varepsilon}\int_Q u_t^2 dxdt \le \frac{1}{\varepsilon}\int_Q f^2 + \frac{C_0^2 k_0 m_0}{\varepsilon}\sum_{i=1}^{n}\int_Q u_{x_it}^2 dxdt;$$

Using inequality (9), we have

$$I_f \le \frac{1}{\varepsilon}\int_Q f^2 dxdt + \frac{C_0^2 k_0 m_0}{2\varepsilon}\frac{1}{\varepsilon^2}\int_Q f^2 dxdt = \frac{1}{\varepsilon}\left(1 + \frac{C_0^2 k_0 m_0}{2\varepsilon^2}\right)\int_Q f^2 dxdt\,;$$

The Newton-Leibniz representation (N-L) using (7) gives:

$$u(x,t) = \int_0^t u_\tau(x,\tau)d\tau + \frac{\lambda\alpha}{1-\lambda\alpha}\int_0^T u_t(x,t)dt;$$

So we have

$$I_1 + I_2 + I_3 \leq \left(\frac{1}{\varepsilon} + \frac{C_0^2 k_0 m_0}{2\varepsilon^3}\right) \int_Q f^2 dxdt; \tag{12}$$

If in (11) we assume $u = \Delta u$, then we obtain:

$$\int_Q (\Delta u)^2 dxdt \leq k_0 \int_Q (\Delta u_t)^2 dxdt \leq \frac{2k_0}{\varepsilon}\left(\frac{1}{\varepsilon} + \frac{C_0^2 k_0 m_0}{2\varepsilon^3}\right) \int_Q f^2 dxdt; \tag{13}$$

It remains to obtain an estimate for the first term, i.e. for $u_{tt}$. To do this, we use the following well-known inequality:

$$\left|(a_1 + \cdots + a_m)^2 \leq m(a_1^2 + \cdots + a_m^2)\right|$$

$$\int_Q u_{tt}^2 dxdt = \int_Q (\Delta u - cu + \varepsilon \Delta u_t + f)^2 dxdt \leq \leq (4 \int_Q (\Delta u)^2 + 4C_0^2 \int_Q u^2 + 4\varepsilon \int_Q (\Delta u)^2 + 4 \int_Q f^2) dxdt.$$

For all terms of the right-hand side of the last inequality, estimates are obtained for fixed $\varepsilon$, i.e.

$$\int_Q u_{tt}^2 dxdt \leq N(\varepsilon) \int_Q f^2 dxdt. \tag{14}$$

Next, let's bonus the continuation method with respect to the parameter for a fixed $\varepsilon$.
Now erase $\lambda$, i.e. consider the boundary value problem for $\lambda$=1.

$$\begin{cases} u(x,0) = \alpha u(x,T), \\ u_t(x,0) = \beta u_t(x,T). \end{cases}$$

Obtaining the first a priori estimate uniform in $\varepsilon$ *:*

$$\frac{1-\beta^2}{2}\int_\Omega u_t^2(x,T)dx + \frac{1-\alpha^2}{2}\sum_{i=1}^n \int_\Omega u_{xi}^2(x,T)dx +$$

$$+\frac{1}{2}\int_\Omega \left[c(x,T) - a^2 c(x,0)\right]u^2(x,T)dx - \frac{1}{2}\int_\Omega c_t u^2 dxdt + \varepsilon \sum_{i=1}^n \int_Q u_{x_i t}^2 dxdt =$$

$$= \int_Q f u_t dxdt \leq \frac{\delta^2}{2}\int_Q u_t^2 dxdt + \frac{1}{2\delta^2}\int_Q f dxdt. \tag{15}$$

Hence, in particular, we have

$$\frac{1-\beta^2}{2}\int_\Omega u_t^2(x,T)dx \leq \frac{\delta^2}{2}\int_Q u_t^2 dxdt + \frac{1}{2\delta^2}\int_Q f^2 dxdt. \tag{16}$$

$(5) \times (A-t)u_t$, where $A$- is some positive number, $A > T$. For example, you can take $A = 2T$, then $A - t > A - T = T > 0$;

$$\int_Q (A-t)u_{tt}u_t dxdt - \int_Q (A-t)\Delta u u_t dxdt +$$

$$+ \int_Q (A-t)cuu_t dxdt - \varepsilon \int_Q (A-t)\Delta u_t u_t dxdt = \int_Q (A-t) f u_t dxdt;$$

Suppose that condition $\frac{\partial}{\partial t}\left[(A-t)c(x,t)\right]\leq 0$, is satisfied, then we have

$$\frac{A-T}{2}\int_{\Omega}u_t^2(x,T)dx+\frac{1}{2}\int_{Q}u_t^2dxdt+\frac{A-T}{2}\sum_{i=1}^{n}\int_{\Omega}u_{x_i}^2(x,T)dx+\frac{1}{2}\sum_{i=1}^{n}\int_{Q}u_{x_i}^2dxdt$$

$$+\frac{A-T}{2}\int_{\Omega}c(x,T)u^2(x,T)dx-\frac{1}{2}\int_{Q}((A-T)c)_t u^2dxdt+\varepsilon\sum_{i=1}^{n}\int_{Q}(A-t)\,u_{x_it}^2dxdt=$$

$$=\int_{Q}f\,(A-t)\,u_t dxdt+\frac{A}{2}\int_{\Omega}u_t^2(x,0)dx+\frac{A}{2}\sum_{i=1}^{n}\int_{\Omega}u_{x_i}^2(x,0)dx+\frac{A}{2}\int_{\Omega}c(x,0)u^2(x,0)dx \quad (17)$$

We use the boundary conditions (7), (8) and (16) we have

$$\int_{\Omega}u_t^2(x,T)dx\leq\frac{\delta^2}{1-\beta^2}\int_{Q}u_t^2dxdt+\frac{1}{\delta^2(1-\beta^2)}\int_{Q}f^2dxdt.$$

As a result, we get

$$\sum_{i}I_i\leq K_1\delta^2\int_{Q}u_t^2dxdt+\frac{K_2}{\delta^2}\int_{Q}f^2dxdt,$$

where $K_1$, $K_2$ - independent of $\varepsilon$.

We choose, in particular, $K_1\delta^2=\frac{1}{4}$, $\delta^2=\frac{1}{4K_1}$. Then we get

$$\frac{1}{4}\int_{Q}u_t^2dxdt+\frac{1}{2}\sum_{i=1}^{n}u_{x_i}^2dxdt+\varepsilon\sum_{i=1}^{n}\int_{Q}u_{x_it}^2dxdt\leq 4K_1K_2\int_{Q}f^2dxdt.$$

Hence, in particular, we have

$$\int_{Q}u_t^2dxdt\leq 16K_1K_2\int_{Q}f^2dxdt,\ \sum_{i=1}^{n}\int_{Q}u_{x_i}^2dxdt\leq 8K_1K_2\int_{Q}f^2dxdt,\ \varepsilon\sum_{i=1}^{n}\int_{Q}u_{x_it}^2dxdt\leq 4K_1K_2\int_{Q}f^2dxdt.$$

Adding all the terms, we get

$$\int_{Q}u_t^2dxdt+\sum_{i=1}^{n}\int_{Q}u_{x_i}^2dxdt+\varepsilon\sum_{i=1}^{n}\int_{Q}u_{x_it}^2dxdt\leq K_0\int_{Q}f^2dxdt,$$

where $K_0$ - is independent of $\varepsilon$. As required.

Thus, the following theorem is proved.

**Theorem.** Let the conditions be satisfied: $c(x,T)-\lambda^2\alpha^2c(x,0)\geq 0$, $c_t(x,T)\leq 0$, $f, f_t\in L_2(Q)$. Then the boundary value problem (1) - (4) cannot have more than one solution in space $W_2^{2,2}(Q)$.

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## КӨПӨЛШЕМДІ ШЕКТЕЛГЕН ОБЛЫСТА ГИПЕРБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН ЛОКАЛЬДЫ ЕМЕС ШЕТТІК ЕСЕПТЕРДІҢ ШЕШІМДІЛІГІ

**Аннотация** Бұл жұмыста шектелген көпөлшемді цилиндрлі облыста $Q = \Omega \times (0,T), \quad \Omega \subset R^n$, $S = \Gamma \times (0,T), \Gamma = \partial\Omega \quad 0 < t < T < +\infty$ келесі гиперболалық теңдеу үшін

$$u_{tt} - \Delta u + c(x,t)u = f(x,t), \qquad (x,t) \in Q \tag{1}$$

келесі түрдегі локалды емес шеттік есеп

$$u(x,t)\big|_S = 0, \tag{2}$$

$$u(x,0) = \alpha u(x,T), \tag{3}$$

$$u_t(x,0) = \beta u_t(x,T) \tag{4}$$

қарастырылады, мұндағы $f(x,t) - x \in \overline{\Omega}$, $t \in [0,T]$ берілген функциялар $\alpha, \beta \in R$ - сандар.

Азғындалған квазигиперболалық теңдеулер үшін шеттік есептердің шешімділігін зерттеу В.Н. Враговтың жұмыстарынан бастау алады.

Сызықты параболалық теңдеулер үшін интегралдық шарты бар, тақ ретті дифференциалдық теңдеулер үшін, стационарлы емес кейбір теңдеулердің класында локальды емес шеттік есептердің шешімділігінің мәселесі, соңғы жылдары, А.И. Кожановтың жұмыстарында зерттелген.

$$u_{tt} - \Delta u + c(x,t)u - \varepsilon \Delta u_t = f(x,t) \in L_2(Q) \tag{5}$$

$$u(x,t)\big|_S = 0, \tag{6}$$

$$u(x,0) = \lambda\alpha u(x,T), \tag{7}$$

$$u_t(x,0) = \lambda\beta u_t(x,T) \tag{8}$$

(1)-(4) есебінің регулярлы шешімін зерттеу үшін қосалқы (5)-(8) регулярланған есебін қарастырамыз. Осы алынған қосалқы (5)-(8) есеп үшін: $\lambda$ параметрі бойынша жалғастыру әдісін, сәйкес функционалдық кеңістіктерде априорлы бағалаулар әдістерін қолданамыз. $\varepsilon$-нан бірқалыпты тәуелді бағалауларда $\varepsilon \to 0$ ұмтылдырыу арқылы, зерттеліп отырған есептің регулярлы шешімінің бар болуы және жалғыздығы туралы теорема дәлелденеді.

**Түйін сөздер:** гиперболалық теңдеу, шеттік есептер, регулярлы шешімдер, бар болуы, жалғыздығы.

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## РАЗРЕШИМОСТЬ КРАЕВЫХ ЗАДАЧ С НЕЛОКАЛЬНЫМИ УСЛОВИЯМИ ДЛЯ МНОГОМЕРНЫХ ГИПЕРБОЛИЧЕСКИХ УРАВНЕНИЙ

**Аннотация.** В данной работе исследуется разрешимость новых нелокальных краевых задач для гиперболических уравнений в многомерной ограниченной области.

В ограниченной цилиндрической области $Q = \Omega \times (0,T), \quad \Omega \subset R^n$, $S = \Gamma \times (0,T), \Gamma = \partial\Omega \quad 0 < t < T < +\infty$ рассмотрим гиперболическое уравнение

$$u_{tt} - \Delta u + c(x,t)u = f(x,t), \qquad (x,t) \in Q \tag{1}$$

со следующими нелокальными по времени краевыми условиями

$$u(x,t)|_S = 0, \tag{2}$$

$$u(x,0) = \alpha u(x,T), \tag{3}$$

$$u_t(x,0) = \beta u_t(x,T) \tag{4}$$

где $c(x,t)$, $f(x,t)$ – заданные функции, определенные при $x \in \overline{\Omega}$, $t \in [0,T]$, $\alpha, \beta \in R$ -некоторые числа.

Исследование разрешимости краевых задач для вырожденных квазигиперболических уравнений (по аналогии с квазиэллиптическими уравнениями) началось с работ В.Н. Врагова.

Исследования нелокальных задач с интегральными условиями для линейных параболических уравнений, для дифференциальных уравнений нечетных порядков и для некоторых классов нестационарных уравнений в последние время активно исследуются в работах А.И. Кожанова.

Для исследования нелокальной краевой задачи (1)-(4) рассмотрим следующую регуляризованную задач

$$u_{tt} - \Delta u + c(x,t)u - \varepsilon \Delta u_t = f(x,t) \in L_2(Q) \tag{5}$$

$$u|_S = 0 \tag{6}$$

$$u(x,0) = \lambda \alpha u(x,T), \tag{7}$$

$$u_t(x,0) = \lambda \beta u_t(x,T). \tag{8}$$

Для этой регуляризованной задачи (5)-(8) мы применим метод продолжения по параметру. Для задачи (5)-(8) получим априорные оценки в соответствующих функциональных пространствах. И в конце переходим к пределу $\varepsilon \to 0$.

Таким образом, для изучаемой задачи доказывается теорема существования и единственности регулярного решения.

**Ключевые слова:** гиперболическое уравнение, краевые задачи, регулярные решения, существование, единственность.

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# GAUGE EQUIVALENCE BETWEEN THE Γ-SPIN SYSTEM AND (2+1)-DIMENSIONAL TWO-COMPONENT NONLINEAR SCHRODINGER EQUATION

**Abstract.** At present, the question of studying multidimensional nonlinear evolution equations in the framework of the theory of solitons is very relevant. Their usefulness is confirmed by numerous scientific publications, articles and many international conferences. One of the results of these works is the conclusion that, each (1+1)-dimensional soliton equation corresponds to several (2+1)-dimensional integrable and nonintegrable extensions. This led to the intensive development of an important subclass of nonlinear evolution equations of the theory of integrable spin systems. The simplest example of an integrable spin system is the equation Myrzakulov-I (M-I). The M-I equation is a (2+1)-dimensional integrable generalization of the well-known Landau-Lifshitz equation and for y = x it is reduced to it. In this paper, we consider the Γ-spin system. This spin system corresponds to the 2-layer M-I equation. A matrix Lax representation for the aforementioned spin system in symmetric space is proposed. The main result of this work is the establishment of gauge equivalence between the Γ-spin system and the (2+1)-dimensional two-component non-linear Schrödinger equation.

**Key words.** Γ-spin system, 2-layer M-I equation, (2+1)-dimensional two-component nonlinear Schrodinger equation, gauge equivalence.

**Introduction**

Modern mathematics and physics are fluttering intensively. Domestic fundamental science is not lag behind. We can distinguish such directions as differential equations of various orders (see, for example, [1]), computational methods and Riemannian metrics [3]. No less interesting topic is mathematical modeling of problems [4]. In particular, one can note such works as [4] where the (2 + 1) -dimensional integrable Fokas-Lenels equation was considered. Also in [5] the work of A.A. Zhadyranova investigated the solutions of the Witten-Dijkgraf-E. Verlinda-G. Werlinde equations (WDVV).

The basis of research in the field of mathematical and theoretical physics is the knowledge of integrable nonlinear evolution equations (NEE). The solitons theory of integrable NEE of (1+1)-dimension is well developed. This is evidenced by the large number of works devoted to this field by well-known modern mathematicians and physicists, such as Ablovets M., Zakharov V.E., Laks P.D., Lipovsky V.D., Manakov S.V., Ishimori U., Sigur H., Shabbat A.B. and others (for example Refs.[6,7]). In recent decades, multidimensional, in particular (2+1) and (3+1)-dimensional NEE have been intensively studied. The interest in them is justified by the fact that with their help many physical processes are described in optics, radiophysics, in the dynamics of continuous media, particle physics and other fields. The more famous among (2+1)-dimensional integrable equations are the nonlinear Schrodinger equation (NLSE), the Ishimori equation, the sine-Gorden equation, the Kadomtsev-Petviashvili equation, the Zakharov-Manakov equation, the Davy-Stewartson equation, M-I etc [8-13]. Theories of solving problems for two-dimensional integrable equations are more complicated, since (2+1)-dimensional connected with systems of linear partial differential equations with variable coefficients. Spin systems are a significant section of the NEE.

(2+1)-dimensional two-component nonlinear Schrodinger equation has forms

$$iq_{1t}+q_{1yx}-v_1q_1-\omega_1q_2=0\,, \quad (1)$$

$$iq_{2t}+q_{2yx}-v_2q_2-\omega_2q_1=0\,, \quad (2)$$

$$ir_{1t}-r_{1yx}+v_1r_1+\omega_2r_2=0\,, \quad (3)$$

$$ir_{2t}-r_{2yx}+v_2r_2+\omega_1r_1=0\,, \quad (4)$$

$$v_{1x}-(2r_1q_1+r_2q_2)_y=0\,, \quad (5)$$

$$v_{2x}-(r_1q_1+2r_2q_2)_y=0\,, \quad (6)$$

$$\omega_{1x}-\left(q_1r_2\right)_y=0\,, \quad (7)$$

$$\omega_{2x}-\left(r_1q_2\right)_y=0\,, \quad (8)$$

where $q_i=q_i(x,y,t)$, $r_i=r_i(x,y,t)$ are the complex functions and $v_i=v_i(x,y,t)$, $w_i=w_i(x,y,t)$, $(i=1{,}2)$ are the real functions. It is integrated by the method of the inverse scattering method, thus, for its there is a Lax representation. Its Lax representation is given in the form

$$\Phi_x=U_1\Phi\,, \quad (9)$$

$$\Phi_t=2\lambda\Phi_y+V_1\Phi\,, \quad (10)$$

$\lambda$-spectral parameter and $\Phi=(\phi_1,\phi_2,\phi_3)$. Here the matrix operators $U_1$ and $V_1$, accordingly, have the forms

$$U_1=-i\lambda\Sigma+Q, \quad (11)$$

$$Q=\begin{pmatrix}0 & q_1 & q_2\\ r_1 & 0 & 0\\ r_2 & 0 & 0\end{pmatrix},\ \Sigma=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

and

$$V_1=i\begin{pmatrix}-\partial_x^{-1}\left(r_1q_1+r_2q_2\right)_y & q_{1y} & q_{2y}\\ -r_{1y} & \partial_x^{-1}\left(r_1q_1\right)_y & \partial_x^{-1}\left(r_1q_2\right)_y\\ -r_{2y} & \partial_x^{-1}\left(q_1r_2\right)_y & \partial_x^{-1}\left(q_2r_2\right)_y\end{pmatrix}.$$

**Lax representation for the $\Gamma$-spin system corresponding to the two-layer M-I equation**

$\Gamma$-spin system which accord to the two-layer M-I equation is given by

$$i\Gamma_t+\frac{1}{2}\left[\Gamma,\Gamma_y\right]_x+2i\left(z\Gamma\right)_x=0, \quad (12)$$

$$z_x=\frac{i}{12}tr\left(\Gamma\left[\Gamma_x,\Gamma_y\right]\right) \quad (13)$$

here $\left[\Gamma_x,\Gamma_y\right]=\Gamma_x\Gamma_y-\Gamma_y\Gamma_x$ and called commutator. We introduce the following notation for $\Gamma$

$$\Gamma=g^{-1}\Sigma g\,,\quad \Gamma^2=I \quad (14)$$

and

$$\Gamma=\begin{pmatrix}\Gamma_{11} & \Gamma_{12} & \Gamma_{13}\\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23}\\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33}\end{pmatrix}\in su(3).$$

Elements of the $\Gamma$ matrix respond some restrictions

$$\Gamma_{33} = -(1 + \Gamma_{11} + \Gamma_{22}),\ \Gamma_{ij} = \overline{\Gamma}_{ji}\ (i, j = 1,2,3)$$

**Theorem 1** The Lax representation for the $\Gamma$-spin system (12)-(13) is given in the form

$$\Psi_x = U_2\Psi, \tag{15}$$

$$\Psi_t = 2\lambda\Psi_y + V_2, \tag{16}$$

where

$$U_2 = -i\lambda\Gamma,\ V_2 = 2i\lambda z\Gamma + \frac{\lambda}{2}[\Gamma, \Gamma_y]. \tag{17}$$

**Proof 1.** From the compatibility condition the system (15) and (16), the matrices $U_2(x, y, t, \lambda)$ and $V_2(x, y, t, \lambda)$ satisfy the condition of zero curvature

$$U_{2t} - V_{2x} + [U_2, V_2] - 2\lambda U_{2y} = 0. \tag{18}$$

Then substituting expressions (15), (16) and (17) in the condition of zero curvature (18) obtain

$$-i\lambda\Gamma_t - 2i\lambda(z\Gamma)_x - \frac{\lambda}{2}[\Gamma, \Gamma_y]_x - i\lambda[\Gamma, V_2] + 2i\lambda^2\Gamma_y = 0,$$

$$[\Gamma, V_2] = \left[\Gamma, 2i\lambda z\Gamma + \frac{\lambda}{2}[\Gamma, \Gamma_y]\right] = \frac{\lambda}{2}[\Gamma, [\Gamma, \Gamma_y]] = 2\lambda\Gamma_y,$$

$$i\Gamma_t + \frac{1}{2}[\Gamma, \Gamma_y]_x + 2i(z\Gamma)_x = 0.$$

Finally, we derive

$$i\Gamma_t + \frac{1}{2}[\Gamma_x, \Gamma_y] + \frac{1}{2}[\Gamma, \Gamma_{xy}] + 2iz_x\Gamma + 2iz\Gamma_x = 0$$

$$\frac{1}{2}\left(\Gamma[\Gamma_x, \Gamma_y] + [\Gamma_x, \Gamma_y]\Gamma\right) + 4iz_x I = 0$$

$$z_x = \frac{i}{12}tr\left(\Gamma[\Gamma_x, \Gamma_y]\right).$$

So we have obtained sought-for $\Gamma$-spin system corresponding to the two-layer M-I equation

$$i\Gamma_t + \frac{1}{2}[\Gamma, \Gamma_y]_x + 2i(z\Gamma)_x = 0, \tag{19}$$

$$z_x = \frac{i}{12}tr\left(\Gamma[\Gamma_x, \Gamma_y]\right). \tag{20}$$

Thus we proved theorem 1.

**Gauge equivalence between the $\Gamma$-spin system and (2+1)-dimensional two-component NLSE**

In this section, will estabilish the gauge equivalence between the (2+1)-dimensional two component NLSE (1)-(8) and the $\Gamma$-spin system (12)-(13).

**Theorem 2.** The (2+1)-dimensional two component NLSE (1)-(8) and the $\Gamma$-spin system (12)-(13) are gauge equivalent to each other.

To prove this theorem, we introduce some concepts from the classical theory of gauge equivalence.

**Definition 1.** Equations admitting the Lax representation

$$\Psi_x = U_j\Psi,\ \ \Psi_t = V_j\Psi,\ \ j = 1,2$$

or satisfying the condition of zero curvature

$$U_{jt} - V_{jx} + [U_j, V_j] = 0,\ \ j = 1,2$$

are called integrable equations.

**Definition 2.** Integrable equations are called gauge equivalents if they are related by a transformation $\Phi_1 = g^{-1}\Phi_2$, $U_1 = gU_2 g^{-1} + g_x g^{-1}$, $V_1 = gV_2 g^{-1} + g_t g^{-1}$ with the matrix function $g$ not conditional on pseudo-differential symbols by other independent variables of operators included in $U$ and $V$.

**Proof 2.** Let pass to the immediate proof of the theorem. First, consider the following gauge transformation

$$\Psi = g^{-1}\Phi,\ \ g = \Phi_{\lambda=0},$$

where $\Phi$ is the solution to the system of the corresponding NLSE (1)-(8), and $\Psi$ is the solution to the system corresponding to the $\Gamma$-spin system (14)-(15), and also $g(x,y,t)$ is an arbitrary $3\times 3$ matrix function that is a solution to system (9)-(10) at $\lambda = 0$. The derivative of the function $\Psi$ with respect to $x$ is

$$\Psi_x = \left(g^{-1}\Phi\right)_x = g^{-1}\Phi_x - g^{-1}g_x g^{-1}\Phi \tag{21}$$

then using then substituting the expressions (9), (10) and (11) in (20) we get

$$\Psi_x = g^{-1}\left(-i\lambda\Sigma\Phi + Q\Phi\right) - g^{-1}Qgg^{-1}\Phi = -i\lambda g^{-1}\Sigma\Phi = -i\lambda g^{-1}\Sigma g\Psi = -i\lambda\Gamma\Psi = U_2\Psi$$

Next we take the derivative of $\Psi$ by $t$

$$\Psi_t = -g^{-1}V_1 gg^{-1}\Phi + g^{-1}\left(2\lambda\Phi_y + V_1\Phi\right) = 2\lambda g^{-1}\Phi_y = 2\lambda g^{-1}\left(g\Psi\right)_y = 2\lambda g^{-1}g_y\Psi + 2\lambda\Psi_y.$$

in total

$$\Psi_x = U_2\Psi, \Psi_t = 2\lambda\Psi_y + 2\lambda g^{-1}g_y\Psi.$$

The last formula depends on $\Psi$, we need to express $\Psi$ in terms of $\Gamma$. From the expression (14) follows

$$\Gamma_y = \left(g^{-1}\Sigma g\right)_y = \left[\Gamma, g^{-1}g_y\right]$$

on the other hand

$$\Gamma_y = \left(g^{-1}\Sigma g\right)_y = g^{-1}\left[\Sigma, g_y g^{-1}\right]g.$$

Next we introduce notational convention

$$g_y g^{-1} = i\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \tag{22}$$

then from the equation (22) follows

$$\Gamma_y = 2ig^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{21} & 0 & 0 \\ -c_{31} & 0 & 0 \end{pmatrix}g. \tag{23}$$

We also have that

$$\Gamma\Gamma_y = 2ig^{-1}\Sigma gg^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{21} & 0 & 0 \\ -c_{31} & 0 & 0 \end{pmatrix}g = 2ig^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & 0 \\ c_{31} & 0 & 0 \end{pmatrix}g \tag{24}$$

$$\Gamma_y\Gamma = 2ig^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{21} & 0 & 0 \\ -c_{31} & 0 & 0 \end{pmatrix}gg^{-1}\Sigma g = -2ig^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & 0 \\ c_{31} & 0 & 0 \end{pmatrix}g. \tag{25}$$

Considering (24) and (25) we get

$$\left[\Gamma, \Gamma_y\right] = 4ig^{-1}\begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & 0 \\ c_{31} & 0 & 0 \end{pmatrix}g.$$

Now we need to express $g^{-1}g_y$ in terms of the matrix $\Gamma$. For this, we represent the matrix $g_y$ in the following form

$$g_y = i\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} g = Wg$$

and $b_{11} = z$, $b_{22} = b_{33} = -z$. Here $z = z(x, y, t)$ is unknown function. Consequently $g^{-1}g_y = g^{-1}Wg$.

$$\Gamma_y = \left[\Gamma, g^{-1}g_y\right] = \left[g^{-1}\Sigma g, g^{-1}Wg\right] = g^{-1}[\Sigma, W]g = 2g^{-1}\begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{21} & 0 & 0 \\ -b_{31} & 0 & 0 \end{pmatrix} g,$$

$$\left[\Gamma, \Gamma_y\right] = 4ig^{-1}\begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix} g,$$

$$g^{-1}\begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix} g = \frac{1}{4i}\left[\Gamma, \Gamma_y\right],$$

$$g^{-1}g_y = ig^{-1}Wg = izg^{-1}\Sigma + ig^{-1}\begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix} g = iz\Gamma + \frac{1}{4}\left[\Gamma, \Gamma_y\right] .$$

Substituting the resulting expression in (14) we obtain

$$\Phi_t = 2\lambda\Psi_y + 2\lambda g^{-1}g_y\Psi = 2\lambda\Psi_y + 2\lambda\left(iz\Gamma + \frac{1}{4}\left[\Gamma, \Gamma_y\right]\right) = 2\lambda\Psi_y + V_2\Psi .$$

Finally, the Lax representation for the $\Gamma$-spin system has the form

$$\Psi_x = U_2\Psi, \tag{26}$$

$$\Psi_t = 2\lambda\Psi_y + V_2\Phi, \tag{27}$$

where

$$U_2 = -i\lambda\Gamma, \tag{28}$$

$$V_2 = 2i\lambda z\Gamma + \frac{\lambda}{2}\left[\Gamma, \Gamma_y\right]. \tag{29}$$

So, we derived the Lax representation for the spin system (14) - (15). The system compatibility condition (26) - (29) $(\Psi_{xt} = \Psi_{tx})$ gives

$$U_{2t} - V_{2x} + \left[U_2, V_2\right] - 2\lambda U_{2y} = 0$$

or

$$i\Gamma_t + \frac{1}{2}\left[\Gamma, \Gamma_y\right]_x + 2i(z\Gamma)_x = 0,$$

$$z_x = \frac{i}{12}tr\left(\Gamma\left[\Gamma_x, \Gamma_y\right]\right).$$

Theorem 2 is proved.

**Conclusions**

In this paper, we considered the $\Gamma$-spin system corresponding to the 2-layer equation Myrzakulov-I. Farther a matrix form of the Lax representation was proposed for the equation under consideration in the symmetric space $su(n+1)/s(u(1)\oplus u(n))$ for the case $n=3$. This kind of Lax representation expands the possibilities of studying the $\Gamma$-spin system. In particular, using the matrix form of the Lax representation for the $\Gamma$-spin system, we established the gauge equivalence of this equation to a (2+1)-dimensional two-component NLSE. The obtained result is used for further research of spin systems when finding different soliton solutions.

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## Г-СПИН ЖҮЙЕСІ МЕН (2+1)-ӨЛШЕМДІ ЕКІ КОМПОНЕНТТІ СЫЗЫКТЫ ЕМЕС ШРЕДИНГЕР ТЕҢДЕУІНІҢ АРАСЫНДАҒЫ КАЛИБРОВТІ ЭКВИВАЛЕНТТІЛІК

**Аннотация.** Қазіргі уақытта көпөлшемді емес сызықты эволюция теңдеулерін солитондар теориясы шеңберіндегі зерттеу өте өзекті болып табылады. Олардың пайдалылығы көптеген ғылыми жарияланымдармен, мақалалармен және көптеген халықаралық конференциялармен расталады. Осы жұмыстардың нәтижелерінің бірі-әрбір (1+1)-мөлшемді солитондық теңдеу бірнеше (2+1)-мөлшемді интегралданатын және интегралданбайтын кеңейтімдерге сәйкес келеді. Бұл сызықтық емес эволюциялық теңдеулер теориясының маңызды тобын интегралданатын спиндік жүйелер теориясының қарқынды дамуына әкелді. Спиндік жүйелер немесе Ландау-Лифшиц теңдеуілері математика мен физикадағы маңызды ұғымдардың бірі. Мысалы, геометрияда олар ортогональ түрінде Гаусс-Вайнардин теңдеуімен анықталады. Физикада спиндік жүйелер магнитиктердегі сызықтық емес процестерді зерттеуде кеңінен қолданылады. Интегралданатын және интегралданбайтын спиндік жүйелердің жаңа класстарының құрылулары мен зерттеулері (әсіресе көп өлшемділік жағдайында) жаратылыстану ғылымдарының қажеттілігі мәселесінің өзектілігінен туындады. Интегралды спиндік жүйелер теориясын дамытуда маңызды рөлді М.Лакшмананның, В.Е. Захарова, Р.Мырзакулов, Л. А. Тахтаджян және басқалары атқарды. Интегралданатын спин жүйесінің қарапайым мысалы - Мырзакулов-I (M-I) теңдеуі. M-I теңдеуі - бұл танымал Ландау-Лифшиц теңдеуінің (2+1)-мөлшемді интегралданатын жалпылауы, ал y = x үшін өзі шығады. Бұл теңдеудің интегралдылығы кері шашырау есебі әдісіне дәлелденеді. Кері шашырау есебі әдісін әдетте сызықтық есептерді шешу үшін қолданылатын Фурье түрлендіру әдісін жалпылау деп қарастыруға болады. Бұл мақалада Г-спиндік жүйені қарастырамыз. Бұл спин жүйесі екі қабатты M-I теңдеуіне сәйкес келеді. Жоғарыда аталған спиндік жүйеге симметриялы кеңістіктегі Лакс ұсынысының (ЛҰ) матрицалық түрі алынды. Бұл жұмыстың негізгі нәтижесі - Г-спиндік жүйе мен (2+1) -өлшемді екі компонентті сызықты емес Шредингер теңдеуі (СЕШТ) арасындағы калибровтық эквиваленттілікті құру. Солитондар теориясында калибровтық эквиваленттілік маңызды рөл атқарады. Интегралданатын (1+1) -өлшемді сызықтық емес теңдеулер үшін калибровтық эквиваленттілік ұғымын Захаров В.Е., Михайлов А.В., Тахтаджаана Л.А. енгізді. Калибровтық эквиваленттілік ұғымы сызықтық емес эволюцияның теңдеулердің (СЕЭТ) ЛҰ-ның бар болуымен тығыз байланысты. ЛҰ болуы СЕЭТ-дің интегралдануы үшін қажетті шарт екені белгілі. Демек, калибровтық эквиваленттілік интегралданбайтын, бірақ ЛҰ бар СЕЭТ арасында орын алады. Захаров В.Е., Тахтаджяна Л.А. жұмысында. Ландау-Лифшиц теңдеуі мен СЕШТ (1+1)-өлшемді тартымдылығы бар жағдай үшін калибровтық эквиваленттілік анықталды. Липовский В.Д., Широкова А.В., Михалева В.Г. еңбектерінде. (2+1)-өлшемді солитондық теңдеулердің белгілі өкілдері - Ишимори теңдеуі және Дэви-Стюартсон теңдеуі арасында дәлелденді. Сондай-ақ, Р.Мырзакулов пен оның тобының жұмыстарында кейбір спин жүйелері мен СЕШТ арасында эквиваленттік орнатылғанын атап өтеміз. Мысалы, СЕШТ изотроптық теңдеуге M-III, Захаров теңдеуі M-IX, екі компонентті Камас-Холм теңдеуі Гейзенберг ферромагниттік теңдеуіне эквивалент екендігін көрсеткен. Осы жұмыс жоғарыда аталған жұмыстарға ұқсас орындалған.

**Түйін сөздер.** Спин жүйесі, M-I 2-қабатты M-I теңдеу, (2+1)-өлшемді екі компонентті сызықты емес Шредингер теңдеуі, калибровті эквиваленттілік.

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## КАЛИБРОВОЧНАЯ ЭКВИВАЛЕНТНОСТЬ МЕЖДУ Г-СПИН СИСТЕМОЙ И (2 + 1)-МЕРНЫМ ДВУХКОМПОНЕНТНЫМ НЕЛИНЕЙНЫМ УРАВНЕНИЕМ ШРЕДИНГЕРА

**Аннотация.** В настоящее время вопрос изучения многомерных нелинейных эволюционных уравнений в рамках теории солитонов является весьма актуальным. Их полезность подтверждается многочисленными научными публикациями, статьями и многими международными конференциями. Одним из результатов этих работ является вывод о том, что каждое (1+1)-мерное солитонное уравнение соответствует нескольким (2+1)-мерным интегрируемым и неинтегрируемым расширениям. Это привело к интенсивному развитию важного подкласса нелинейных эволюционных уравнений теории интегрируемых спиновых систем. Важные значения в математике и физике имеют спиновые системы. К примеру, в геометрии они отожествляются с уравнением Гаусса - Вейнгардена в его ортогональной форме. В физике спиновые системы успешно применяются в изучении нелинейных процессов в магнетиках. Построения и исследования новых классов интегрируемых и неинтегрируемых спиновых систем (особенно в многомерии) были вызваны актуальностью проблемы потребности естественных наук. Важную роль в становлении теории интегрируемых спиновых систем сыграли работы М. Лакшманана, В.Е. Захарова, Р. Мырзакулова, Л. А. Тахтаджяна и др. Простейшим примером интегрируемой спиновой системы является уравнение Мырзакулов-I (M-I). Уравнение M-I представляет собой (2+1) -мерное интегрируемое обобщение известного уравнения Ландау-Лифшица, и для y = x оно сводится к нему. Интегрируемость этого уравнение сводится к методу обратной задачи рассеяния. Метод обратной задачи рассеяния можно рассматривать как обобщение метода преобразования Фурье, который обычно применяется для решения линейных задач. В этой статье мы рассмотрим Г-спин систему. Эта спиновая система соответствует двухслойному уравнению M-I. Предложено матричное представление Лакса (ПЛ) $\Phi_x = U(x, y, t, \lambda)\Phi$, $\Phi_t = V(x, y, t, \lambda)\Phi$ для вышеупомянутой спиновой системы в симметричном пространстве. Основным результатом этой работы является установление калибровочной эквивалентности между Г-спиновой системой и (2+1)-мерным двухкомпонентным нелинейным уравнением Шредингера (НУШ). Калибровочная эквивалентность играет важную роль в теории солитонов. Захаровым В.Е., Михайловым А. В., Тахтаджяна Л.А. было введено понятие калибровочной эквивалентности для вполне интегрируемых (1+1)-мерных нелинейных уравнений. Понимание калибровочной эквивалентности тесно сязана со существованием ПЛ для нелинейных эволюционных уравнении (НЭУ). Хорошо известно, что существование представления Лакса является необходимым условием интегрируемости НЭУ. Следовательно, калибровочная эквивалентность также существует между неинтегрируемыми, но обладающими ПЛ НЭУ. В работе Захарова В.Е., Тахтаджяна Л.А. было установлена калибровочная эквивалентность уравнения Ландау-Лифщица и НУШ с притяжением для (1+1)-мерного случая. В работах Липовского В.Д., Широкова А.В., Михалева В.Г. были доказаны эквивалентность между наиболее известными представителями (2+1)-мерных солитонных уравнения - уравнение Ишимори и уравнение Дэви-Стьюартсона. Отметим также, что в работах Мырзакулова Р. и его группы были установлены эквивалентности между некоторыми спиновыми системами и НУШ. К примеру, НУШ калибровочно эквивалентное к изотропному уравнению M-III, уравнение Захарова к M-IX, двухкомпонентное уравнения Камассы-Холма к обобщенному уравнению ферромагнетика Гейзенберга, и так далее. Данная работа выполнена аналогично вышеупомянутым работам.

**Ключевые слова:** Спин система, 2-слойное уравнение M-I, (2+1)-мерное двухкомпонентное нелинейное уравнение Шредингера, калибровочная эквивалентность.

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# INTRODUCING METHOD OF GENERALIZED DERIVATIVE CONCEPT IN MATHEMATICS

**Abstract.** In this paper, we consider a technique for introducing the concept of a generalized derivative function of one variable. Some assertions of combinatorics are given and proved; the concept of the derivative of the natural order is introduced using the limit of the sequence. Using the above statements, the concept of a fractional derivative is introduced. The basic properties of the fractional derivative are formulated and proved. Examples are given

**Key words:** mathematical analysis, derivative, combination, limit transition, fractional derivative.

## Introduction

Mathematical analysis is a field of mathematics related to the concepts of function, derivative and integral.

The great English physicist, astronomer and mathematician Isaac Newton and the German mathematician and philosopher Gottfried Leibniz completed the construction of differential and integral calculi by the end of the 17th century. The discovery of differential and integral calculus was the beginning of a period of rapid development of mathematics.

Mathematics continues to develop rapidly. Various generalizations of the concepts of function, derivatives, and integral have a particular interest. For example, mathematicians, including Leibniz, Euler, Liouville, and Riemann, dealt with generalization of the concept of a derivative. Generalized functions and their derivatives find various applications in real processes of the economy and production [1-4].

Our goal is to develop a methodology for introducing various definitions of a generalized derivative function of one variable.

To achieve this goal, we first give some statements of combinatorics, introduce the concept of a derivative of the natural order using the limit of the sequence. Using these statements, we introduce the concept of a fractional derivative

## 1. Auxiliary definitions and formulas

As it is known [5], the number of combinations of $n$ elements by $k$ is equal

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad n \ge k . \tag{1}$$

Here it is convenient to assume that $0! = 1$. The following well-known equalities are justly:

$$C_n^k + C_n^{k+1} = C_{n+1}^{k+1}; \tag{2}$$

$$C_n^0 \cdot C_m^k + C_n^1 \cdot C_m^{k-1} + ... + C_n^{k-1} \cdot C_m^1 + C_n^k \cdot C_m^0 = C_{n+m}^k, \quad m, n \ge k . \tag{3}$$

In particular, in $n = m$ formula (3) takes the form:

$$\sum_{i=0}^{k} C_n^i C_n^{k-i} = C_{2n}^k, \quad n > k \tag{4}$$

Obviously, that the formula (1) one can be wtitten in the form of

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!}. \quad (5)$$

Now, we will consider a generalization of formulas (2), (3), (5). We notice that the right-hand side of formula (5) is defined for any real values $n$. Thus, by definition, we assume that

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!}, \quad (6)$$

where $\tau$ is a real number. For example:

$$C_{-2}^k = \frac{(-2-0)(-2-1)(-2-2)...(-2-(k-1))}{k!} = (-1)^k(k+1),\ C_{-1}^k = (-1)^k$$

*Lemma 1*. For any real values of $\tau$, the equality

$$C_\tau^k + C_{\tau+1}^k = C_{\tau+1}^{k+1} \quad (7)$$

*Proof:* Using (4), we have

$$C_\tau^k + C_\tau^{k+1} = \frac{\tau(\tau-1)...(\tau-(k-1))}{k!} + \frac{\tau(\tau-1)...(\tau-(k-1))(\tau+1)}{(k+1)!} =$$
$$= \frac{\tau(\tau-1)...(\tau-(k-1))}{k!}\cdot\left(1+\frac{\tau-k}{k+1}\right) = \frac{\tau(\tau-1)...(\tau-(k-1))(\tau+1)}{(k+1)!} =$$
$$= \frac{(\tau+1-0)(\tau+1-1)(\tau+1-2)..(\tau+1-k)}{(k+1)!} = C_{\tau+1}^{k+1}.$$

Lemma 1 is proved.

*Lemma 2*. For any real values of $\tau$, the equality

$$C_\tau^0 C_\tau^k + C_\tau^1 C_\tau^{k-1} + ... + C_\tau^{k-1} C_\tau^1 + C_\tau^k C_\tau^0 = C_{2\tau}^k \quad (8)$$

*Proof.* We use the following statement [6].

If the polynomials $P(x)$ and $Q(x)$, whose degrees do not exceed $n$, have equal values for more than $n$ different of values unknowns, then $P(x) = Q(x)$.

We write the equalities $P(\tau) = \sum_{i=0}^{k} C_\tau^i C_\tau^{k-i}$, $Q(\tau) = C_{2\tau}^k$, where $P(\tau)$ and $P(\tau)$ are polynomials of degree $k$.

By virtue of formula (4), the polynomials $P(\tau)$ and $Q(\tau)$ have equal values at $\tau = n > k$. Then, by based on the above statement, it is easy to verify the validity of formula (8) for any real values $\tau$. Lemma 2 is proved.

**2. About a definition of a natural order derivative**

For a function of one variable, we define a derivative of the natural order in a slightly different way. Let the function $f(x)$ be defined and continuous on the interval $[a, b], a < 0,\ b > 0$, at that $f(x) = 0$ for $x < 0$, where $x$ is a fixed point of this interval.

We divide the segment $[0,\ x]$ into equal $n$ parts by points $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$. We define the increment of the function $y = f(x)$ at the point $x$ in the form $\Delta y = f(x) - f(x - \frac{x}{n})$. As it is known, if the ratio

$$\frac{\Delta y}{\frac{x}{n}}=\frac{f(x)-f(x-\frac{x}{n})}{\frac{x}{n}}$$

has a limit is at $n\to\infty$, then this limit is called the derivative of the function $f(x)$ at the point $x$ and is denoted by:

$$f'(x)=\lim_{n\to\infty}\frac{f(x)-f\left(x-\frac{x}{n}\right)}{\frac{x}{n}}.$$

Now we define the increment $\Delta^2 y$:

$$\Delta^2 y=f(x)-f(x-\frac{x}{n})-\left[f(x-\frac{x}{n})-f(x-\frac{2x}{n})\right]=$$

$$=f(x)-2f(x-\frac{x}{n})+f(x-\frac{2x}{n})=C_2^0 f(x)+C_2^1 f(x-\frac{x}{n})+C_2^2 f(x-\frac{2x}{n})=$$

$$=\sum_{k=0}^{2}(-1)^k\cdot C_2^k f\left(x-\frac{kx}{n}\right),$$

etc. continuing this process, we obtain:

$$\Delta^m y=\sum_{k=0}^{m}(-1)^k C_m^k f(x-\frac{kx}{n}). \quad (9)$$

Let $m$ be a fixed positive integer. We choose a positive integer $n$ so that $n>m$. Then the formula (9) will take the form:

$$\Delta^m y=\sum_{k=1}^{n}(-1)^k C_m^k\cdot f\left(x-\frac{kx}{n}\right), \quad (10)$$

as $C_m^k=0;\ k>m$. Using (10), we obtain

$$f^{(m)}(x)=\lim_{n\to\infty}\frac{\Delta^m y}{\left(\frac{x}{n}\right)^m}=\lim_{n\to\infty}\left(\frac{x}{n}\right)^{-m}\sum_{k=0}^{n}(-1)^k C_m^k\cdot f\left(x-\frac{kx}{n}\right). \quad (11)$$

As an example, we consider the function $f(x)=x^2$:

$$(x^2)''=\lim_{n\to\infty}\left(\frac{x}{n}\right)^{-2}\left[x^2-2(x-\frac{x}{n})^2+(x-\frac{2x}{n})^2\right]=\lim_{n\to\infty}\frac{2\frac{x^2}{n^2}}{\frac{x^2}{n^2}}=2,\ \text{где}\ C_2^k=0,\ k=\overline{3,n}.$$

**3. Definition of a fractional order derivative. Properties**

Based on the fact that the expression $C_\tau^k$ according to (6) is defined for any real values of $\tau$, then the right-hand side of formula (10) is determined for any real values of $\tau$.

Now, using [7], for any real number $\tau$ we define the derivative of the $\tau$ th order:

$$f^{(\tau)}(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{x}{n}\right)^{\tau}}\sum_{k=0}^{n}(-1)^{k}c_{\tau}^{k}f\left(x-\frac{kx}{n}\right). \quad (12)$$

Since any real number is an infinite decimal fraction, we call the derivative (14) fractional.

For example, given that $C_{-1}^{k}=(-1)^{k}$, from (12) the formula we find:

$$f^{(-1)}(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{x}{n}\right)^{-1}}\sum_{k=0}^{n}f\left(x-\frac{kx}{n}\right)=\lim_{n\to\infty}\sum_{k=0}^{n}f\left(x-\frac{kx}{n}\right)\cdot\left(\frac{x}{n}\right) \quad (13)$$

Example: We define the (-1)st order derivative for function $f(x)=x$. From (13) we get:

$$f^{(-1)}(x)=\lim_{n\to\infty}\frac{x}{n}\sum_{k=0}^{n}\left(x-\frac{kx}{n}\right)=\lim_{n\to\infty}\frac{x}{n}\left[(x-0)+\left(x-\frac{x}{n}\right)+\left(x-\frac{2x}{n}\right)+\cdots+\left(x-\frac{kx}{n}\right)\right]=$$

$$=\lim_{n\to\infty}\frac{x}{n}\left[(x-1)x-\frac{x}{n}\cdot\frac{(n+1)n}{2}\right]=\lim_{nn\to\infty}\left[\frac{x^{2}(n+1)}{n}-\frac{x^{2}(n+1)n}{2n^{2}}\right]=x^{2}-\frac{x^{2}}{2}=\frac{x^{2}}{2}.$$

In this way, $(x)^{-1}=\frac{x^{2}}{2}$.

Now we give the main properties of the $\tau$ th derivative for any real of values of $\tau$.

Theorem 1. Right

$$(\alpha\cdot f(x)+\beta\cdot g(x))^{(\tau)}=\alpha\cdot f^{(\tau)}(x)+\beta\cdot g^{(\tau)}(x), \quad (14)$$
$$\left[f^{(\tau_1)}(x)\right]^{(\tau_2)}=f^{(\tau_1+\tau_2)}(x).$$

To establish relations (14), it suffices to use representation (12) of the derivative of an arbitrary function.

From formula (15), we note that the sum $\sum_{л=0}^{m}f\left(x-\frac{kx}{n}\right)\cdot\frac{x}{n}$ represents the integral sum of the function $f(x)$ for a given partition $\left\{\frac{kx}{n}\right\}$, $k=\overline{1,n}$ of the segment $[0;x]$, $x\in[0,x]$.

Since the function $f(x)$, being continuous on the segment $[0,x]$, is integrable on $[0,x]$, therefore, the limit (13) gives us a definite integral $\lim_{n\to\infty}\sum_{k=0}^{n}f\left(x-\frac{kx}{n}\right)\cdot\frac{x}{n}=\int_{0}^{x}f(t)dt$ and thus, we obtain

$$f^{(-1)}(x)=\int_{0}^{x}f(t)dt. \quad (15)$$

Theorem 2. For the natural value m, the derivative of the (–m)th order function is determined by the formula:

$$f^{(-m)}(x)=\frac{1}{(m-1)!}\int_{0}^{x}f(t)(x-t)^{m+1}dt. \quad (16)$$

Proof: By hypothesis, the function $f(x)$ is continuous at any point $x \in [a,b]$. Therefore, the function $f^{(-1)}(x) = g(x) = \int_0^x f(t)dt$ is also continuous in the same $x \in [a,b]$. Therefore, for the point $x \in [0,x]$ we can find

$$f^{(-2)}(x) = \int_0^x f(t)(x-t)dt .$$

Next, by induction, we establish that

$$f^{(-k)}(x) = \frac{1}{(k-1)!}\int_0^x f(t)(x-t)^{k-1}dt . \quad (17)$$

Theorem 2 is proved.

**Conclusions**

Thus, in this paper, we propose a technique for introducing the concept of a fractional derivative. Using the limit of the sequence, the notion of a derivative of the natural order is introduced, the definition of a fractional derivative is given for any real values of x. The basic properties of a fractional derivative are proved. Examples are given.

Practice has shown that this approach of introducing the concept of a generalized derivative contributes to the effective assimilation by students of various definitions of generalized functions.

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**МАТЕМАТИКА КУРСЫНДА ЖАЛПЫЛАНҒАН ТУЫНДЫ ҰҒЫМЫН БЕРУ ӘДІСТЕМЕСІ**

Қазіргі математикада функция, туынды және интеграл ұғымдарының әртүрлі жалпылама тұжырымдары ерекше қызығушылық тудыруда. Мысалы, туынды ұғымның жалпылауымен математиктер, оның ішінде Лейбниц, Эйлер, Лиувиль және Риман айналысқан. Жалпыланған функциялар мен олардың туындылары экономика мен өндірістің нақты процестерінде әртүрлі қолданыстарын табуда. Бұл жұмыстың мақсаты бір айнымалы функцияның жалпыланған туындысының әр түрлі анықтамаларын енгізу әдістемесін жасау болып табылады. Аталған мақсатқа жету үшін бұл мақалада алдымен комбинаториканың тұжырымдарын, тізбек шегі ұғымын пайдалана отырып, натурал ретті туынды ұғымы енгізілген. Осы тұжырымдарды қолдана отырып, бөлшек туынды деген ұғым енгізіледі.

Комбинаторикада $n$ элементтен $k$ бойынша алынған терулер саны

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!} \quad (1)$$

формуласымен анықталатыны белгілі. (1) формуланың оң жағы кезкелген $n=\tau$ нақты саны саны үшін де анықталатынын байқаймыз. Онда анықтама бойынша

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!} . \quad (2)$$

формуласын қабылдауымызға болады. Осы формуланы пайдалана отырып бір айнымалы функцияның натурал ретті туындысын сәл басқаша енгізелік. $f(x)$ функциясы $[a,\, b],\ \ a<0,\ \ b>0$ аралығында анықталсын және осы интервалдың кезкелген нақты $x$ нүктесі үшін $f(x)=0$, мұнда $x<0$. $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$ нүктелерімен $[0,\ x]$ кесіндісін $n$ бірдей бөліктерге бөлелік. Сонда

$$\frac{\Delta y}{\frac{x}{n}} = \frac{f(x) - f(x-\frac{x}{n})}{\frac{x}{n}}$$

қатысының $n \to \infty$ -да шегі бар болса, онда бұл шекті $f(x)$ функциясының $x$ нүктесіндегі туындысы деп атаймыз. Сонымен:

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f\left(x - \frac{x}{n}\right)}{\frac{x}{n}}; \quad f^{(m)}(x) = \lim_{n\to\infty} \frac{\Delta^m y}{\left(\frac{x}{n}\right)^m} = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-m} \sum_{k=0}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right). \qquad (3)$$

Сонда (2) формулаға сәйкес $C_\tau^k$ өрнегі кезкелген $\tau$ нақты саны үшін анықталатын болғандықтан (3) формуланың оң жақтары кезкелген $\tau$ нақты саны үшін де анықталатын болады. Олай болса, кезкелген $\tau$ нақты саны үшін $f(x)$ функциясының $x$ нүктесіндегі $\tau$ –ші ретті туындысы

$$f^{(\tau)}(x) = \lim_{n\to\infty} \frac{1}{\left(\frac{x}{n}\right)^\tau} \sum_{k=0}^{n} (-1)^k c_\tau^k f\left(x - \frac{kx}{n}\right). \qquad (4)$$

Формуласымен анықталады. Ал кезкелген нақты сан шексіз ондық бөлшек болғандықтан, (4) –ші туындыны бөлшек ретті туынды деп атаймыз .

Теорема . $m$ натурал саны үшін $f(x)$ функциясының $(-m)$-ші ретті туындысы:

$$f^{(-m)}(x) = \frac{1}{(m-1)!} \int_0^x f(t)(x-t)^{m+1} dt \cdot$$

формуласымен анықталады.

**Түйін сөздер:** математикалық анализ, туынды, теру, шекке көшу, бөлшек ретті туынды

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## МЕТОДИКА ВВЕДЕНИЯ ПОНЯТИЯ ОБОБЩЕННОЙ ПРОИЗВОДНОЙ В КУРСЕ МАТЕМАТИКИ

**Аннотация.** В современной математике особый интерес представляют различные обобщения понятий функции, производных и интеграла. Например, вопросами об обобщении понятия производной занимались математики, в том числе Лейбниц, Эйлер, Лиувилл и Риман. Обобщенные функции и их производные находят различные применения в реальных процессах экономики и производства.

Цель настоящей работы – разработка методики введения различных определений обобщенной производной функции одной переменной.

Для достижения этой цели вначале приведены некоторые утверждения комбинаторики, введены понятие производной натурального порядка с помощью предела последовательности. Используя эти утверждения, введены понятие производной дробного порядка.

В комбинаторике число сочетаний из $n$ элементов по $k$ определяется по формуле

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!}. \qquad (1)$$

Заметим, что правая часть формулы (1) определена при любых вещественных значениях $n = \tau$ . Тогда по определению можем положить, что

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!}. \qquad (2)$$

Используя эту формулу определим для функции одной переменной производную натурального порядка несколько иным образом.

Пусть функция $f(x)$ определена и непрерывна на промежутке $[a, b]$, $a < 0$, $b > 0$ , причем $f(x) = 0$ при $x < 0$ , где $x$ -фиксированнная точка этого интервала.

Отрезок $[0,\ x]$ разобьем на $n$ равных частей точками $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$. Тогда, если отношение

$$\frac{\Delta y}{\frac{x}{n}} = \frac{f(x) - f(x - \frac{x}{n})}{\frac{x}{n}}$$

имеет предел при $n \to \infty$, то этот предел называется производной функции $f(x)$ в точке $x$ и обозначается:

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f\left(x - \frac{x}{n}\right)}{\frac{x}{n}}, \quad f^{(m)}(x) = \lim_{n\to\infty} \frac{\Delta^m y}{\left(\frac{x}{n}\right)^m} = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-m} \sum_{k=0}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right) \qquad (3)$$

На основании того, что выражение $C_\tau^k$ согласно (2) определено при любых вещественных значениях $\tau$, то правая часть формулы (3) определяется при любых вещественных значениях $\tau$.

Следовательно, для любого вещественного числа $\tau$ производная $\tau$–го порядка функции $f(x)$ в точке $x$ определяется в виде:

$$f^{(\tau)}(x) = \lim_{n\to\infty} \frac{1}{\left(\frac{x}{n}\right)^\tau} \sum_{k=0}^{n} (-1)^k c_\tau^k f\left(x - \frac{kx}{n}\right). \qquad (4)$$

Так как любое вещественное число это бесконечная десятичная дробь, то производную (4) назовем дробным .

Теорема . Для натурального значения $m$ производная функции $f(x)$ $(-m)$-го порядка определяется по формуле:

$$f^{(-m)}(x) = \frac{1}{(m-1)!} \int_0^x f(t)(x-t)^{m+1} dt \cdot$$

**Ключевые слова:** математический анализ, производная, сочетание, предельный переход, дробная производная.

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# DIURNAL DEPENDENCE OF THE FREQUENCY OF OCCURRENCE OF TRAVELING IONOSPHERIC DISTURBANCES OVER ALMATY

**Abstract.** We studied the diurnal dependence of the frequency of occurrence of large-scale traveling ionospheric disturbances (LSTID’s) at mid-latitudes, which are a manifestation of atmospheric gravitaty waves (AGWs) generated in polar regions during geomagnetic disturbances. A significant amount of vertical sounding data of the ionosphere obtained at the Institute of the Ionosphere (Almaty 76 ° 55'E, 43 ° 15'N) in 2000-2007 was analyzed on a digital PARUS ionosonde connected to a computer designed to collect, store and process ionograms in digital form.

Further processing included the calculation of the altitude distributions of electron density

(*N* (*h*) profiles) by the Titeridge method and deriving from them variations of a number of *F*-region parameters (electron density at fixed heights $N_h(t)$; density at the maximum of the layer $N_mF$ (t) ; the heights of the maximum of the layer $N_mF(t)$, etc.). The ionozonde provided a readout accuracy of

*h* '(*t*) ~ 2.5 km and a readout accuracy of $f_oF$ ~ 0.05 MHz. It was shown that from session to session the number of waves observed during the night changed. In constructing the duirnal dependence of the frequency of occurrence of traveling ionospheric disturbances, all waves were taken into account.

A diurnal dependence of the frequency of occurrence of traveling ionospheric disturbances over Almaty was constructed from the measurement data. For this, a visual control of the behavior of a number of parameters of the F-region was carried out, identification of the LSTID’s, determination of the time of their onset and duration.

It is shown that the predominant beginning of the development of the LSTID’s is close to the moment of local midnight. 87% of traveling ionospheric disturbances were observed in the interval 20:00-04:00 LT. The distribution of the frequency of occurrence of the LSTID’s in the time of day coincides with the dependence of the substorm frequency on world time, explained on the basis of diurnal variations in the angle of inclination of the Earth's magnetic axis to the Sun-Earth line. Ion drag and the dependence of the level of auroral activity, and, consequently, the intensity of the generation of LSTID’s, on world time determines that the most favorable conditions for the distribution of LSTID’s are created over Almaty at night.

**Key words:** ionosphere, vertical sounding, diurnal dependence of the frequency of occurrence of large-scale traveling ionospheric disturbances.

## Introduction

Large-scale traveling ionospheric disturbances (LSTIDs) are caused by atmospheric gravity waves (AGWs), generated in the polar regions during geomagnetic storms [1], when the fast enhancement of auroral electrojects leads to the heating of the atmosphere. The process of rapid expansion and further compression of the atmosphere generates AGWs that propagates towards the equator and originates an LSTIDs on the way of their propagation. For many years, the propagation of AGWs in the neutral atmosphere and their ionospheric manifestation have been studied both experimentally and theoretically. The results of these studies are presented in the review works [1, 2]. The typical parameters of LSTIDs in the F region of the ionosphere are represented by the following values: the periods lie within the range from 40 min to 3 h; the horizontal wave lengths are 1000–3000 km; and the phase velocities are 400–1000 m/s. It is believed that generation and propagation of AGWs play an important role in the transfer of energy from the magnetosphere to the low latitude ionosphere. Although the LSTIDs have been studied for several decades, some fundamental problems remain little studied. Such problems include the question of the diurnal dependence of the frequency of their appearance at mid-latitudes.

**Description of the equipment, and observation results**

Nighttime observations of LSTIDs in the F region of the ionosphere are carried out at the Institute of Ionosphere (Almaty 76°55′ E, 43°15′ N) with a PARUS digital ionosonde connected to a computer for the collection, storage, and processing of ionograms in digital form. The information required for calculating various parameters of LSTIDs is read from ionograms using the semiautomatic method. Ionospheric sounding is conducted every 5 min. Nighttime measurement sessions last ~8–12 *h*, depending on the season. Ionograms provide the values of virtual heights of radio signal reflection h'(t) at several fixed operating frequencies of sounding and the values of critical frequencies foF. The further processing involves the estimation of the height distributions of electron densities ($N(h)$ profiles) by the Titheridge method [3] and obtaining the variations in several parameters of the *F* region based on them (the electron density at fixed heights $N_h(t)$; the density at layer maximum $N_mF(t)$; the heights of layer maximum $h_mF(t)$, etc.).

The ionosonde ensures the reading accuracies $h'(t)$ ~2.5 km and $f_oF$ ~ 0.05 MHz. Nighttime was chosen for observations because LSTIDs with big amplitudes of variations in ionospheric parameters at midlatitudes are usually observed at night [4]. For the period 2000-2007 we carried out 1166 night observations, while 581 nights were characterized by wave activity [5]. The variations in $N_h(t)$ at a series of heights made it possible to determine the form of a height profile of amplitudes $A(h)$ with the $A_m$ maximum absolute amplitude. For analysis, we selected observation sessions that recorded LSTIDs with a relative amplitude ($\delta h$) exceeding 25% at a height corresponding to $A_m$. Here, $\delta h = A(h)/N(h)$, where $A(h)$ is the absolute amplitude of a wave at height $h$ and $N(h)$ is the value of the undisturbed electron density at a given height.

Figure 1 shows a typical time behavior of a number of parameters after the onset of large magnetic disturbances. From the figure it follows that the beginning of the LSTIDs falls on the time interval 20:00-22:00, and the end on the interval 02: 00-04: 00. The lower curve corresponds to the height of the base of the layer ($h$ = 150 km). The upper (bold) curve corresponds to the variations of $N_mF$ ($t$) at the maximum of the $h_mF$ layer. Variations in the electron density shown in the figures demonstrate a feature characteristic of most of the sessions in which LSTIDs were observed. The peculiarity lies in the fact that the LSTIDs in the $N_mF$ (t) variations are manifested much weaker than in the $N_h$ ($t$) variations at fixed heights located below the height of the layer maximum. The reasons for this altitudinal dependence of the ionosphere response to AGW passage were considered in [6].

It should be noted that the number of waves observed during the night changed from session to session. In the example shown in figure 1, four waves are present. In constructing the diurnal dependence of the frequency of occurrence of traveling ionospheric disturbances, we took into account all the waves.

Earlier [7], a good correlation of LS TIDs with auroral substorms was proved; therefore, good similarity of their diurnal dependences should be expected. The long-term component in auroral disturbances, representing the dependence of the intensity and number of substorms on world time, was noted in [8, 9]. It became obvious that the beginning of the auroral substorms, as follows from the behavior of the auroral electrode jet index (*AL*-index), was the most frequent between 13:00 and 19:00 UT. More than 30% of the reported peaks in the *AL*-index accounted for a relatively narrow time domain of 13: 00–16: 00 UT [9]. The dependence of the substorm frequency on world time was explained on the basis of diurnal variations in the angle of inclination of the Earth's magnetic axis to the Sun-Earth line.

We analyzed the diurnal dependence of the frequency of occurrence of traveling ionospheric disturbances over Almaty according to measurements for 2000-2007. For this, a visual control of the behavior of a number of parameters of the *F*-region was carried out, identification of the LS TIDs, determination of their start time and duration. It is nessesary to notice that high probabilities of the formation of nighttime enhancements in NmF2 and the passage of LS TIDs mean a high probability of their simultaneous presence over the observation point, leading to a necessity to distinguish these two events. We performed selection LS TIDs as made it in our work [10].

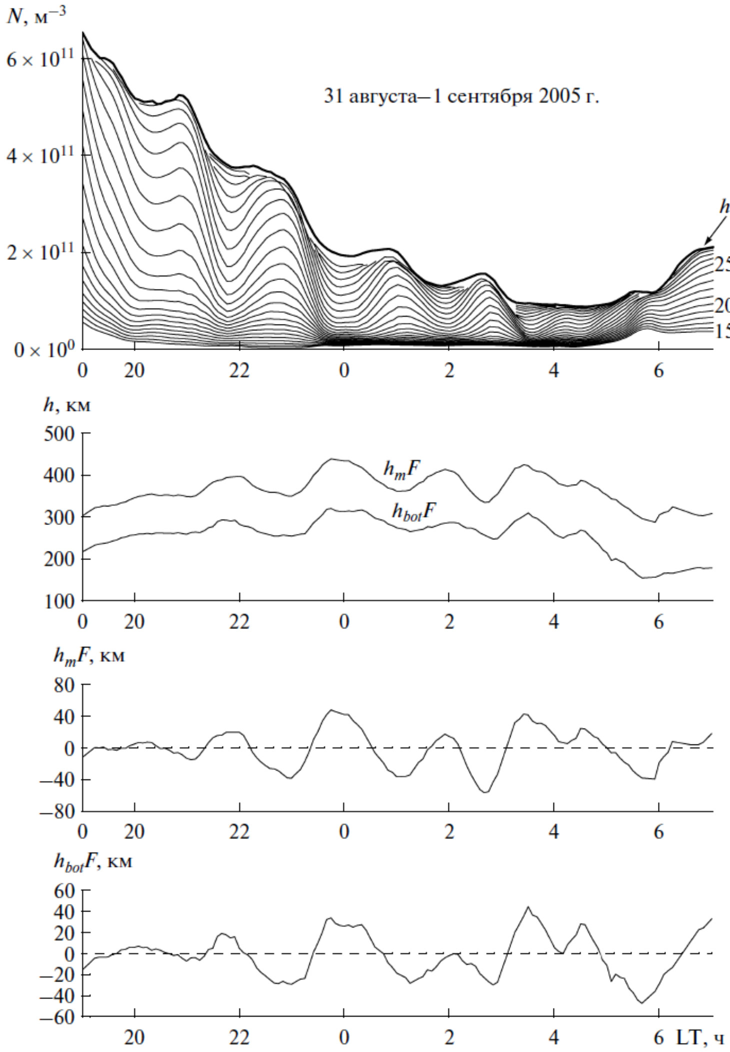


Figure 1 - Variations of the *F*-region parameters during the night with high magnetic activity (August 31 – September 1, 2005): top panel — electron density $N(t)$ at a series of heights with a distance between adjacent heights of 10 km; the second panel - the heights of the maximum of the F-region $h_mF$ and its base $h_{bot}F$; the third and fourth from the top of the panel are $h_mF$ and $h_{bot}F$ with an exclusive trend

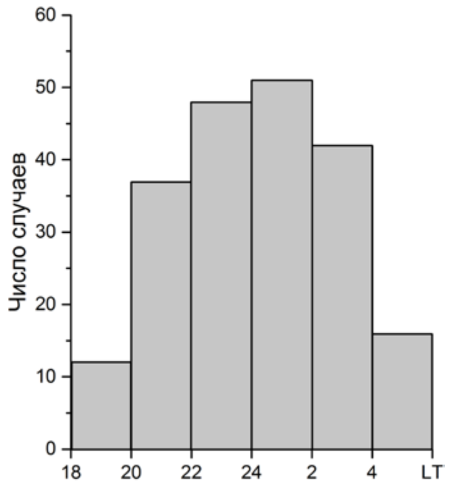


Figure 2 - diurnal dependence of the frequency of occurrence of large-scale traveling ionospheric disturbances over Almaty

According to measurements for 2000-2007. we have constructed a diurnal dependence of the frequency of occurrence of traveling ionospheric disturbances (figure 2). From the figure it follows that the predominant beginning of the development of the LS TIDs is close to the moment of local midnight. 87% of the traveling ionospheric disturbances were observed in the interval 20:00-04:00 LT. Taking into account the difference between world time and local Almaty time is 5 hours, we find that the overwhelming majority of the LS TIDs was observed in the interval 15:00-23:00 UT.

Let us consider why the LS TIDs are local in time near the midnight. In [11], the response of the ionosphere to a large magnetic storm was studied using data from the global network of vertical sounding stations and the Cosmos-900 satellite. It has been shown that the ionospheric response, which is the LS TIDs with a quasiperiod of ~ 3-4 hours, is observed throughout the globe, while the amplitudes of the LS TIDs in the night observation sector are several times higher than the amplitudes observed in the illuminated half of the Earth.

The planetary nature of the propagation of large-scale LS TIDs was also studied in [7] for a period of high solar activity, where it was shown that the probability of observing LS TIDs in the Australian-Asian longitude sector is significantly higher than the probability for the European and American sectors. This longitudinal effect is explained on the basis of the mechanism of ion drag of AGW, which takes maximum values at the illuminated time of day, and the diurnal dependence of auroral activity with a maximum of 13–19 UT. While neutral particles move freely across the line of the geomagnetic field, ions rotate around the field line and have difficulty crossing the field lines. This difference between the mobility of neutral particles and the mobility of ions limits the movement of neutral particles in the AGW in the collision of neutral particles and ions, leading to the attenuation of the AGW. This mechanism is called the ion drag effect. The magnitude of ion drag depends on the frequency of collisions of neutrals with ions, which is linearly related to the densities of neutrals and ions. Since the density of ions in the daytime is much higher than the nighttime density, the AGW attenuation on the daytime side of the Earth noticeably exceeds the attenuation on the nighttime side.

Ion drag and the dependence of the level of auroral activity, and, consequently, on the intensity of the generation of LS TIDs, on world time determines that the most favorable conditions for the distribution of LS TIDs are created over Almaty at night.

**Conclusion**

On the basis of a graphical representation of the reaction of the parameters of the F2 layer in the 23rd cycle of solar activity to the passage of large-scale traveling ionospheric disturbances (LS TIDs), the diurnal dependence of the frequency of the appearance of LS TIDs over Almaty was studied. It is shown that the frequency has pronounced maxima in the interval 20: 00-04: 00 local time. The distribution of the frequency of occurrence of the LS TIDs in time of day coincides with the dependence of the substorm frequency on world time, which is explained on the basis of diurnal variations in the angle of inclination of the Earth's magnetic axis to the Sun-Earth line.

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**АЛМАТЫ ҚАЛАСЫНЫҢ ҮСТІНЕН ҚОЗҒАЛАТЫН ИОНОСФЕРАЛЫҚ АУЫТҚУЛАРДЫҢ ПАЙДА БОЛУ ЖИІЛІГІНІҢ ТӘУЛІКТІК ТӘУЕЛДІЛІГІ**

**Аннотация.** Орташа ендіктегі ірі масштабты қозғалатын ионосфералық ұйтқулардың (ІМҚИҰ) пайда болу жиілігінің, геомагниттік ауытқулар кезінде полярлық аудандарда генерацияланатын атмосфералық гравитациялық толқындардың (АГТ) көріністері болып табылатын тәуліктік тәуелділігі зерттелді.

Ионосфераны тік зондтау деректерінің 2000-2007 жылдары алынған елеулі көлемі талданды. Ионосфера институтында (Алматы 76°55' E , 43°15' N ) сандық ионозонд "ПАРУС" компьютермен ұштасқан, ионограммаларды сандық түрде жинауға, сақтауға және өңдеуге арналған. ІМҚИҰ әр түрлі параметрлерін есептеу үшін қажетті ақпарат ионограммен жартылай автоматты әдіспен есептелді. Ионосфераны зондтау әрбір 5 мин. жүргізілді. Түнгі өлшеу сеанстарының ұзындығы маусымға байланысты өзгерді және ~ 8-12 сағ құрады. Ионограммалардан маңызы бар қаланың қолданыстағы биіктерге көрсету *h'(t)* радиосигналды бірқатар белгіленген жұмыс жиілік зондтау және маңызы бар сыни жиілік $f_oF$ есептелінді. Одан әрі өңдеу Титеридж әдісімен электронды тығыздықтың $N(h)$ -профильдердің) биіктік таралуын есептеуді және олардың негізінде F-аймақтың бірқатар параметрлерінің вариацияларын алуды ($N_h(t)$ тіркелген биіктіктердегі электрондық тығыздықты; $N_mF(t)$ қабатының максимумындағы тығыздықты; $N_mF(t)$ қабатының максимум биіктігін және т.б.) қамтиды. Ионозонд *h'(t)* ~ 2.5 км оқу дәлдігін және $f_oF$ ~ 0.05 МГц оқу дәлдігін қамтамасыз етті.

Сеанстан сеансқа түнгі уақытта байқалатын толқындардың саны өзгергені көрсетілген.

Қозғалатын ионосфералық ауытқулардың пайда болу жиілігінің тәуліктік тәуелділігін құру кезінде барлық толқындар ескерілді. Өлшеу деректері бойынша Алматы үстінен қозғалатын ионосфералық ұйтқулардың пайда болу жиілігінің тәуліктік тәуелділігі салынды. Ол үшін F-облыстың бірқатар параметрлерінің мінез-құлқына визуалды бақылау, ІМҚИҰ сәйкестендіру, олардың басталу уақыты мен ұзақтығын анықтау жүзеге асырылды. ІМҚИҰ дамуының басым бастауы жергілікті түн жаруға жақын екендігі көрсетілген. 87% қозғалатын ионосфералық 20:00-04:00 LT аралығында қалыптан байқалды. ІМҚИҰ пайда болу жиілігінің тәулік уақытында таралуы жер магниттік осінің Күн-Жер сызығына көлбеу бұрышының тәуліктік вариациясы негізінде түсіндірілетін қосалқы ағын жиілігінің әлемдік уақыттан тәуелділігіне сәйкес келеді. Иондық тежелу және авроральді белсенділік деңгейінің тәуелділігі, демек, ІМҚИҰ генерациясының қарқындылығы әлемдік уақыттан бастап ІМҚИҰ тарату үшін аса қолайлы жағдайлар Алматының үстінен түнгі уақытта құрылатынын анықтайды.

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## СУТОЧНАЯ ЗАВИСИМОСТЬ ЧАСТОТЫ ПОЯВЛЕНИЯ ПЕРЕМЕЩАЮЩИХСЯ ИОНОСФЕРНЫХ ВОЗМУЩЕНИЙ НАД АЛМАТЫ

**Аннотация**. Изучена суточная зависимость частоты появления крупномасштабных перемещающихся ионосферных возмущений (КМ ПИВ) на средних широтах, являющихся проявлением атмосферных гравитационных волн (АГВ), генерируемых в полярных районах во время геомагнитных возмущений. Проанализирован значительный объем данных вертикального зондирования ионосферы, полученных в 2000-2007 гг. в Институте ионосферы (Алматы 76°55'E, 43°15'N) на цифровом ионозонде «ПАРУС», сопряженным с компьютером, предназначенным для сбора, хранения и обработки ионограмм в цифровом виде. Информация, необходимая для расчетов разнообразных параметров КМ ПИВ, считывалась с ионограмм полуавтоматическим методом. Зондирование ионосферы проводилась каждые 5 мин. Длина ночных сеансов измерений изменялась в зависимости от сезона и составляла ~ 8–12 ч. С ионограмм считывались значения действующих высот отражения $h'(t)$ радиосигнала на ряде фиксированных рабочих частот зондирования и значения критических частот $f_oF$. Дальнейшая обработка включала в себя расчет высотных распределений электронной плотности ($N(h)$-профилей) методом Титериджа и получение на их основе вариаций ряда параметров $F$-области (электронной плотности на фиксированных высотах $N_h(t)$; плотности в максимуме слоя $N_mF(t)$; высоты максимума слоя $N_mF(t)$ и др.). Ионозонд обеспечивал точность считывания $h'(t)$ ~ 2.5 км и точность считывания $f_oF$ ~ 0.05 МГц. Показано, что от сеанса к сеансу менялось количество волн, наблюдаемых в течение ночи. При построении суточной зависимости частоты появления перемещающихся ионосферных возмущений учитывались все волны.

Была построена суточная зависимость частоты появления перемещающихся ионосферных возмущений над Алматы по данным измерений. Для этого осуществлялся визуальный контроль поведения ряда параметров F-области, идентификация КМПИВ, определение времени их начала и продолжительности.

Показано, что преобладающее начало развития КМПИВ близко к моменту местной полуночи. 87% перемещающихся ионосферных возмущений наблюдалось в интервале 20:00-04:00 LT. Распределение частоты появления КМПИВ во времени суток совпадает с зависимостью частоты суббурь от мирового времени, объясняемой на основе суточных вариаций угла наклона земной магнитной оси к линии Солнце-

Земля. Ионное торможение и зависимость уровня авроральной активности, а, следовательно, и интенсивности генерации КМ ПИВ, от мирового времени определяет то, что наиболее благоприятные условия для распространения КМ ПИВ создаются над Алматы в ночные часы.

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# AN INITIAL-BOUNDARY VALUE PROBLEM FOR A HIGHER-ORDER PARTIAL DIFFERENTIAL EQUATION

**Abstract.** The initial-boundary value problem for higher-order partial differential equations is considered. We study the existence of its classical solutions, and also propose a method for finding approximate solutions. Paper establishes sufficient conditions for the existence and uniqueness of the classical solution of the problem under consideration. Introducing a new unknown function, we reduce the considered problem to an equivalent problem consisting of a nonlocal problem for second-order hyperbolic equations with functional parameters and integral relations. An algorithm for finding an approximate solution to the problem under study is proposed and its convergence is proved. Sufficient conditions for the existence of a unique solution to an equivalent problem with parameters are established. The conditions for the unique solvability of the initial-boundary value problem for higher-order partial differential equations are obtained in terms of the initial data. Solvability of the initial-boundary value problem for higher-order partial differential equations is connected with solvability of the nonlocal problem for second-order hyperbolic equations.

**Keywords**: the higher order partial differential equations, initial-boundary value problem, nonlocal problem, hyperbolic equations of second order, solvability, algorithm.

*Introduction.* It is well known that initial-boundary value problems for higher-order partial differential equations belong to one of the most important classes of problems in mathematical physics. To study various boundary value problems for higher-order partial differential equations, along with classical methods of mathematical physics, such as the Fourier method, the Green's function method, the Poincare metric concept, the method of differential inequalities, and other methods of the qualitative theory of ordinary differential equations are also often applied. Based on these methods, the solvability conditions of the considered boundary value problems were obtained and ways to solve them were proposed in [1-14]. However, the search for effective signs of the unique solvability of initial-boundary value problems, an analogue of multipoint boundary value problems for higher-order partial differential equations, still remains relevant. The article by T. I. Kiguradze and T. Kusano is one of the first works to fill this gap [4]. This paper establishes an equivalence between the well-posedness of the initial-boundary value problem for a higher-order hyperbolic equation and the existence of only trivial solutions of the corresponding family of homogeneous boundary value problems for ordinary differential equations. Based on it, a criterion is established for the well-posedness of initial-boundary value problems for one class of partial differential equations of higher-order hyperbolic type. It is known that by means of substitution, an ordinary differential equation of higher order can be reduced to a system of ordinary differential equations of the first order. Using the methods of the qualitative theory of differential equations, the solvability conditions for the resulting system can be formulated in terms of the fundamental matrix of the differential part or the right side of the system. A similar approach can be applied to higher-order hyperbolic equations with two independent variables, and by replacement, the equations can be reduced to a system of second order hyperbolic equations with mixed derivatives. Then, using well-known methods for solving boundary value problems for systems of hyperbolic equations with mixed derivatives, the solvability conditions can be established in different terms.

Mathematical modeling of many problems of physics, mechanics, chemistry, biology, and other sciences has resulted into the necessity of studying multipoint and nonlocal boundary value problems for higher-order partial differential equations of hyperbolic type. Applying the methods of the qualitative theory of differential equations directly to these problems, we can establish the conditions for their solvability [4, 8, 14]. Multipoint and nonlocal boundary value problems for high-order partial differential equations of hyperbolic type by replacement are reduced to nonlocal boundary value problems for systems of second-order hyperbolic equations. The theory of nonlocal boundary value problems for systems of second-order hyperbolic equations has been developed in many papers. To date, various solvability conditions for nonlocal boundary value problems for hyperbolic equations have been obtained.

The criteria for the unique solvability of some classes of linear boundary value problems for hyperbolic equations with variable coefficients were obtained relatively recently [15-21]. In [15], a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations by introducing new unknown functions is reduced to a problem consisting of a family of boundary value problems with an integral condition for systems of ordinary differential equations and functional relations. It is established that the well-posedness of a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations is equivalent to the well-posedness of a family of two-point boundary value problems for a system of ordinary differential equations. In terms of the initial data, a criterion is obtained for the well-posedness of a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations.

In present paper, we consider a higher-order partial differential equation defined in a rectangular domain. The boundary conditions for the time variable are specified as a combination of values from the partial derivatives of the desired solution in rows $t = t_j$, $j=\overline{1,l}$. We also study the existence and uniqueness of the classical solution to the initial-boundary value problem for a higher-order partial differential equation and its applications.

*1. Methods.* To solve the problem under consideration, we use the method of introducing additional functional parameters [15-33] and reduce the original problem to an equivalent problem consisting of a nonlocal problem for a second-order hyperbolic equation with functional parameters and integral relations. We establish sufficient conditions for the unique solvability of the problem under study in terms of the initial data. Algorithms for finding a solution to an equivalent problem are constructed. The conditions for the unique solvability of the initial-boundary-value problem for a system of fourth-order partial differential equations are established in terms of the coefficients of the system and the boundary matrices. Separately, the result is given for an initial periodic-time boundary value problem. Note that in [34–36] a similar approach was applied to the initial-boundary value problem and the nonlocal problem for a system of partial differential equations of the third and fourth orders.

*2. Statement of problem.* At the domain $\Omega=[0,T]\times[0,\omega]$, we consider the initial-boundary value problem for the higher-order partial differential equation of the following form:

$$\frac{\partial^{m+1}u}{\partial t\partial x^m}=\sum_{i=1}^{m}\left\{A_i(t,x)\frac{\partial^i u}{\partial x^i}+B_i(t,x)\frac{\partial^i u}{\partial t\partial x^{i-1}}\right\}+C(t,x)u+f(t,x),\ (t,x)\in\Omega, \tag{1}$$

$$\sum_{j=1}^{l}\sum_{i=1}^{m}\left\{P_{ij}(x)\frac{\partial^i u(t,x)}{\partial x^i}+S_{ij}(x)\frac{\partial^i u(t,x)}{\partial t\partial x^{i-1}}\right\}\Big|_{t=t_j}=\varphi(x),\ x\in[0,\omega], \tag{2}$$

$$u(t,0)=\psi_0(t),\ \frac{\partial u(t,x)}{\partial x}\Big|_{x=0}=\psi_1(t),\ \ldots,\ \frac{\partial^{m-1}u(t,x)}{\partial x^{m-1}}\Big|_{x=0}=\psi_{m-1}(t),\ t\in[0,T], \tag{3}$$

where $u(t,x)$ is an unknown function, the functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ are continuous on $\Omega$, the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\varphi(x)$ are continuous on $[0,\omega]$, $0\le t_1<t_2<\ldots<t_{l-1}<t_l\le T$, the functions $\psi_s(t)$, $s=\overline{0,m-1}$, are continuously differentiable on $[0,T]$. The initial data satisfy the matching condition.

A function $u(t,x) \in C(\Omega, R)$ having partial derivatives $\frac{\partial^{p+r}u(t,x)}{\partial t^r \partial x^p} \in C(\Omega, R)$, $p = \overline{1,m}$, $r = 0,1$, is called a classical solution to problem (1) - (3) if it satisfies equation (1) for all $(t,x) \in \Omega$, and the initial-boundary conditions (2), (3).

We will investigate the questions of the existence and uniqueness of classical solutions to the initial-boundary value problem for a higher-order partial differential equation (1) - (3) and the construction of its approximate solutions. For these purposes, we apply the method of introducing additional functional parameters proposed in [15–33] for solving various nonlocal problems for systems of hyperbolic equations with mixed derivatives. The considered problem is reduced to a nonlocal problem for second-order hyperbolic equations, including additional functions, and integral relations. An algorithm for finding an approximate solution to the problem under study is proposed and its convergence is proved. Sufficient conditions for the existence of a unique classical solution to problem (1) - (3) are obtained in terms of the initial data.

*3. Scheme of the method and reduction to equivalent problem.* We introduce new unknown functions

$$v(t,x) = \frac{\partial^{m-1}u(t,x)}{\partial x^{m-1}},\ v_1(t,x) = u(t,x),\ v_2(t,x) = \frac{\partial u(t,x)}{\partial x},\ \ldots,\ v_{m-1}(t,x) = \frac{\partial^{m-2}u(t,x)}{\partial x^{m-2}} \quad (4)$$

and re-write problem (1)--(3) in the following form:

$$\frac{\partial^2 v}{\partial t \partial x} = A_m(t,x)\frac{\partial v}{\partial x} + B_m(t,x)\frac{\partial v}{\partial t} + A_{m-1}(t,x)v + f(t,x) +$$

$$+\sum_{r=1}^{m-2} A_r(t,x)v_{r+1}(t,x) + \sum_{s=1}^{m-1} B_r(t,x)\frac{\partial v_r(t,x)}{\partial t} + C(t,x)v_1(t,x),\ (t,x) \in \Omega, \quad (5)$$

$$\sum_{j=1}^{l}\left\{P_{m,j}(x)\frac{\partial v(t,x)}{\partial x} + S_{m,j}(x)\frac{\partial v(t,x)}{\partial t} + P_{m-1,j}(x)v(t,x)\right\}\Big|_{t=t_j} = \varphi(x) -$$

$$+\sum_{j=1}^{l}\left\{\sum_{r=1}^{m-2} P_{r,j}(x)v_{r+1}(t,x) + \sum_{s=1}^{m-1} S_{s,j}(x)\frac{\partial v_s(t,x)}{\partial t}\right\}\Big|_{t=t_j},\ x \in [0,\omega], \quad (6)$$

$$v(t,0) = \psi_{m-1}(t),\ t \in [0,T], \quad (7)$$

$$v_s(t,x) = \sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!} + \int_0^x \frac{(x-\xi)^{m-1-s}}{(m-1-s)!}v(t,\xi)d\xi,\ s = \overline{1,m-1},\ (t,x) \in \Omega. \quad (8)$$

Here the conditions (3) are taken into account in (9).

Differentiating (8) by $t$, we obtain

$$\frac{\partial v_s(t,x)}{\partial t} = \sum_{p=0}^{s-1}\dot{\psi}_k(t)\frac{x^p}{p!} + \int_0^x \frac{(x-\xi)^{m-1-s}}{(m-1-s)!}\frac{\partial v(t,\xi)}{\partial t}d\xi,\ s = \overline{1,m-1},\ (t,x) \in \Omega. \quad (9)$$

A system of functions $(v(t,x), v_1(t,x), v_2(t,x), \ldots, v_{m-1}(t,x))$, where the function $v(t,x) \in C(\Omega, R)$ has partial derivatives $\frac{\partial v(t,x)}{\partial x} \in C(\Omega, R)$, $\frac{\partial v(t,x)}{\partial t} \in C(\Omega, R)$, and $\frac{\partial^2 v(t,x)}{\partial t \partial x} \in C(\Omega, R^n)$, functions $v_s(t,x) \in C(\Omega, R)$ have partial derivatives $\frac{\partial v_s(t,x)}{\partial t} \in C(\Omega, R)$, $s = \overline{1,m-1}$, is called a solution to problem (5)--(8) if it satisfies the second-order hyperbolic equation of (5) for all $(t,x) \in \Omega$, boundary conditions (6) and (7) and integral relations (8).

For fixed $v_s(t,x)$, $s = \overline{1,m-1}$, problem (5)--(7) is a nonlocal problem for the hyperbolic equation with respect to $v(t,x)$ on $\Omega$. Integral relations (8) allow us to determine unknown functions $v_s(t,x)$, $s = \overline{1,m-1}$ for all $(t,x) \in \Omega$.

*4. Algorithm.* We determine the unknown function $v(t,x)$ from the nonlocal problem for hyperbolic equations (5)--(7). Unknown functions $v_s(t,x)$, $s=\overline{1,m-1}$, will be found from integral relations (8).

If we know the functions $v_s(t,x)$, $s=\overline{1,m-1}$, then from the nonlocal problem (5)--(7) we find the function $v(t,x)$. And, conversely, if we know the function $v(t,x)$, then from the integral conditions (8) we find the functions $v_s(t,x)$, $s=\overline{1,m-1}$. Since both functions $v(t,x)$, $v_s(t,x)$, $s=\overline{1,m-1}$, are unknown, then to find a solution to problem (5)--(8) we use an iterative method. The solution to problem (5)--(8) is the system of functions $(v^*(t,x), v_1^*(t,x), v_2^*(t,x), \ldots, v_{m-1}^*(t,x))$, which we defined as the limit of the sequence of systems $(v^{(k)}(t,x), v_1^{(k)}(t,x), v_2^{(k)}(t,x), \ldots, v_{m-1}^{(k)}(t,x))$, $k=0,1,2,\ldots$, according to the following algorithm:

*Step 0.* 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=\sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!}$ and $\frac{\partial v_s(t,x)}{\partial t}=\sum_{p=0}^{s-1}\dot{\psi}_k(t)\frac{x^p}{p!}$, $s=\overline{1,m-1}$. From nonlocal problem (5)--(7) we find the initial approximation $v^{(0)}(t,x)$ and its partial derivatives $\frac{\partial v^{(0)}(t,x)}{\partial x}$ and $\frac{\partial v^{(0)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$;

2) From integral relations (8) and (9) under $v(t,x)=v^{(0)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(0)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(0)}(t,x)$ and $\frac{\partial v_s^{(0)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$.

*Step 1.* 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=v_s^{(0)}(t,x)$ and $\frac{\partial v_s(t,x)}{\partial t}=\frac{\partial v_s^{(0)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$. From nonlocal problem (5)--(7) we find the first approximation $v^{(1)}(t,x)$ and its partial derivatives $\frac{\partial v^{(1)}(t,x)}{\partial x}$ and $\frac{\partial v^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

2) From integral relations (8) and (9) under $v(t,x)=v^{(1)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(1)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(1)}(t,x)$ and $\frac{\partial v_s^{(1)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$.

And so on.

Step $k$. 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=v_s^{(k-1)}(t,x)$ and $\frac{\partial v_s(t,x)}{\partial t}=\frac{\partial v_s^{(k-1)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$. From nonlocal problem (6)--(7) we find the $k$-th approximation $v^{(k)}(t,x)$ and its partial derivatives $\frac{\partial v^{(k)}(t,x)}{\partial x}$ and $\frac{\partial v^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$:

$$\frac{\partial^2 v^{(k)}}{\partial t\partial x}=A_m(t,x)\frac{\partial v^{(k)}}{\partial x}+B_m(t,x)\frac{\partial v^{(k)}}{\partial t}+A_{m-1}(t,x)v^{(k)}+f(t,x)+$$

$$+\sum_{r=1}^{m-2}A_r(t,x)v_{r+1}^{(k-1)}(t,x)+\sum_{s=1}^{m-1}B_r(t,x)\frac{\partial v_r^{(k-1)}(t,x)}{\partial t}+C(t,x)v_1^{(k-1)}(t,x),\ (t,x)\in\Omega, \qquad (10)$$

$$\sum_{j=1}^{l}\left\{P_{m,j}(x)\frac{\partial v^{(k)}(t,x)}{\partial x}+S_{m,j}(x)\frac{\partial v^{(k)}(t,x)}{\partial t}+P_{m-1,j}(x)v^{(k)}(t,x)\right\}\Big|_{t=t_j}=\varphi(x)-$$

$$+\sum_{j=1}^{l}\left\{\sum_{r=1}^{m-2}P_{r,j}(x)v_{r+1}^{(k-1)}(t,x)+\sum_{s=1}^{m-1}S_{s,j}(x)\frac{\partial v_s^{(k-1)}(t,x)}{\partial t}\right\}\Big|_{t=t_j},\ x\in[0,\omega], \tag{11}$$

$$v^{(k)}(t,0)=\psi_{m-1}(t),\ t\in[0,T]. \tag{12}$$

2) From integral relations (8) and (9) under $v(t,x)=v^{(k)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(k)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(k)}(t,x)$ and $\frac{\partial v_s^{(k)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$:

$$v_s^{(k)}(t,x)=\sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}v^{(k)}(t,\xi)d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega. \tag{13}$$

$$\frac{\partial v_s^{(k)}(t,x)}{\partial t}=\sum_{p=0}^{s-1}\dot\psi_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}\frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega. \tag{14}$$

Here $k=1,2,3,\ldots$.

*5. The main results.* The following theorem provides conditions for the feasibility and convergence of the constructed algorithm, as well as conditions for the existence of a unique solution to problem (5)--(8). The functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ are continuous on $\Omega$, the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\varphi(x)$ are continuous on $[0,\omega]$, the functions $\psi_s(t)$, $s=\overline{0,m-1}$, are continuously differentiable on $[0,T]$.

**Theorem 1.** *Let*

*i) the functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ be continuous on $\Omega$;*

*ii) the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\phi(x)$ be continuous on $[0,\omega]$;*

*iii) the functions $\psi_s(t)$, $s=\overline{0,m-1}$, be continuously differentiable on $[0,T]$;*

*iv) the function* $Q(x)=\sum_{j=1}^{l}P_{m,j}(x)\exp\left[\int_0^{t_j}A_m(\tau,x)d\tau\right]\neq 0$ *for all $x\in[0,\omega]$.*

*Then the nonlocal problem for the hyperbolic equation with parameters and integral conditions (5)--(8) has a unique solution $(v^*(t,x),v_1^*(t,x),v_2^*(t,x),\ldots,v_{m-1}^*(t,x))$ as a limit of sequences $(v^{(k)}(t,x),v_1^{(k)}(t,x),v_2^{(k)}(t,x),\ldots,v_{m-1}^{(k)}(t,x))$ determined by the algorithm proposed above for* $k=0,1,2,\ldots$.

The proof of Theorem 1 is similar to the proof of Theorem 1 in [36].

The equivalence of problems (5)—(8) and (1)—(3) implies

**Theorem 2.** *Let conditions i) - iv) of Theorem 1 be fulfilled.*

*Then the initial-periodic boundary value problem for the higher-order partial differential equation (1)--(3) has a unique classical solution* $u^*(t,x)$.

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## ЖОҒАРҒЫ РЕТТІ ДЕРБЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН БАСТАПҚЫ- ШЕТТІК ЕСЕП ТУРАЛЫ

**Аннотация.** Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есептер математикалыө физика мәселелерінің барынша маңызды кластарына жататыны жақсы белгілі. Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін әралуан есептерді зерттеу үшін, математикалық изиканың классикалық әдістерімен, мысалға Фурье әдісі, Грин функциялары әдісі сияқты, қатар, көп жағдайда Пуанкаренің метрикалық концепциясы, дифференциалдық теңсіздіктер әдісі және басқа да жәй дифференциалдық теңдеулердің сапалық теориясының әдістері қолданылды. Осы әдістер негізінде қарастырылып отырған шеттік есептердің шешілімділік шарттары алынды және оларды шешу тәсілдері ұсынылды. Осыған қарамастан, жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есептердің бірмәнді шешілімділігінің тиімді белгілерін іздеу мәселесі әлі де ашық әрі өзекті болып отыр. Бұрынырақ жоғарғы ретті гиперболалық теңдеу үшін бастапқы-шеттік есептің қисындылығы мен жоғарғы ретті жәй дифференциалдық теңдеу үшін сәйкес біртекті шеттік есептер әулетінің тек тривиалды шешімдерінің бар болуының арасындағы пара-парлық орнатылған болатын. Соған негізделе отырып, жоғарғы ретті гиперболалық тектес дербес туындылы дифференциалдық теңдеулердің бір класы үшін бастапқы-шеттік есептердің қисындылық критерийі тағайындалды. Жоғарғы ретті жәй дифференциалдық теңдеуді алмастырулар көмегімен бірінші ретті жәй дифференциалдық теңдеулер жүйесіне келтіруге болатыны баршаға белгілі. Дифференциалдық теңдеулердің сапалық теориясының әдістерін пайдалана отырып, алынған жүйенің шешілімділік шарттары осы жүйенің дифференциалдық бөлігінің фундаменталдық матрицасы терминінде тұжырымдалуы мүмкін. Осыған ұқсас тәсілді екі айнымалылы жоғарғы ретті гиперболалық теңдеулерге қолдануға болады және олар аралас туындылы екінші ретті гиперболалық теңдеулер жүйесіне келтірілуі мүмкін. Сонда, аралас туындылы гиперболалық теңдеулер жүйелері үшін шеттік есептерді шешудің белгілі әдістерін пайдалана отырып, шешілімділік шарттары әртүрлі терминдерде тағайындалуы мүмкін.

Жаратылыстанудың көптеген есептерін математикалық моделдеу жоғарғы ретті гиперболалық тектес дербес туындылы теңдеулер үшін көпнүктелі және бейлокал шеттік есептерді зерттеу қажеттілігіне алып келді. Дифференциалдық теңдеулердің сапалық теориясының әдістерін осы есептерге тікелей қолдану арқылы олардың шешілімділігін орнатуға болады. Жоғарғы ретті гиперболалық тектес дербес туындылы теңдеулер үшін көпнүктелі және бейлокал шеттік есептер алмастыру жолымен екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептерге келтіріледі. Екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептер теориясы көптеген жұмыстарда дамытылған. Бүгінгі кезеңде екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептердің шешілімділігінің әралуан шарттары алынған. Айнымалы коэффициенттері бар гиперболалық теңдеулер үшін сызықты шеттік есептердің бірмәнді шешілімділігінің критерийлері салыстырмалы түрде жақында алынды. Авторлардың біреуінің жұмысында гиперболалық теңдеулер жүйелері үшін интегралдық шарты бар бейлокал шеттік есеп жаңа белгісіз функциялар енгізу арқылы жәй дифференциалдық теңдеулер жүйелері үшін интегралдық шарты бар шеттік есептер әулеті мен функционалдық қатынастардан тұратын есепке келтіріледі. Гиперболалық теңдеулер жүйесі үшін бейлокал есептің қисынды шешілімділігі жәй дифференциалдық теңдеулер жүйесі үшін шеттік есептер әулетінің қисынды шешілімділігіне пара-пар екені орнатылды. Гиперболалық теңдеулер жүйелері үшін интегралдық шарты бар бейлокал шеттік есептің қисынды шешілімділігі критерийі бастапқы берілімдер терминінде алынды.

Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есеп қарастырылады. Оның классикалық шешімдерінің бар болуы мәселелері мен жуық шешімдерін табу әдістері зерттелген. Жаңа белгісіз функциялар енгізу жолымен зерттеліп отырған есеп гиперболалық теңдеулер үшін параметрлері бар бейлокал есептен және интегралдық қатынастардан тұратын пара-пар есепке келтірілген. Зерттеліп отырған есептің жуық шешімін табу алгоритмдері ұсынылған және олардың жинақтылығы дәлелденген. Параметрлері бар пара-пар есептің жалғыз шешімінің бар болуының жеткілікті шарттары тағайындалған. Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін бастапқы-шеттік есептің бірмәнді шешілімділігінің шарттары бастапқы берілімдер терминінде алынған.

**Түйін сөздер**: Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер, бастапқы-шеттік есеп, бейлокал есеп, екінші ретті гиперболалық теңдеулер, шешілімділік, алгоритм.

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# О НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧЕ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ВЫСОКОГО ПОРЯДКА

**Аннотация**. Хорошо известно, что начально-краевые задачи для дифференциальных уравнений в частных производных высокого порядка относятся к наиболее важным классам задач математической физики. Для исследования различных задач для дифференциальных уравнений в частных производных высокого порядка, наряду с классическими методами математической физики, таких как метод Фурье, метод функций Грина, также часто применялись метрическая концепция Пуанкаре, метод дифференциальных неравенств и другие методы качественной теории обыкновенных дифференциальных уравнений. На основе этих методов были получены условия разрешимости рассматриваемых краевых задач и предложены способы их решения. Несмотря на это, поиск эффективных признаков однозначной разрешимости начально-краевых задач для дифференциальных уравнений в частных производных высокого порядка все еще остается открытым. Ранее была установлена эквивалентность между корректностью начально-краевой задачи для гиперболического уравнения высокого порядка и существованием только тривиальных решений соответствующего семейства однородных краевых задач для обыкновенных дифференциальных уравнений высокого порядка. Основываясь на этом, установлен критерий корректности начально-краевых задач для одного класса дифференциальных уравнений в частных производных гиперболического типа высокого порядка. Известно, что с помощью замены обыкновенное дифференциальное уравнение высокого порядка может быть сведено к системе обыкновенных дифференциальных уравнений первого порядка. Используя методы качественной теории дифференциальных уравнений, условия разрешимости полученной системы могут быть сформулированы в терминах фундаментальной матрицы дифференциальной части системы. Аналогичный подход может быть применен к гиперболическим уравнениям высокого порядка с двумя независимыми переменными и они могут быть сведены к системе гиперболических уравнений второго порядка со смешанными производными. Тогда, используя известные методы решения краевых задач для систем гиперболических уравнений со смешанными производными, условия разрешимости могут быть установлены в различных терминах.

Математическое моделирование многих задач естествознания привело к необходимости изучения многоточечных и нелокальных краевых задач для уравнений в частных производных высокого порядка гиперболического типа. Применяя методы качественной теории дифференциальных уравнений непосредственно к этим задачам, можно установить условия их разрешимости. Многоточечные и нелокальные краевые задачи для уравнений с частными производными высокого порядка гиперболического типа путем замены сводятся к нелокальным краевым задачам для систем гиперболических уравнений второго порядка. Теория нелокальных краевых задач для систем гиперболических уравнений второго порядка развита во многих работах. К настоящему времени получены различные условия разрешимости нелокальных краевых задач для гиперболических уравнений. Критерии однозначной разрешимости некоторых классов линейных краевых задач для гиперболических уравнений с переменными коэффициентами были получены сравнительно недавно. В работе одного из авторов нелокальная краевая задача с интегральным условием для систем гиперболических уравнений путем введения новых неизвестных функций сводится к задаче, состоящей из семейства краевых задач с интегральным условием для систем обыкновенных дифференциальных уравнений и функциональных отношений. Установлено, что корректная разрешимость нелокальной задачи для системы гиперболических уравнений эквивалентна корректной разрешимости семейства краевых задач для системы обыкновенных дифференциальных уравнений. Получен критерий корректной разрешимости нелокальной краевой задачи с интегральным условием для систем гиперболических уравнений в терминах исходных данных.

Рассматривается начально-краевая задача для дифференциальных уравнений в частных производных высокого порядка. Исследуются вопросы существования ее классических решений и предлагаются методы нахождения приближенных решений. Установлены достаточные условия существования и единственности классического решения рассматриваемой задачи. Введя новые неизвестные функции мы сводим исследуемую задачу к эквивалентной задаче, состоящей из нелокальной задачи для гиперболических уравнений второго порядка с функциональными параметрами и интегральных соотношений. Предложены алгоритмы нахождения приближенного решения исследуемой задачи и доказана их сходимость. Установлены достаточные условия существования единственного решения эквивалентной задачи с параметрами. Условия однозначной разрешимости начально-краевой задачи для дифференциальных уравнений в частных производных высокого порядка получены в терминах исходных данных.

**Ключевые слова**: дифференциальные уравнения в частных производных высокого порядка, начально-краевая задача, нелокальная задача, гиперболические уравнения второго порядка, разрешимость, алгоритм.

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# SPATIAL TECHNICAL CURVES AS SMOOTH LINES OF THE ARCS NORMCURVES

**Abstract.** In solving applied problems one often encounters with the construction of spatial curves for a number of pre-set conditions. As a rule, the constructed curve is set by a set of pre-calculated or experimentally obtained conditions (points, tangents, values of curvature and torsion at these points). The horizontal and frontal projections (plan and profile) are calculated independently. As a result, the outline will consist of arcs of spatial curves no lower than the fourth order with unpredictable differential properties. The arcs of circular norm-curved spaces are devoid of this drawback.

The article discusses some theoretical questions of curve theory for a rational choice of smooth contour components. It is proposed to use arcs of cubic circles as components of a spatial smooth one-dimensional contour. A cubic circle, being a special case of a cubic ellipse, intersects an improper plane at a real point and at two cyclic points. Such curves have better differential properties from the standpoint of the monotonicity of changes in the values of the angles of inclination of the tangents, curvature, and torsion, which significantly affects the dynamic qualities of the contour being constructed.

**Key words:** normcurves, descriptive geometry, conical surface, method of rotating the plane, one-dimensional spatial outline, smoothness, component, intersection line, butt point.

The variety of technical problems of designing the various communications as a geometric component includes into itself the design of a spatial curve. As a rule, it is set by many pre-calculated or experimentally obtained geometric conditions (points, tangents, values of curvature and torsion at these points, etc.) [4,11,13]. At that, the horizontal and frontal projections are independently constructed, or its plan and profile are designed as one-dimensional contours. As a result, the desired curve is obtained as a line of intersection of two composite projecting cylindrical surfaces. The main advantage of this method is its simplicity, which is achieved by bringing a three-dimensional problem to the solution of two flat problems and using the normcurves of the plane as components. Normcurves are algebraic curves whose order is equal to the dimension of the space. A second-order curve as a normcurve of the plane is given by two points *A* and *B*, tangents $t_A$, $t_B$ at these points, and an engineering discriminant *d=BC:NC*, where $T = t_A \cap t_B$, *TC* - is the median of the triangle *ATB*.

In modern graphic packages and computer-aided design systems, spatial curves are set parametrically, which makes calculations easy. They are represented as composite lines (Hermite curves, Beziers, splines of various types). The equations of their components have the form

$$x = f_1(p), \quad y = f_2(p), \quad z = f_3(p)$$

where cubic polynomials [2,5,7,9,10,12,14] are taken as functions $f_1, f_2, f_3$ . Graphs of these functions do not allow you to visually evaluate the geometric and differential properties of the curve they describe. As an example, on figure 1 shows the construction of projections $y = \varphi_1(x), z = \varphi_2(x)$ in three-dimensional space *Oxyz* of a parametrically given curve $x = f_1(p), y = f_2(p), z = f_3(p)$ of a four-dimensional space, where $f_1(p), f_2(p), f_3(p)$– are the second-degree polynomials. Analysis of projection $y = \varphi_1(x), z = \varphi_2(x)$ of second-order parabolas shows that the original parametric given curve is a fourth-order spatial curve.

Thus, the simplicity of graphical construction of the components of the conture during manual construction and the simplicity of computational procedures in computer-aided design systems ultimately lead to the complication of the simulation result. In our case, this is expressed in an unjustified increase of the order of the components of the contour and their number from the geometric positions. This conclusion follows from the fact that the simplest spatial curves of the third order (normcurves of space) are not used as the components of the one-dimensional spatial contours.

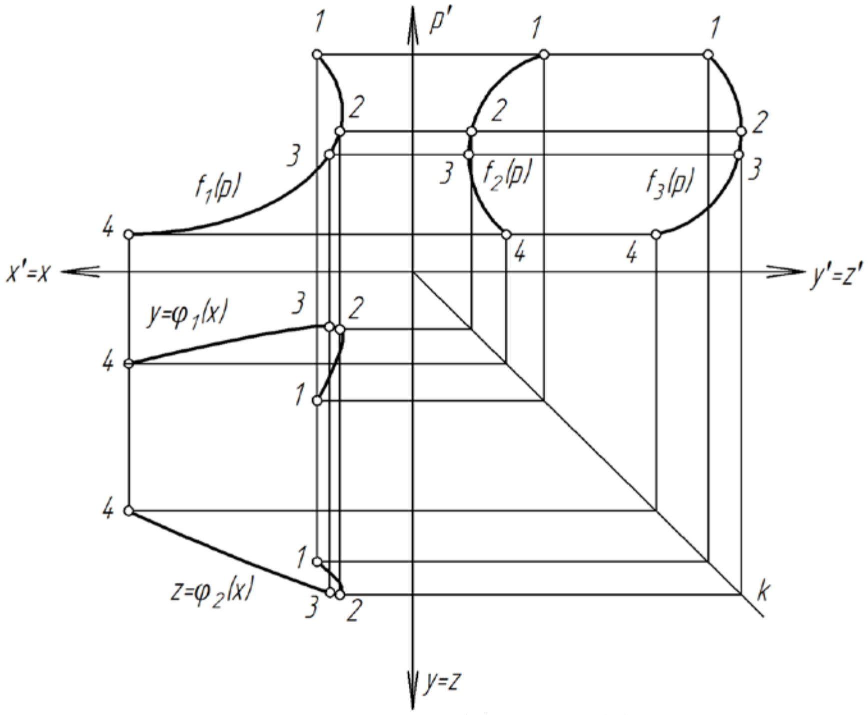


Figure 1 – The construction of projections $y = \varphi_1(x),\ z = \varphi_2(x)$ in three-dimensional space $Oxyz$ of a parametrically defined curve of four-dimensional space $Ox'y'z'p'$

In this regard, the purpose of this publication is to present the theoretical foundations of the method proposed by the authors for modeling spatial technical curves in the form of smooth one-dimensional contours from arcs of normcurves.

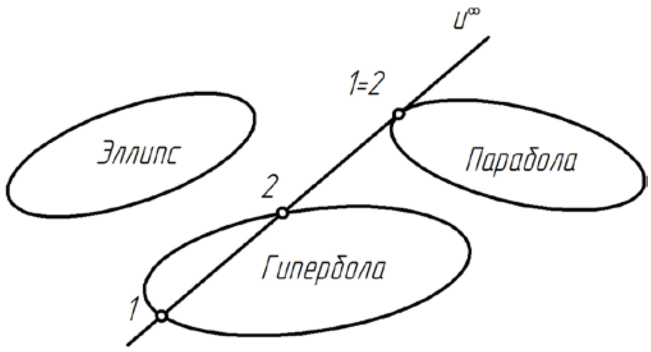


Figure 2 – The position of the second-order curves relative to the improper straight line

Since there is no information in the General technical literature about normcurves, we will first cover the questions necessary for further presentation. As the normcurves of plane (second-order curves) are classified by their position relative to the non-own straight line $u^\infty$ (figure 2), so and normcurves $k^3$ of three-dimensional space are classified by their position relative to the non-own plane $\lambda^\infty$ (figure 3) [1,6].

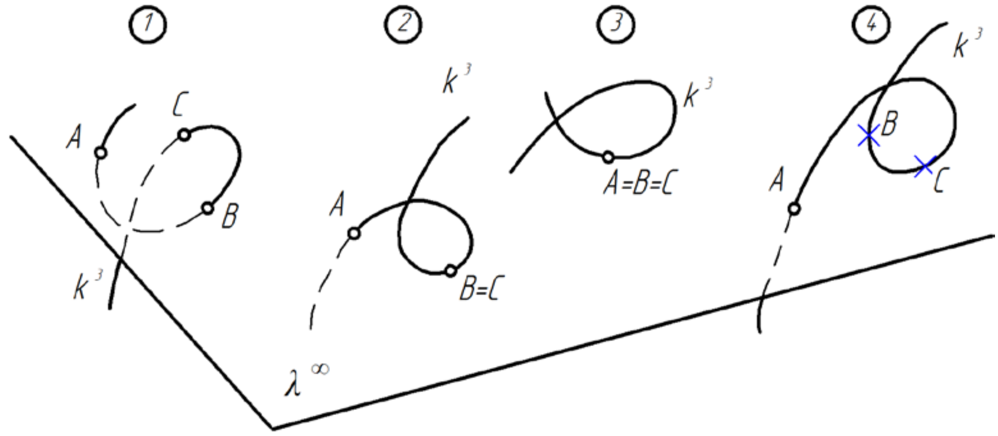


Figure 3 – The position of the norm curves of three-dimensional space relative to improper plane

1. $k^3$ - cubic hyperbola intersects $\lambda^\infty$ at three real different points.

2. $k^3$ – a hyperbolic parabola that intersects $\lambda^\infty$ at a real point *A* and touches $\lambda^\infty$ at two coinciding real points *B*=*C*.

3. $k^3$ - the cubic parabola intersects $\lambda^\infty$ at three coinciding points *A*=*B*=*C* ($\lambda^\infty$ is the touching plane of the curve $k^3$).

4. $k^3$ - cubic ellipse, intersects $\lambda^\infty$ at a real point *A* and at two imaginary-conjugate points *B* and *C*. A special case of a cubic ellipse is a cubic circle, for which the points *B* and *C* will be cyclic.

The algorithm for constructing normcurves of three-dimensional space in theoretical plan is obvious. Two second-order linear surfaces (quadrics) with one common generatrix *l* intersect along a fourth – order spatial curve that splits into their common generatrix *l* and the remainder - the desired norm curve $k^3$. The constructions will be simple if we take as these quadrics the conical surfaces $\beta, \delta$ with guide circles *b* and *d* belonging to the same plane $\gamma$ (figure 4). In this case, the normcurve $k^3$ will be a circular curve (a cubic circle) that has the best dynamic properties among normcurves [3, 8, 9, 10].

In figure 4, conical surfaces $\beta(S_1, b), \delta(S_2, d)$ with a common generatrix $l(S_1 S_2)$, whose guiding circles *b* and *d* belong to the plane $\gamma$, intersect in a cubic circle $k^3$, where $K_0 = k^3 \cap \gamma$. The curve $k^3$ is constructed by the method of a rotating plane, that is, a beam of planes $l(\alpha_i)$ with an axis *l*: $\alpha_i \cap \beta = S_1 B_i$, $\alpha_i \cap \delta = S_2 D_i$, $S_1 B_i \cap S_2 D_i = K_i$, is used as intermediaries where $K_i \in k^3$. It passes through the vertices $S_1, S_2$ of the conical surfaces $\beta$ and $\delta$, since the tangent plane $\tau_d(S_2, t_d)$ of the surface $\delta$ intersects the surface $\beta$ along its generator $S_1 1$, where $1 = t_d \cap b$, and the tangent plane $\tau_b(S_1, t_b)$ intersects the surface $\delta$ along the generator $S_2 2$, where $2 = t_b \cap d$.

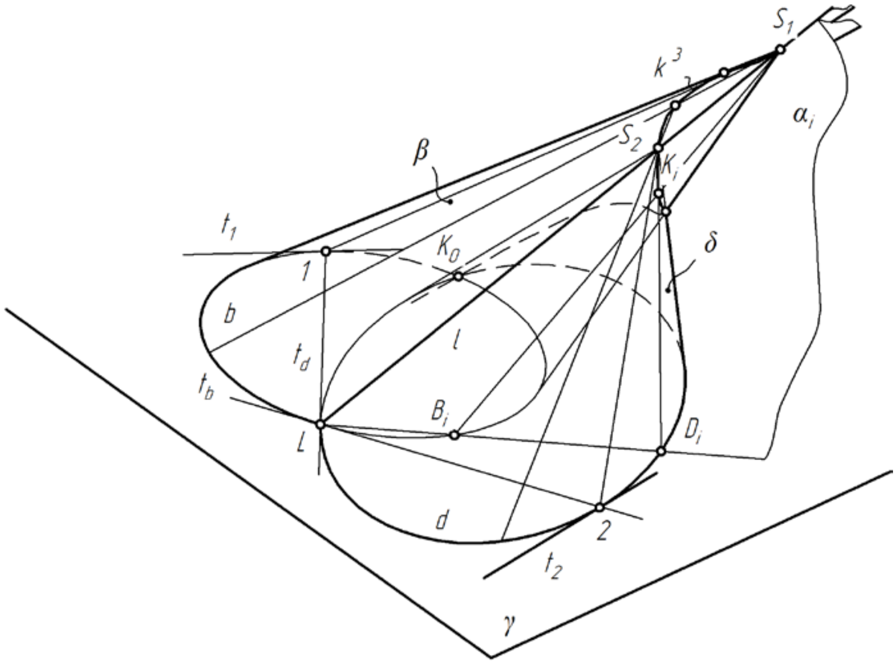


Figure 4 – Construction of a norm-curved three-dimensional space as a curve of intersection of two conical surfaces with a common generatrix

Thus, the incidence of the constructed curve $k^3$ to the vertices $S_1, S_2$ of the conical surfaces $\beta$ and $\delta$ allows to use them as butt joints in the preparation of a spatial one-dimensional contour.

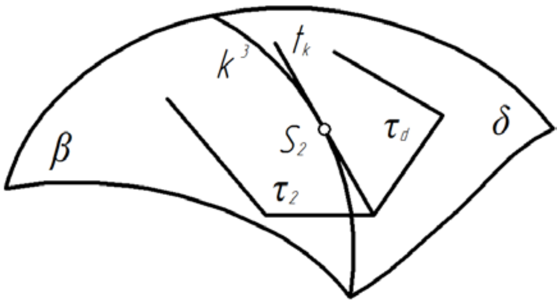


Figure 5 – Building a tangent at a butt point

To construct a smooth one-dimensional contour, it is required to specify the tangents $t_{S_1}$ and $t_{S_2}$ in the butt points $S_1$ and $S_2$, respectively. The tangent line $t_k$ at any point, for example $S_2$, of the line $k^3$ of intersection of surfaces $\beta$ and $\delta$ is constructed as the line of intersection of two tangent planes $\tau_d(S_2, t_d), \tau_2(S_2, t_2)$ drawn at a point $S_2$ to the surfaces $\beta$ and $\delta$ (Fig. 4, 5). Here the point 2 - is different from the point *L* of intersection of the tangent $t_b$ with the guide *d* of the conical surface $\delta$.

From the foregoing, the sequence of constructing the component of a smooth one-dimensional spatial contour specified by an ordered array of points $S_i$ with fixed tangents $t_{Si}$, from arcs of cubic circles $k^3$, follows:

1) two adjacent butt points $S_1$ and $S_2$ with the tangents $t_{S_1} \in S_1$, $t_{S_2} \in S_2$ specified in them are taken as the vertices of the auxiliary conical surfaces $\beta$ and $\delta$;

2) in a certain plane $\gamma$, which is chosen from the condition of simplicity of computational procedures (as a rule, a coordinate plane $Oxy$), the points $L = S_1S_2 \cap \gamma$, $1 = t_{S_1} \cap \gamma$, $2 = t_{S_2} \cap \gamma$ and straight lines $t_b(L2)$, $t_d(L1)$ of intersections $\gamma$ with planes, are built;

3) points $L, 1$ and tangent $t_1 \in 1$ define a circle $b$ - a guide of the first conical surface $\beta$; points $L, 2$ and tangent $t_2 \in 2$ define a circle $d$ - a guide of the second conical surface $\delta$; the

arc $S_1S_2$ of the normcurve $k^3$ of intersection of conical surfaces $\beta$ and $\delta$ will be the desired component of the constructed one-dimensional contour; in points $S_1$ and $S_2$ it touches these lines $t_{S_1} \in S_1, t_{S_2} \in S_2$.

Thus, a method is proposed for constructing a spatial one-dimensional smooth contour, the components of which are the arcs of a circular normal curve - a cubic circle. As known from the theory of curves, such curves have better differential properties from the standpoint of the monotonicity of changes in the values of the angles of inclination of the tangents, curvature, and torsion, which significantly affects the dynamic qualities of the contour being constructed. This is achieved through the use as constituent arcs of curves of the smallest possible order, while in all known methods, the contour components are the arcs of non-circular curves of the forth or above order.

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## НОРМҚИСЫҚТАР ДОҒАСЫНАН ҚҰРАЛҒАН ЖАНАМАЛАР РЕТІНДЕГІ КЕҢІСТІКТІК ТЕХНИКАЛЫҚ ҚИСЫҚТАР

**Аннотация.** Геометриялық компонент ретінде коммуникациялардың барлық түрлерін жобалаудың әр түрлі техникалық мәселелері кеңістіктік қисықтардың құрылысын қамтиды. Әдетте, олар бұрын есептелген немесе тәжірибе жүзінде алынған геометриялық жағдайлардың жиынтығымен анықталады (нүктелер, тангенстер, қисықтық және бұралу мәндері және т.б.). Бұл жағдайда көлденең және фронталь проекциялар (жоспар және профиль) дербес есептеледі. Нәтижесінде қалаған қисық екі құрама проекциялаушы цилиндрлік беттердің қиылысу сызығы ретінде алынады. Бұл әдістің басты артықшылығы - оның үшөлшемді есепті екі жазықтық есепті шешуге және қалыпты қисық сызықты компоненттер ретінде қолдану арқылы қол жетімділігі. Сонымен, контур құрастырушыларын қолмен жобалаудың графикалық жолмен тұрғызудың қарапайымдылығы және есептеу техникасының автоматтандырылған жүйелеріндегі есептеу процедураларының қарапайымдылығы сайып келгенде, контур құрастырушыларының және олардың дәрежелерінің геометриялық тұрғыдан негізсіз күрделенуіне әкеледі. Бұл тұжырым үшінші дәрежелі ең қарапайым кеңістіктік қисықтар (кеңістіктің норм. қисықтары) бір өлшемді кеңістіктік контурлардың құрастырушылары ретінде пайдаланылмайтынынан шығады. Нәтижесінде контур алдын-ала болжанбайтын дифференциалдық қасиеттері бар кем дегенде төртінші ретті кеңістіктік қисықтардың доғаларынан тұрады. Сондықтан тегіс кеңістіктік контурларды құру әдістерін әзірлеу - кезек күттірмейтін мәселе. Осыған байланысты, мақаланың мақсаты қарапайым қисық доғаларынан тегіс бір өлшемді контурлар түрінде кеңістіктік техникалық қисықтарды модельдеу үшін авторлар ұсынған әдістің теориялық негіздерін ұсыну болып табылады.

Тегіс контур компоненттерін ұтымды таңдау үшін мақалада қисықтар теориясының кейбір теориялық сұрақтары қарастырылады. Үш өлшемді кеңістіктің норм. қисық сызықтары шексіз алыс (дұрыс емес) жазықтыққа қатысты орналасуы бойынша жіктеледі деп атап өтілген. Сонымен қатар, норм. қисық сызықтарға қатысты келесі жағдайлар да болуы мүмкін:

- меншіксіз жазықтықты үш нақты нүктеде кесіп өтетін үшінші дәрежелі гипербола;

- меншіксіз жазықтықты нақты нүктеде кесіп өтіп, түйіндескен екі нақты нүктеде жанап өтетін гиперболалық парабола;

- меншіксіз жазықтықты түйіндескен үш нүкте арқылы қиып алатын үшінші дәрежелі парабола (меншіксіз жазықтық - қисыққа жанасатын жазықтық);

- меншіксіз жазықтықты нақты нүктеде және екі нақсыз түйісу нүктесінде қиып өтетін үшінші дәрежелі эллипс.

Үшінші дәрежелі шеңбер доғаларын кеңістіктік тегіс бір өлшемді контурдың компоненттері ретінде пайдалану ұсынылады. Кубтық эллипстің ерекше жағдайы бола отырып, куб шеңбері меншіксіз жазықтықты нақты нүкте мен екі циклдық нүктеде қиып өтеді. Мұндай қисық сызықтар контурдың динамикалық қасиеттеріне едәуір әсер ететін жанаманың көлбеулік бұрышы, қисықтық және бұралу мәндерінің өзгеруінің монотондылығы тұрғысынан жақсырақ дифференциалды қасиеттерге ие.

**Түйін сөздер**: нормқисық, сызба геометрия, конустық бет, айналмалы жазықтық әдісі, бір өлшемді кеңістіктік жанама қисық, тегістік, қиылысу сызығы, түйіндесу нүктесі..

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## ПРОСТРАНСТВЕННЫЕ ТЕХНИЧЕСКИЕ КРИВЫЕ КАК ГЛАДКИЕ ОБВОДЫ ИЗ ДУГ НОРМКРИВЫХ

**Аннотация.** Многообразие технических задач проектирования всевозможных коммуникаций в качестве геометрической составляющей включает в себя конструирование пространственных кривых. Как правило, они задаются множеством предварительно рассчитанных или экспериментально полученных геометрических условий (точек, касательных, значений кривизны и кручений в данных точках и т.д.). При этом горизонтальная и фронтальная проекции (план и профиль) рассчитываются независимо. В результате искомая кривая получается как линия пресечения двух составных проецирующих цилиндрических поверхностей. Главным достоинством такого способа является его простота, которая достигается приведением трехмерной задачи к решению двух плоских задач и использованием в качестве составляющих нормкривых плоскости. Таким образом, простота графического построения составляющих обвода при ручном его конструировании и простота вычислительных процедур в системах автоматизированного проектирования в конечном итоге ведут к усложнению результата моделирования, что выражается в необоснованном с геометрических позиций повышении порядков составляющих обвода и их количества. Этот вывод следует из того факта, что в качестве составляющих одномерных пространственных обводов не используются самые простые пространственные кривые третьего порядка (нормкривые пространства). В результате обвод состоит из дуг пространственных кривых не ниже четвертого порядка с непрогнозируемыми дифференциальными свойствами. Поэтому разработка способов конструирования гладких пространственных обводов является актуальной задачей. В связи с этим целью настоящей публикации является изложение теоретических основ предлагаемого авторами способа моделирования пространственных технических кривых в виде гладких одномерных обводов из дуг нормкривых.

Для рационального выбора составляющих гладкого обвода в статье рассматриваются некоторые теоретические вопросы теории кривых. Отмечается, что нормкривые трехмерного пространства классифицируются по их положению относительно бесконечно удаленной (несобственной) плоскости. При этом возможны случаи, когда нормкривая является

- кубической гиперболой, пересекающей несобственную плоскость в трех действительных различных точках;
- гиперболической параболой, пересекающей несобственную плоскость в действительной точке и касающееся в двух совпавших действительных точках;
- кубической параболой, которая пересекает несобственную плоскость в трех совпавших точках (несобственная плоскость является соприкасающейся плоскостью кривой);
- кубическим эллипсом, пересекающим несобственную плоскость в действительной точке и в двух мнимо-сопряженных точках.

В качестве составляющих пространственного гладкого одномерного обвода предлагается использовать дуги кубических окружностей. Кубическая окружность, являясь частным случаем кубического эллипса, пересекает несобственную плоскость в действительной точке и в двух циклических точках. Такие кривые обладают лучшими дифференциальными свойствами с позиций монотонности изменения значений углов наклона касательных, кривизны и кручения, что существенно влияет на динамические качества конструируемого обвода.

**Ключевые слова.** нормкривая, начертательная геометрия, коническая поверхность, способ вращающееся плоскости, одномерный пространственный обвод, гладкость, составляющая, линия пересечения, стыковая точка.

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# ON ONE METHOD OF RESEARCH OF MULTIPERIODIC SOLUTION OF BLOCK-MATRIX TYPE SYSTEM WITH VARIOUS DIFFERENTIATION OPERATORS

**Abstract.** There is researched the problem of existence and integral representation of a unique multiperiodic solution in all independent variables of a linear system with constant coefficients and with various differentiation operators in the direction of a vector field. Based on the Cauchy characteristics method, a methodology is developed for constructing solutions of initial problem for a linear system with constant coefficients and various special differentiation operators along two straight lines of the independent variables space, where integration characteristics are determined using a projector. It is given a methodology for constructing a matrix of homogeneous block-triangular system, as well as a matricant of a homogeneous linear system in the general case when a Jordan block is split into the sum of two sub-blocks. The Cauchy problems for linear homogeneous and nonhomogeneous systems with integral representation are solved using this methodology. At the same time, the introduced projectors for determining characteristics were of significant importance. Along with the construction of general solutions of linear systems with two differentiation operators, a theorem on the conditions of multiperiodicity of their solutions is proved. On their basis, in noncritical case, the theorem on existence and uniqueness of a multiperiodic solution of linear nonhomogeneous system is proved and its integral representation is given. The developed methodology has the perspective of extending the results obtained to the quasilinear case of system under consideration, as well as to the cases of a system with $n$ various differentiation operators and multiperiodic matrices with partial derivatives of the desired vector function.

**Key words:** multiperiodic solution, method of characteristics, projection operators, differentiation operators by vector fields, integral representation.

**Introduction.** In solving many problems of modern science and technology, we often have to deal with oscillatory processes that are described by partial differential equations. Thusfore, research of oscillatory processes described by single and multifrequency periodic solutions of differential equations systems has important theoretical and applied value. It is known that basis of oscillatory solutions theory of differential equations originates from classical works of A.Lyapunov, A.Poincare, N.N.Bogolyubov, N.M.Krylov, Yu.A.Mitropolsky, A.M.Samoilenko, A.N.Kolmogorov, V.I.Arnold, Yu.Moser and et al. Methods for integrating systems of quasilinear differential equations with identical main part are described in fundamental works [1-6]. It is known that basis of the theory of almost periodic and multiperiodic solutions of partial differential equations systems is laid down in the works [4-10]. The works of many authors has been devoted to finding effective signs of solvability and constructing constructive methods for researching problems for systems of differential equations. we note only [11,12]. The research of periodic, both time and space variables of the wave motion of the particle flow, non-stationary flows of compressible liquids and gases [2], and linear gas whose molecules have different velocity values that change each other during collisions, described by a system of partial differential equations [13], is of considerable interest in the continuum mechanics theory. Note that the integration of quasilinear differential equations systems with different main parts is one of the little-studied problems in the partial differential equations theory. Therefore, the development of methods for solving multiperiodic solutions problems of such systems is at the initial stage of its development. Some ideas of these works methods, based on researches [14-17], are extended in [18,19] to study multiperiodic solutions problems of quasilinear equations systems with various differentiation operators along their characteristics. The questions of multiperiodic solutions of quasilinear equations systems with various differentiation operators are studied in [5] in terms of the matricant when the matrix of coefficients is block-diagonal. In [20] in the case of a triangular matrix and in [18] the block-matrix method is used to construct the matricant, and in [19] the problems of multiperiodic solutions are researched by introducing projection operators [21]. Numerous applications of problems for differential equations with various differentiation operators and the necessity to expand the class of multiperiodic functions solvable in space, the creation of new approaches to solving such problems represents a new scientific interest. The article proposes a method for researching a multiperiodic solution of system with various differentiation operators, constructing matricant of a linear system in general case, when splitting a Jordan block into the sum of two sub-blocks, introducing projectors to determine characteristics, and establishing conditions for the existence of $(\theta,\omega)$-periodic solutions of the system under consideration and their integral representations.

**1. Problem statement.** We consider linear system

$$Dx = Ax + f(\tau,t), \tag{1.1}$$

where $x=(x_1,x_2)$ is unknown vector function; $x_i$ are $n_i$-vector, $n_1+n_2=n$, $D=(D_1,D_2)$ is differentiation operator with various components

$$D_1 = \frac{\partial}{\partial\tau} + \left\langle a_1, \frac{\partial}{\partial t}\right\rangle, \tag{1.1$'$}$$

$$D_2 = \frac{\partial}{\partial\tau} + \left\langle a_2, \frac{\partial}{\partial t}\right\rangle, \tag{1.1$''$}$$

$Dx=(D_1x_1,D_2x_2)$, $\left\langle a_i, \frac{\partial}{\partial t}\right\rangle$ is scalar product of vectors $a_i$ and $\frac{\partial}{\partial t}$; $a_1 \neq a_2$ are constant $m$-vectors; $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1},...,\frac{\partial}{\partial t_m}\right)$, $A=\lfloor A_{ij}\rfloor_{i,j=1,2}$ is constant block $n\times n$-matrix with blocks $A_{ij}$ of dimension $n_i \times n_j$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}, \tag{1.2}$$

$f(\tau,t)=(f_1(\tau,t), f_2(\tau,t))$ is given $n$-vector function with vector components $f_i(\tau,t)$ of dimension $n_i$, $i=\overline{1,2}$, $(\tau,t)$ are independent variables, $\tau\in(-\infty,+\infty)=R$, $t=(t_1,...,t_m)\in R^m$.

We set the problem of developing a methodology for integration, establishing the conditions for the existence of $(\theta,\omega)$-periodic solutions of system (1.1) and their integral representations. In connection with introduction of operator with various components (1.1$'$)-(1.1$''$), system (1.1) is represented by blocks of matrix and functions of input data, which require a new approach to the issue of its integration.

**2. Methodology for constructing matricant of a homogeneous block-triangular system**

For this purpose, we consider homogeneous system

$$Dx = Ax, \tag{2.1}$$

corresponding to the system (1.1), which we describe in the section of differentiation operators

$$\begin{aligned} D_1x_1 &= A_{11}x_1 + A_{12}x_2, \\ D_2x_2 &= A_{21}x_1 + A_{22}x_2. \end{aligned} \tag{2.2}$$

The problem of constructing a matricant $X(\tau)$ of system (2.1), or (2.2), in accordance with the division into blocks of $A=\lfloor A_{ij}\rfloor_{i,j=1,2}$. At first, we consider the block-triangular case when $A_{12}=O_{12}$.

Therefore,

$$A = \begin{pmatrix} A_{11} & O_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (2.3)$$

In this case, we write the system (2.2) as

$$D_1 x_1 = A_{11} x_1, \quad (2.4_I)$$

$$D_2 x_2 = A_{21} x_1 + A_{22} x_2 . \ (2.4_{II}) \quad (2.4)$$

Using [4-6, 8], we construct a matricant $X_{11}(\tau)$ of system $(2.4_I)$ with condition $X_{11}(0) = E_1$, based on

$$X_{11}(\tau) = E_1 + \int_0^\tau A_{11} X_{11}(s) ds, \quad (2.5_I)$$

and then we define a solution $X_{22} = X_{22}(\tau)$ with initial condition $X_{22}(0) = E_2$ from the matrix equation $(2.4_{II})$, where $E_1$ and $E_2$ are identity matrices, $E = diag[E_1, E_2]$ is $n$-matrix.

It is obvious that such initial problem is equivalent to integral matrix equation

$$X_{22}(\tau) = E_2 + \int_0^\tau X_{22}(\tau - s) A_{21} X_{11}(s) ds . \ (2.5_{II})$$

This solution $X_{22}(\tau)$ can be built using the same methodology as the matrix $X_{11}(\tau)$ is built.

Thus, we have a matricant $X(\tau)$ of system (2.4) in the form

$$X(\tau) = diag[X_{11}(\tau), X_{22}(\tau)]. \quad (2.5)$$

**Lemma.** *If the matrix $A$ has form* (2.3), *then the matricant $X(\tau)$ of system* (2.4) *is represented as* (2.5), *where the diagonal blocks are defined by the integral equations* $(2.5_I)$ *and* $(2.5_{II})$.

**3. Construction of matricant of a homogeneous linear system in the general case**

By replacement

$$x = By \quad (3.1)$$

with a nondegenerate constant $n$-matrix $B$, we bring the system (2.1) to the form

$$Dy = Jy, \ J = diag(J_1, ..., J_k), \quad (3.2)$$

$J_j$ are blocks of Jordan matrix $J$ an order $l_j$ with subdiagonal units, $j = \overline{1,k}$, $l_1 + ... + l_k = n_1 + n_2 = n$, according to components $D_1$, $D_2$ of operator $D$, the unknown function $y$ has coordinates $y_1$, $y_2$.

In connection with equality $l_1 + ... + l_k = n_1 + n_2$, two cases should be distinguished:

I. $l_1 + ... + l_{k_1} = n_1$, $l_{k_1+1} + ... + l_{k_1+k_2} = n_2$, where $k_1 + k_2 = k$.

II. $l_1 + ... + l_{k_1-1} + l_{k_1'} = n_1$, $l_{k_1''} + l_{k_1+1} + ... + l_{k_1+k_2} = n_2$, where $l_{k_1'} + l_{k_1''} = l_{k_1}$, $l_{k_1'} > 0$, $l_{k_1''} > 0$, $k_1 + k_2 = k$.

In the case I, the system (3.2) has the form

$$D_1 y_1 = J' y_1, \ \ J' = diag\left(J_1, ..., J_{k_1}\right), \quad (3.3_I')$$

$$D_2 y_2 = J'' y_2, \ \ J'' = diag\left(l_{k_1+1} ... + l_{k_1+k_2}\right). \ (3.3_I'') \quad (3.3_I)$$

Then its matricant is defined in the diagonal form

$$Y(\tau) = diag(Y_1(\tau), Y_2(\tau)) \quad (3.4)$$

with $n_i$-blocks $Y_i(\tau)$ that are constructed using known methodology for constructing matricant.

In the case II, the Jordan block $J_{k_1}$ is split into the sum of two sub-blocks $J'_{k_1}$, $J''_{k_1}$ and the connecting sub-block $J'''_{k_1}$ an orders $l'_{k_1}$, $l''_{k_1}$, $l'_{k_1} + l''_{k_1} = l_{k_1}$ and the system (3.2) is represented as

$$D_1 y_1 = J' y_1, \quad (3.3_{II}')$$

$$D_1 y'_{k_1} = J'_{k_1} y'_{k_1}\,,\ D_2 y''_{k_1} = J'''_{k_1} y'_{k_1} + J''_{k_1} y''_{k_1}\,, \qquad (3.3''_{II})\ (3.3_{II})$$

$$D_2 y_2 = J'' y_2\,, \qquad (3.3'''_{II})$$

where $\left(y'_{k_1}, y''_{k_1}\right) = y_{k_1}$, $J' = diag\left(J_1,..., J_{k_1-1}\right)$, $J_k = diag\left(J'_{k_1}, J''_{k_1}\right)$, $J'' = diag\left(J_{k_1+1},..., J_{k_1+k_2}\right)$, $k_1 + k_2 = k$ .

The matricants $Y_1(\tau)$ and $Y_3(\tau)$ of systems ($3.3'_{II}$) and ($3.3'''_{II}$) are constructed using the known methodology for constructing matricants of systems with single differentiation operator. In order to determine the matricant of system ($3.3''_{II}$), corresponding to the block $J_{k_1}$, what is represented in sub-blocks $J'_{k_1}$, $J''_{k_1}$ and $J'''_{k_1}$, we write in an easy-to-understand form

$$\begin{pmatrix} D_1 u \\ D_2 v \end{pmatrix} = \begin{pmatrix} J'_{k_1} & 0 \\ J'''_{k_1} & J''_{k_1} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix},\ w = y'_{k_1}\,,\ v = y''_{k_1}\,, \qquad (3.5)$$

$$J'_{k_1} = \begin{pmatrix} \mu & 0 & 0 & ... & 0 & 0 \\ 1 & \mu & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \mu & 0 \\ 0 & 0 & 0 & ... & 1 & \mu \end{pmatrix},\ J''_{k_1} = \begin{pmatrix} \mu & 0 & 0 & ... & 0 & 0 \\ 1 & \mu & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \mu & 0 \\ 0 & 0 & 0 & ... & 1 & \mu \end{pmatrix},\ J'''_{k_1} = \begin{pmatrix} 0 & ... & 0 & 1 \\ 0 & ... & 0 & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & 0 \end{pmatrix}.$$

Assuming $\lambda_k = \mu$ write down the system (3.5) in coordinate form:

$$\begin{array}{ll} D_1 w_1 = \mu\, w_1 & D_2 v_1 = \mathbf{w}_{k'_1} + \mu\, v_1 \\ ............................ & ........................ \\ D_1 w_{k'_1} = w_{k'_1-1} + \mu\, \mathbf{w}_{k'_1} & D_2 v_{k'_1+k''_1} = v_{k'_1+k''-1} + \mu\, v_{k'_1+k''} \end{array} \qquad (3.6)$$

Since the system (3.5) and, therefore, the system (3.6) has a block-triangular form, we construct its matricant $Y_2(\tau)$ based on the proved lemma. So, in case II, we have a matricant of the form

$$Y(\tau) = diag\left[Y_1(\tau), Y_2(\tau), Y_3(\tau)\right]. \qquad (3.7)$$

Therefore, from the transformation (3.1) we have the matricant of the system (2.1)

$$X(\tau) = BY(\tau)B^{-1}, \qquad (3.8)$$

where $Y(\tau)$ is determined by the relations (3.4) and (3.7). Thus, in this section it specifies a peculiar approach to constructing a matrix of a homogeneous linear block-matrix system (2.2) and, consequently, a system (2.1). Let's formalize the result in the form of the following theorem.

**Theorem 1.** *The matricant of linear homogeneous system* (2.1) *with block constant matrix* (1.2) *has the form* (3.8), *where the matrix* $Y(\tau)$ *has form* (3.4) *or* (3.7).

**4. Solutions of linear systems with various operators and their integral representations**

From the characteristic system

$$\frac{dt}{d\tau} = a_i\,,\ i = 1,2 \qquad (4.1)$$

we have solutions $t = t^0 + a_i\left(\tau - \tau^0\right) \equiv h_i\left(\tau, \tau^0, t^0\right)$, $i = 1,2$ with arbitrary initial data $\left(\tau^0, t^0\right) \in R \times R^m$. Let's assume that characteristic variable $h = h\left(\tau, \tau^0, t^0\right)$ changes on set of above defined characteristics $H = \left\{h_1\left(\tau, \tau^0, t^0\right), h_2\left(\tau, \tau^0, t^0\right)\right\}$. Note that $h_i\left(\tau^0, \tau, t\right) = t^0$, $i = 1,2$ are the first integrals of (4.1). Functions $u\left(h_i\left(\tau^0, \tau, t\right)\right)$ are zeros of operators $D_i$, respectively; where $u(t) \in C_t^{(e)}\left(R^m\right)$, $e = (1,...,1)$ is $m$-vector.

Next, the matricant $X(\tau)$ of system (2.1) is represented using block matrices $X_{ij}(\tau)$ in the form

$$X(\tau)=\begin{pmatrix} X_{11}(\tau) & X_{12}(\tau) \\ X_{21}(\tau) & X_{22}(\tau) \end{pmatrix},\ X_{ij}(\tau) \text{ are } n_i \times n_j\text{-matrices.} \quad (4.2)$$

Let the operators $P_i$ act on function $u(t)$ defined on one of two characteristics $t=h_i(\tau,\tau^0,t^0)$ as follows

$$P_i u(h(\tau,\tau^0,t^0))=u(h_i(\tau,\tau^0,t^0)),\ i=1,2,\ h(\tau,\tau^0,t^0)\in H. \quad (4.3)$$

Operators $P_i$ can be called projectors that define a function on the corresponding characteristic.

We introduce the operator $P$ associated with projectors $P_1$ and $P_2$ acting on matricant $X(\tau)$ on the right by following relation $X(\tau)P=\lfloor X_{ij}(\tau)P_i\rfloor$, $i=1,2$, where the blocks $X_{ij}(\tau)$ and projectors $P_i$ are defined by the formulas (4.2) and (4.3). We set the problem of constructing a solution $x$ of the system (2.1) with initial condition $x|_{\tau=\tau^0}=u(t)$, $u(t)\in C_t^{(e)}(R^m)$, $e=(1,...,1)$ is $m$-vector.

By checking directly, you can make sure that the problem has a unique solution of the form

$$x(\tau,t)=X(\tau-\tau^0)Pu(h(\tau^0,\tau,t)),\ h(\tau,\tau^0,t^0)\in H. \quad (4.4)$$

Here $X(\tau)$ is matricant of system (2.1), $P$ is projector. Thus, the following theorem is proved.

**Theorem 2.** *The unique solution of linear homogeneous system* (2.1) *with various differentiation operators* $D_1$ *and* $D_2$*, satisfying initial condition* $x|_{\tau=\tau^0}=u(t)$ *is determined by the relation* (4.4).

Let the vector functions $f(\tau,t)$ have smoothness property of the order $(0,e)=(0,1,...,1)$:

$$f(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m). \quad (4.5)$$

**Theorem 3.** *Under condition* (4.5)*, the unique solution* $x$ *of linear nonhomogeneous system* (1.1) *that satisfies initial condition* $x|_{\tau=\tau^0}=u(t)$, $u(t)\in C_t^{(e)}(R^m)$ *is determined by the relation*

$$x(\tau,t)=X(\tau-\tau^0)Pu(h(\tau^0,\tau,t))+\int_{\tau^0}^{\tau}X(\tau-s)Pf(s,h(s,\tau,t))ds. \quad (4.6)$$

**Proof.** It is obvious that the second term of equality (4.6) under condition (4.5) is a solution of a nonhomogeneous system (1.1), and the first term, in accordance with theorem 2, is a solution of a homogeneous system (2.1) that satisfies the given initial condition. Therefore, the relation (4.6) represents the solution of the system (1.1). Uniqueness follows from theorem 2. Q.E.D.

**5. Multiperiodic solutions of systems with various operators and their integral representations**

Let the vector functions $f_i(\tau,t)$, $i=1,2$ have $(\theta,\omega)$-periodicity and the smoothness property

$$f_i(\tau+\theta,t+q\omega)=f_i(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m),\ q\in Z^m, \quad (5.1)$$

where $(\theta,\omega)=(\theta,\omega_1,...,\omega_m)$ is a period with rationally incommensurable coordinates $\omega_0=\theta,\omega_1,...,\omega_m$.

**Theorem 4.** *Under condition* (5.1) *for* $(\theta,\omega)$*-periodicity of solution* $x(\tau,t)=\varphi(\tau,t,u(h(0,\tau,t)))$ *of system* (1.1) *with initial function* $u(t)$*, it is necessary and sufficient that the functional-difference system*

$$u(t)=\varphi(\theta,t,u(t+\Delta(t))). \quad (5.2)$$

*was solvable in the class* $\omega$*-periodic smooth functions* $u(t)=u(t+q\omega)\in C_t^{(e)}(R^m)$, $q\in Z^m$.

**Proof.** Note that if $x=x(\tau,t)$ is solution of system (1.1), then $y(\tau,t)=x(\tau+\theta,t+q\omega)$, also satisfies the system (1.1), and $z(\tau,t)=y(\tau,t)-x(\tau,t)$ is solution of (2.1). It is obvious that if the initial function

$u(t)$ of solution $x(\tau,t)$ of (1.1) has property $u(t)=u(t+q\omega)\in C_t^{(e)}(R^m)$, $q\in Z^m$, then $x(\tau,t)$ is $\omega$-periodic by $t\in R^m$ and vice versa. In the future, we will assume that this condition always satisfied.

If the initial conditions for these solutions $x(\tau,t)$ and $y(\tau,t)$ for the same

$$x(0,t)=x(\theta,t), \tag{5.3}$$

then these solutions are identically equal, moreover $x(\tau,t)=x(\tau+\theta,t+q\omega)$. Conversely, if the solution $x(\tau,t)$ is $\theta$-periodic by $\tau$, then the condition (5.3) is satisfy. Then by virtue of theorem 3, we have $x(\tau,t)=\varphi(\tau,t,u(h(0,\tau,t)))$, at that $x(0,t)=u(t)$ and $x(\theta,t)=\varphi(\theta,t,u(h(0,\theta,t)))\equiv\varphi(\theta,t,u(t+\Delta(t)))$, $\Delta(t)=h(0,\theta,t)-t$. Therefore, condition (5.3), depending on function $u(t)$, has the form (5.2). Q.E.D.

**Theorem 5.** *Linear system* (1.1) *with various operators* ($1.1'$)-($1.1''$) *under the conditions (5.1) and*

$$\operatorname{Re}\lambda(A)<0, \tag{5.4}$$

*has a unique* $(\theta,\omega)$*-periodic solution* $x^*(\tau,t)$ *with integral representation*

$$x^*(\tau,t)=\int_{-\infty}^{\tau}X(\tau-s)Pf(s,h(s,\tau,t))ds. \tag{5.5}$$

**Proof.** By virtue of structure of the general solution (4.6), the system (5.2) has the form

$$u(t)=X(\theta)Pu(t+\Delta(t))+\psi(t), \tag{5.6}$$

arbitrary term defines by $\psi(t)=\int_0^{\theta}X(\theta-s)Pf(s,h(s,\theta,t))ds$, $\omega$-periodic by $t$ and $\Delta(t)=-a\theta$. If real parts of eigenvalues $\lambda(A)$ of $A$ are negative (5.4), then after $k$ of iterations, system (5.6) can be reduced

$$u(t)=X(k\theta)Pu(t+k\Delta(t))+\psi_k(t) \tag{5.7}$$

with matrix $X(k\theta)$ that is normally bounded by constant $\delta$ from interval $0<\delta<1$. Therefore, we have

$$\|X(k\theta)\|\le\delta<1. \tag{5.8}$$

Then by the method of successive approximations by virtue of conditions (5.4) and (5.8) it is easy to show that system (5.7) and, therefore, system (5.6) has unique smooth $\omega$-periodic solution:

$$u(t)=\int_{-\infty}^{0}X(\theta-s)Pf(s,h(s,\theta,t))ds. \tag{5.9}$$

It's clear that corresponding homogeneous system (2.1) with various differentiation operators ($1.1'$) and ($1.1''$) under the conditions (5.4) has only a zero $(\theta,\omega)$-periodic solution.

Substituting (5.9) in the formula of general solution (4.6) we have integral representation of the unique $(\theta,\omega)$-periodic solution (5.5). Thus, theorem 5 is completely proved.

**Conclusion.** In article on the basis of the method of projection operator multiperiodic it is researched the solution of system with various operators of differentiation, it is built a matricant of linear system in the general case, the splitting Jordan block as a sum of two sub-blocks, it is established the conditions for existence of $(\theta,\omega)$-periodic solutions of the considered system and their integral representations. Note that the developed method can be generalized to the quasilinear case when the coefficients of linear part are multiperiodic. Meanwhile, it will be necessary to use generalizations of methods of works [4-10], [14-16] for the case under consideration.

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# ӘРТҮРЛІ ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛЫ БЛОК-МАТРИЦАЛЫҚ ТҮРДЕГІ ЖҮЙЕНІҢ КӨППЕРИОДТЫ ШЕШІМІН ЗЕРТТЕУДІҢ БІР ӘДІСІ ТУРАЛЫ

**Аннотация.** Дербес туындылы теңдеулер жүйелерімен сипатталатын тербеліс процестерінің көппериодты және периодты дерлік шешімдерін зерттеу дифференциалдық теңдеулер теориясының қазіргі даму кезеңінде үлкен қызығушылық туғызуда. Бұл тұтас орта механика теориясы, сығылатын сұйықтық пен газдардың стационар емес ағысы, бөлшектер ағынының толқындық қозғалысының физикалық-техникалық есептерінің қолданыстарымен байланысты. Деформацияланатын, кристаллдық емес орталардағы процестерді зерттеу периодтылық немесе периодты дерлік қасиеттерді ескеру қажеттілігіне алып келетініне ескере кеткен жөн.

Векторлық өріс бағыты бойынша әртүрлі дифференциалдау операторлы сызықты тұрақты коэффициентті жүйенің барлық айнымалылары бойынша жалғыз көппериодты шешімінің бары және интегралдық бейнесі туралы есеп зерттелген.

Интегралдау сипаттауыштары проектор арқылы анықталатын тәуелсіз айнымалылар кеңістігінің екі түзуі бойында әртүрлі арнайы дифференциалдау операторлы сызықты тұрақты коэффициентті блок-матрицалық түрдегі жүйе үшін бастапқы есептің шешімдерін тұрғызу тәсілі Кошидің сипаттауыштар әдісі негізінде түзілді.

Үшбұрыш-блокты түрдегі біртекті жүйенің матрицантын құру тәсілдемесі келтірілген. Диагональды блоктардың интегралдық матрицалық теңдеулері табылды. Жордан блогы екі блокшаның қосындысына ажырағанда жалпы жағдайдағы сызықты біртекті жүйенің матрицантын құрудың жаңа тәсілдемесі ұсынылған.

Құрылған әдістеме бойынша векторлар өрісі бағыты бойынша әртүрлі арнайы дифференциалдау операторлы біртекті және біртекті емес сызықты жүйелер үшін Коши есебінің жалғыз шешімінің бар болу туралы есебінің сұрағы шешілді және енетін деректер анықталған жатықтыққа ие болғандағы шарт орындалғанда интегралдық бейнелері келтірілді.

Осы тұста дифференциалдау және интегралдау жүргізілетін сәйкес характеристикаларды анықтаушы енгізілген проекторлардың маңызы шешуші болды. Проекторлардың кейбір қасиеттері тағайындалды, оның ішінде проекторлардың екі характеристикалардың бірінде берілген вектор-функцияға және құрылған матрицантқа әсері көрсетілді.

Әртүрлі екі дифференциалдау операторлы сызықты біртекті және біртекті емес жүйелердің жалпы шешімдерін құрумен қатар, периодты жатық функциялар класындағы функционалдық-айырымдық жүйенің шешілімділігінің қажетті және жеткілікті шарттары орындалғанда қарастырылған арнайы дифференциалдау операторлы жүйенің көппериодты шешімінің бар болу шарттары дәлелденді. Алынған нәтиже теорема түрінде тұжырымдалды.

Жоғарыда қарастырылғандардың негізінде, енетін деректер көппериодтылық пен анықталған жатықтық қасиеттеріне қанағаттандырғанда, критикалық емес жағдайда біртекті емес сызықты блок-матрицалық түрдегі жүйенің көппериодты шешімінің бар және жалғыз болуы туралы теорема дәлелденді және оның проекциялау операторларынан тәуелді интегралдық бейнесі келтірілді. Теореманы дәлелдеу барысында көппериодты функциялар класында функционалдық-айырымдық жүйенің шешілімділігі және енгізілген проекциялау операторы қолданылды. Теореманы дәлелдегенде біртіндеп жуықтау әдісі падаланылып, критикалық емес және анықталған жатықтық қасиеттеріне сүйене қарастырылған жүйенің жалғыз жатық көппериодты шешім табылды.

Векторлық өріс бағыты бойынша арнайы дифференциалдау операторлы блок-матрицалық түрдегі сызықты дифференциалдық теңдеулер жүйесінің көппериодты шешімін құру және проектор теориясын қолдану көппериодты функциялар кеңістігінде шешілімді есептер класының кеңеюіне келтіретінін ескереміз.

Бірінші ретті дербес туындылы сызықты теңдеулер жүйесін енетін деректердің матрицалық блоктары мен вектор-функциялары арқылы өрнектелетіндігі тәуелсіз айнымалылар кеңістігінің векторлық өріс бағыты бойынша әртүрлі арнайы сызықты дифференциалдау операторлы жүйенің

уақыт және кеңістік тәуелсіз айнымалылары бойынша көппериодты шешімінің бар болатындығын зерттеудің жаңа тәсілдемесін құруға әкелді.

Қолданылған әдістемемен алынған нәтижелерді қарастырылған жүйенің жалпыланған квазисызықты жағдайында да, сондай-ақ, әртүрлі $n$ дифференциалдау операторлы жүйе жағдайында және белгісіз вектор-функцияның дербес туындыларының жанындағы коэффициенттері көппериодты матрицалар болған кезде де осы әдісті қолданып алуға болады.

**Түйін сөздер:** көппериодты шешім, характеристикалар әдісі, проекциялау операторлары, векторлық өрістер бойынша дифференциалдау операторлары, интегралдық бейне.

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## ОБ ОДНОМ МЕТОДЕ ИССЛЕДОВАНИЯ МНОГОПЕРИОДИЧЕСКОГО РЕШЕНИЯ СИСТЕМЫ БЛОЧНО-МАТРИЧНОГО ВИДА С РАЗЛИЧНЫМИ ОПЕРАТОРАМИ ДИФФЕРЕНЦИРОВАНИЯ

**Аннотация.** На современном этапе развития теории дифференциальных уравнений наибольший интерес представляет исследование многопериодических и почти периодических решений колебательных процессов, описывающихся системами дифференциальных уравнений в частных производных. Это связано с приложениями физико-технических задач теории механики сплошной среды, нестационарных течений сжимаемых жидкостей и газов, волнового движения потока частиц. Заметим, что исследование процессов в деформируемых, некристаллических средах приводит к необходимости учета свойства периодичности или почти периодичности.

Исследована задача о существовании и интегральном представлении единственного многопериодического по всем независимым переменным решения линейной системы с постоянными коэффициентами и с различными операторами дифференцирования по направлению векторного поля пространства независимых переменных.

На основе метода характеристик Коши разработана методика построения решений начальной задачи для линейной системы с постоянными коэффициентами и с различными специальными операторами дифференцирования вдоль двух прямых пространства независимых переменных, где характеристики интегрирования определяются при помощи проектора.

Приведена методика построения матриканта однородной системы блочно-треугольного вида. Найдены интегральные матричные уравнения диагональных блоков. Предложен новый подход построения матриканта линейной однородной системы в общем случае, когда жордановый блок расщепляется на сумму двух подблоков.

По разработанной методике решен вопрос о существовании единственного решения задачи Коши для линейной однородной и неоднородной систем с различными специальными операторами дифференцирования по направлению векторного поля и найдены их интегральные представления, при условии, что входные данные обладают определенной гладкостью. При этом существенное значение имели введенные проекторы по определению соответствующих характеристик, по которым ведутся дифференцирование и интегрирование. Установлены некоторые свойства проекторов, в том числе действие проекторов на вектор-функцию заданных на одной из двух характеристик, а также воздействие на построенный матрикант.

Наряду с построением общих решений линейных систем с двумя операторами дифференцирования доказана теорема об условиях существования многопериодического решения рассматриваемой системы со специальным оператором дифференцирования при необходимом и достаточном условии разрешимости функционально-разностной системы в классе периодических гладких функций.

При предположении многопериодичности и определенной гладкости входных данных на основе вышеизложенного в некритическом случае доказана теорема о существовании и единственности многопериодического решения линейной неоднородной системы блочно-матричного вида и дано его интегральное представление, зависящее от операторов проектирования. В процессе доказательства теоремы воспользовались разрешимостью функционально-разностной системы в классе многопериодических функций и введенным оператором проектирования. В ходе доказательства используя метод последовательных приближений, в силу условий некритичности и гладкости получено единственное гладкое многопериодическое решение рассматриваемой системы.

Отметим, что построение многопериодического решения системы дифференциальных уравнений блочно-матричного вида со специальными операторами дифференцирования по направлению векторного поля с применением теории проектора приводит к расширению класса разрешимых задач в пространстве многопериодических функций.

Представление линейной системы уравнений в частных производных первого порядка через блоки матричной и векторной функций входных данных привело к разработке нового подхода исследования вопроса существования многопериодического решения как по временным так и по пространственным переменным системы с различными операторами дифференцирования по направлению векторного поля пространства независимых переменных.

Разработанная методика имеет перспективу распространения полученных результатов на квазилинейный случай рассматриваемой системы, а также на случаи системы с $n$ различными операторами дифференцирования и многопериодических матриц при частных производных искомой вектор-функции.

**Ключевые слова:** многопериодическое решение, метод характеристик, операторы проектирования, операторы дифференцирования по векторным полям, интегральное представление.

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# DESIGN OF THE THIRD-ORDER CURVES OF THE TYPE THE HYPERBOLISM OF THE HYPERBOLA

**Abstract.** The solution of many engineering problems requires the construction of curves with given characteristics. One way to create the curves is to obtain a new curve by some transformations of an already known curve. This method is especially important because the line of the family of lines belonging to the prototype, is definitely displayed in the relevant lines, of the family of lines of the constructed surface. And knowing the properties of the transformation and the preimage, it is easy to establish the properties of the image.

In this regard, this article proposes an algorithm (method) for constructing a third-order curve, where the initial image is a parabola. The choice of the parabola is based on the fact that among the set of curves of the third order when modeling the technical curves in CAD, the most widely used is a cubic parabola due to its' uniqueness. The essence of the method is as follows. Each focal straight line of parabola crosses the parabola at two points, through which two straight lines (parabolic beams) run parallel to the focal axis, and a set of pairs of these lines form a bundle with an improper center. The set of points of intersection of the bundle of lines with the center at the point P with a projective to thatbundle of straight lines (parabolic beams) forms a curve of the third order, namely "hyperbolism of hyperbola" according to Newton's classification.

**Key words:** Second order curves, curves of the third order, cubic parabola, a number of second-order, bundle of the second order, hyperbolism of conic sections, the hyperbolism of the hyperbola.

**Introduction.** In the design of surfaces, especially non-linear, an important condition is to obtain its model, described by the simplest algebraic equations. This allows you to produce a component a complex surface, minimal cost and high accuracy. The line, line families belonging to the prototype are uniquely mapped on corresponding lines, on line families of the surface being constructed [3]. The study of the curve features and its properties by means of differential geometry is possible only if the curve is expressed in an analytical form, i.e. by an equation. However, in many problems of theoretical and especially of practical nature, before investigating the equation of the curve, it is necessary to make it on the basis of some data that somehow determine this curve and expressed in the problem conditions [5]. Therefore, knowing the properties of the transformation and the prototype, it is not difficult to establish the properties of the image. In this regard, research in this area is considered very relevant.

This article relates to the field of formation of third-order curves. It gives a method based on the use of an auxiliary parabola of the second degree.

Of the many third-order curves, the cubic parabola is the most widely used for modeling the technical curves in CAD because of its unambiguity. The parabola equation of the 3rd degree has the form

$$y = ax^3 + bx^2 + cx + d$$

When considering the projective method of curve formation, the concept of the order of a series of points and a beam of lines is introduced [15, 16]. The notion of a second-order series and a second-order sheaf makes it possible to define with projective method algebraic curves of higher orders and classes. Thus, in General, if two sheaves of first-order lines lie in the same plane and are projective with each other, then the intersection points of their respective lines form a second-order curve, and similarly, if a sheaf of first-order lines and a sheaf of second-order lines lie in the same plane and are projective with each other, then the intersection points of their respective lines form a third-order curve.

In theoretical and engineering practice it is developed and proposed, taking into account the above mentioned, many ways of making the curves of the third order, where the original images are: a bundle of conic sections and projective to it pencil of lines; the points and straight lines; triangle; circle; parabola; conic sections; conic section and triangle, etc. [8,11, 12,13,14,17].

**Materials and methods.** It should be noted that for construction of the third order curve, where parabola is the original image, 28 different approaches (methods) presented [14].

In the monograph of G. S. Ivanov [8, 9, 10] "Construction of technical surfaces" it is proposed to use cubics given in the form

$$y = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2}$$

and obtained by means of cremonic involutions with non-owned center.

Here the involution $I_3$ is given by non-owned point $F^{\infty}$, the axis *Oy* of the reference system, and an invariant cubic decayed into a straight line $d^1$ and a second-order curve $d^2$ (hyperbola with asymptot or a parabola with a vertical axis) (figure 1).

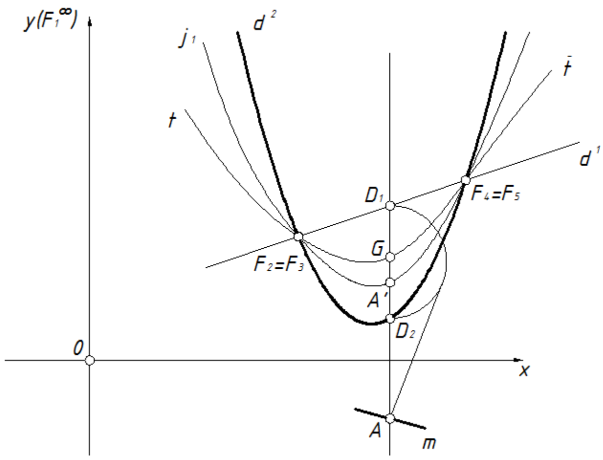


Figure 1 - Involution $I_3$ with non-owned point

We propose an algorithm (method) for constructing a third-order curve, where the initial image is parabola. The essence of the method is that the set of points of intersection of bundle of straight lines with the center at the point P with a projective to it bundle of straight lines (parabolic rays), forms a curve of the third order.

Let us have a parabola $y = ax^2 + bx + c$, with a vertex at the origin, c = 0, and its focal axis coincides with the ordinate axis (figure2) [4,6,7].

The straight lines of the beam P and the straight lines of the beam ***L*** form a bundle of first-order straight lines (figure 3).

The projective correspondence of the first-order beam ***P*** with the center at p and the beam ***L*** (parabolic rays) is established using the beam ***F*** of the focal lines of this parabola.

The position of the straight line of the first-order beam ***P*** with the corresponding straight line of the beam ***F*** of the focal lines can be determined by the angular displacement $\Delta$.

If the straight line of the beam ***P*** of the first order is located at an angle $\alpha$ to the abscissa axis, then the corresponding straight line of the beam ***F*** of focal straight lines is located to the abscissa axis at an angle $\alpha + \Delta$ (figure 2).

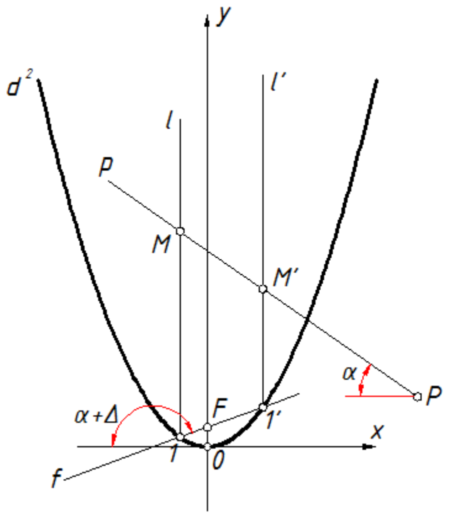


Figure 2 - Intersection points of projective lines

The focal straight line ***f*** crosses the parabola at points ***1, 1'***. Two straight lines ***l, l'*** (two parabolic rays) pass through these points. The line p intersects with these lines at points ***M, M'***.

Each pair of straight lines $l_i$, $l_i'$ of the beam ***L*** is incident to the intersection points of the focal straight line $f_i$ with a given parabola $d^2$.

To each straight $\boldsymbol{p_i}$ of the beam (***P***) corresponds two straight $\boldsymbol{l_i}$ of the beam (***L***) and Vice versa, to each straight $\boldsymbol{l_i}$ of the beam (L) corresponds one straight $\boldsymbol{p_i}$ of the beam (P).

Therefore, the correspondence between the straight of the beams (***P***) and (***L***) [1, 2] - valued.

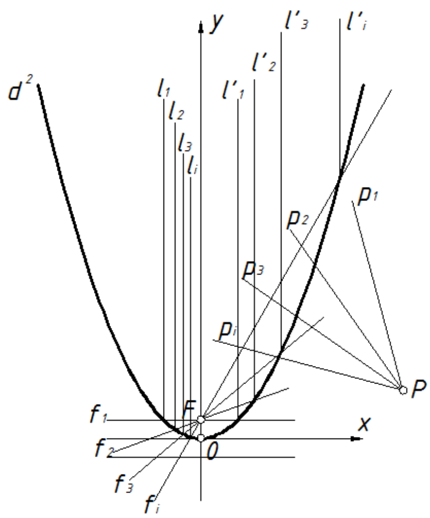

Figure 3 - Building a lot of points

The order of an algebraic curve, as a result of the intersection of the corresponding straight lines of two sheaves, is

$$1 \cdot 1 + 1 \cdot 2 = 3$$

Thus, each straight line $\boldsymbol{p_i}$ of the beam (***P***) intersecting with two straight lines $\boldsymbol{l_i}$ of the beam ( ***L*** ), located on different sides of the axis of symmetry of the parabola, gives a set of points ***Mi***, constituting a curve of the third-order.

**Derivation of the curve equation.** We write the equation of the line passing through the focus F of the parabola

$$y = kx + t$$

here the value $t$ is the deviation of the focus from the origin in the direction of the axis $y$. We calculate it from the parabola equation under the condition $x = 2y$ (according to the geometric definition of a parabola). Then,

$$t = \frac{1 - 2b}{4a}$$

Solving the equations of the parabola and the focal line together, we determine the coordinates $x_1, x_2$ of the points of their intersection.

Substituting the coordinates $x_1, x_2$ in the equation of the line passing through the point P, we calculate the coordinates $y_1, y_2$ of the points of intersection of this line with projectively corresponding to it the lines $l_i$ of the bundle L.

The set of points $\{x_1, y_1\}$, $\{x_2, y_2\}$ computed with the variable $\alpha$ $(0 \le \alpha \le 180^\circ)$ make up a third-order curve of the type of hyperbolism of a hyperbola.

***Results and discussion.*** The nature and shape of the obtained curves show that they belong to the fourth class of Newton's classifications- "hyperbolisms of conical sections".

Hyperbolism of conical section

$$yx^2 + ex = cy + d \quad (a = b = 0)$$

can be considered as a special case of a parabolic hyperbola in which the parabolic asymptote has split into two parallel lines: $x = \pm\sqrt{c}$. Hyperbolisms have three asymptotes: $y = 0, x = \pm\sqrt{c}$ and a double infinitely distant point

Hyperbolisms of the conic sections are divided into three kinds [14]. Studies have found that the position of the center of the bundle (P) determines belonging to a certain genus (Fig.4, 5 – hyperbolism of the hyperbola). And the value of the parameter affects the position of the asymptotes and the shape of the curves themselves.

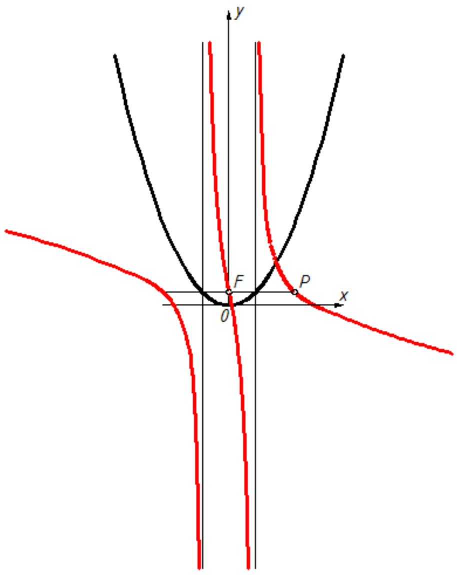


Figure 4 - The center P of the beam P is located outside the band bounded by the asymptotes

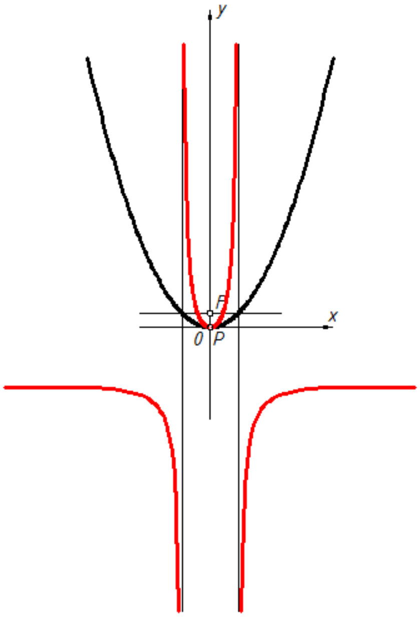


Figure 5 - The center P of the beam P is located between the asymptotes

***Conclusion.*** We have developed an algorithm (method) for constructing a third-order curve, where the initial image is a parabola. In this case, the transformation results in a third-order curve belonging to the fourth class of Newton's classifications- "hyperbolisms of conical sections", namely hyperbolism of hyperbola. Studies have shown that changes in the parameter Δ and the position of the center P of the beam P can be controlled by the shape of the resulting curve.

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## ГИПЕРБОЛАНЫҢ ГИПЕРБОЛИЗМІ УЛГІЛІ ҮШІНШІ ДӘРЕЖЕЛІ ҚИСЫҚТАРДЫ ҚҰРАСТЫРУ

**Аннотация.** Көптеген инженерлік мәселелерді шешу белгілі сипаттамалары бар қисық сызықтарды салуды қажет етеді. Қисық сызықтарды тұрғызудың бір әдісі – алдын-ала белгілі қисықты көптеген белгілі геометриялық түрлендіру әдістерінің бірі арқылы жаңа қисық сызықты алу. Қисықты құрастырудың бұл әдісі ең тиімді болып табылады. Ол жаңа қисықтарды анықтауға арналған сарқылмайтын құралдарды беріп қана қоймайды, сонымен қатар жаңа қисықтың қасиеттерін түрлендірілетін қисықтың кескіні ретінде анықтауға мүмкіндік береді. Сонымен қатар, бұл әдістің ерекше маңыздығы - түрлендірілетін кескінге жататын сызықтар тобы сәйкес сызықтарға, құрастырылатын бет сызықтарының тобына бір қалыпты кескінделеді. Ал түрлендіру әдісінің және бастапқы қисықтың қасиеттерін біле отырып, алынатын сызықтың қасиеттерін анықтау қиын емес. Беттерді, әсіресе қисық сызықты беттерді салу кезінде қарапайым алгебралық теңдеулермен сипатталған оның үлгісін алу маңызды шарт болып табылады. Бұл күрделі бетті бөлшекті құны кем де кем, және жоғары дәлдігімен жасауға мүмкіндік береді. Осыған байланысты, осы саладағы зерттеулер өте өзекті болып саналады.

Бұл мақала үшінші дәрежелі қисықтарды тұрғызу әдістеріне қатысты. Осыған байланысты, бұл мақалада екінші дәрежелі көмекші параболаны қолдануға негізделген үшінші дәрежелі қисық сызықты тұрғызу алгоритмі (әдіс) ұсынылған. Параболаны таңдаудың негізгі себебі - техникалық қисықтарды автоматтандырылған есептеу жүйелерінде модельдеуде екінші және үшінші дәрежелі көп сызықтардың арасында бірегейлік қасиетінің арқасында кеңінен қолданылатын парабола болып табылады. Бұл жерде, түрлендіру нәтижесінде Ньютонның классификациясы бойынша төртінші класқа жататын «конустық қималардың гиперболизмдері» типті үшінші дәрежелі қисық сызық пайда болады - атап айтқанда гиперболаның гиперболизмы.

Ұсынылып отырған әдістің мәні келесідей. Бізге екінші дәрежелі $\boldsymbol{d^2}$ параболасы және Р нүктесінде шоғырланған ***p*** сызықтар шоғы берілген делік. $\boldsymbol{d^2}$ параболаның әрбір фокалдық сызығы осы параболаны екі нүкте арқылы қиып өтеді, ол нүктелер арқылы фокалдық оське параллель екі түзу (параболалық сәулелер) ***l, l'*** өтеді, және осы сызықтардың көптеген жұптары өзіндік емес центрмен шоқ құрайды. Центрі Р, ***p*** сызықтар шоғының оған проективті түзулер шоғымен (параболалық сәулелер) ***l, l'*** қиылысу нүктелерінің жиынтығы $\boldsymbol{d^3}$ үшінші дәрежелі қисық сызықты құрайды, және ол «гиперболаның гиперболизмы» болып табылады. Центрі Р нүктесінде орналасқан бірінші дәрежелі ***p*** түзулер шоғының қосарланған түзулер (параболалық сәулелер) ***l, l'*** шоғымен проективті сәйкестігі осы параболаның фокустық сызықтарының ***f*** шоғы арқылы тағайындалады. Бірінші дәрежелі ***p*** шоғы түзу сызығының оған проективті сәйкес келетін ***f*** фокалдық шоғы сызығының өзара орналасу жағдайы олардың бұрыштық жылжуы Δ арқылы анықталуы мүмкін. Зерттеулер нәтижесінде ***p*** шоғының Р центрінің орнын және Δ параметрін өзгерту арқылы пайда болған қисық сызықтың пішіні басқарылатындығы анықталды. Ұсынылған әдістің аса ерекшелігі - түрлендіру нәтижесінде пайда болған сызық тек қана «гиперболаның гиперболизмы» болып табылады, және басқа қисық сызық болмайды, сондай-ақ, есептеу қисынды алгоритмдердің қарапайымдылығы автоматтандырылған есептеу жүйелерінде мұндай қисықтарды модельдеу кезінде маңызды.

**Түйін сөздер**: Екінші дәрежелі қисық сызықтар, үшінші дәрежелі қисықтар, текшелік парабола, екінші дәрежелі қатарлар, екінші дәрежелі сызықтар шқғы, конус қимасы гиперболизмдері, гиперболаның гиперболизмі.

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## КОНСТРУИРОВАНИЕ КРИВЫХ ТРЕТЬЕГО ПОРЯДКА ТИПА ГИПЕРБОЛИЗМ ГИПЕРБОЛЫ

**Аннотация.** Решение многих инженерных задач требует построения кривых линий с заданными характеристиками. Одним из способов образования кривых является получение новой кривой путем того или иного геометрического преобразования уже известной кривой. Этот способ образования кривых является наиболее эффективным. Он не только даёт неиссякаемые средства для определения новых кривых, но и позволяет определять свойства новой кривой как отражение свойств преобразуемой кривой. Также, этот способ особенно важен потому, что линия, семейства линий, принадлежащих прообразу, однозначно отображаются в соответствующие линии, семейства линий конструируемой поверхности. А зная свойства преобразования и прообраза, нетрудно установить свойства образа. При конструировании поверхностей, особенно нелинейчатых, важным условием является получение ее модели, описываемой простейшими алгебраическими уравнениями. Это позволяет изготовить деталь со сложной поверхностью с минимальными затратами и высокой точностью. В связи с этим, исследования в данной области считаются весьма актуальными.

Данная статья относится к области образования кривых третьего порядка. В связи с этим в данной статье предлагается алгоритм (способ) построения кривой третьего порядка, основанный на использовании вспомогательной параболы второй степени. Выбор параболы основан на том, что из множества кривых второго и третьего порядка при моделировании технических кривых в САПР наиболее широко применяется парабола из-за ее однозначности. При этом в результате преобразования получается кривая третьего порядка, относящаяся к четвертому классу по классификаций Ньютона – «Гиперболизмы конических сечений», а именно гиперболизм гиперболы.

Суть способа заключается в следующем. Пусть мы имеем параболу второго порядка $\boldsymbol{d^2}$ и пучёк прямых ***p*** с центром в точке Р. Каждая фокальная прямая *f* параболы $\boldsymbol{d^2}$ пересекает эту параболу в двух точках, через которые проходят две прямые (параболических лучи) ***l, l'*** параллельно фокальной оси, а множество пар этих прямых образуют пучок с несобственным центром. Множество точек пересечения пучка прямых ***p*** с центром в точке ***P*** с проективным ему пучком прямых (параболических лучей) ***l, l'***, образует кривую третьего порядка $\boldsymbol{d^3}$, а именно «гиперболизм гиперболы». Проективное соответствие пучка ***p*** первого порядка с центром в точке **Р** и пучка (параболических лучей) ***l, l'*** устанавливается с помощью пучка ***f*** фокальных прямых данной параболы. Положение прямой пучка ***p*** первого порядка с проективно соответствующей прямой пучка ***f*** фокальных прямых может определяться угловым смещением Δ. Исследованиями установлено, что изменениями параметра Δ и положением центра ***P*** пучка ***p*** можно управлять формой получаемой кривой. Отличительной особенностью предлагаемого способа является то, что результате преобразования получается

именно «гиперболизм гиперболы», и никакая другая кривая, а так же, простота вычислительных алгоритмов, что важно при моделировании таких кривых в САПР.

**Ключевые слова.** Кривые второго порядка, кривые третьего порядка, кубическая парабола, ряда второго порядка, пучок второго порядка, гиперболизмы конических сечений, гиперболизм гиперболы.

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# NUMERICAL SIMULATION OF HEAT AND MASS TRANSFER AT THE PARTIAL STOP OF FUEL SUPPLYING IN THE CHAMBER OF TPP

**Abstract**. In this article, using numerical methods, studies have been performed to determine the effect of a forced partial stop of coal dust supply (emergency mode) through burners on the main characteristics of the combustion chamber of the BKZ-75 boiler of the Shakhtinskaya thermal power plant. Using computer simulation methods, various modes of supplying pulverized coal into the combustion chamber were investigated. Direct-flow method of supplying air mixture, when only two direct-flow burners work out of four burners, and two are in emergency mode. The vortex method of supplying the mixture, when two vortex burners operate with four torches with a swirl angle of the mixture flow and tilting them to the center of symmetry of the boiler by 30 degrees, and two are in emergency mode. The performed computational experiments made it possible to obtain the main characteristics of the heat and mass transfer process in the combustion chamber: flow aerodynamics, temperature fields, and concentrations of harmful substances (carbon and nitrogen dioxides) in the combustion chamber and at its exit. A comparative analysis was carried out for the two investigated emergency conditions (direct-flow and vortex), on the basis of which it was concluded that in the event of a forced partial stop of the burners, the use of a vortex method of supplying air mixture improves the metabolic processes in the combustion chamber and reduces emissions of harmful substances into the atmosphere.

**Key words.** Computational experiment, numerical simulation, pulverized coal fuel, emergency mode, aerosol mix, aerodynamics, temperature and concentration fields, combustion chamber.

## Introduction

Coal, oil, gas, oil shale, peat, urine, etc. are the main sources of energy; their share is up to 93%. Among the geological fuel and energy resources, the largest reserve in the world belongs to solid fuel, the gross volume of which is estimated at 6.3 trillion tons of standard fuel. Moreover, the volume of solid fuel is 3970 billion tons of fuel equivalent, oil and gas are about 800 billion and 900 billion tons of conventional fuel, respectively [1]. According to expert estimates, the share of coal in the structure of the world fuel and energy balance is about 27%.

Combustion of energy fuel is accompanied by the formation of dust and gas emissions harmful to the environment, the amount of which depends on the technology and modes of combustion of coal dust, as well as on its composition [2-6]. The most important pollutants entering the atmosphere when burning pulverized coal in combustion chambers are particulate matter (ash, dust, soot particles), as well as gas emissions (nitrogen, carbon, sulfur oxides, etc.) [7-15]. Currently, it is necessary to develop “clean” technologies for generating electricity.

The emergency shutdown of the boiler can be in the following cases: when the steam pressure in the boiler rises above the permissible one; due to malfunction of the pressure gauge and all water indicating devices; the presence of significant damage to the elements of the boiler; detection of abnormalities in the operation of the boiler. The scheduled shutdown of the boiler is carried out according to the schedule; a short shutdown of the boiler unit can be caused by a violation of its normal operation due to equipment malfunction or for other reasons that can cause an accident. Short-term shutdown of the boiler unit may be caused by a violation of its normal operation due to equipment malfunction or for other reasons that may cause an accident [16-17].

Below are the results of a study of heat and mass transfer processes at a partial stop of the fuel supply to the combustion chamber of a thermal power plant, which allows us to suggest ways to optimize the combustion process and minimize emissions of harmful substances. To conduct computational experiments, 3D modeling methods and modern computer software packages were used. The real combustion chamber of the operating boiler BKZ-75 of the Shakhtinskaya TPP (Shakhtinsk, Kazakhstan) was chosen as the object of study [18-23], in which high-ash Kazakhstan coal is burned. The results obtained made it possible to determine the effect of swirling of the pulverized coal flow during a forced partial stop of the supply of coal dust through the burners on the main characteristics of the combustion chamber.

**Physical statement of the problem**

For carrying out computational experiments, the boiler combustion chamber was chosen, which is equipped with four dust-coal burners installed two burners from the front and from the rear in one tier [24-25]. Below are a general view of the combustion chamber of the BKZ-75 boiler (figure 1a), its breakdown into control volumes for numerical simulation (figure 1b) and the design of burners (figure 1 c, d). The finite-difference grid (figure 1b) for numerical modeling has steps along the X, Y, and Z axes: 59×32×67, which are 138 355 control volumes.

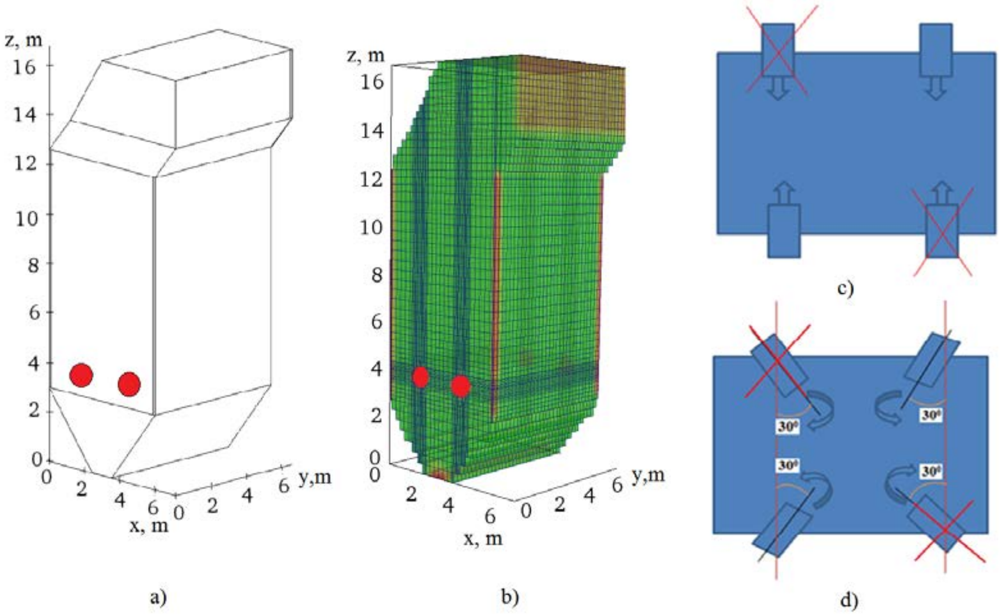


Figure 1 - A general view of the combustion chamber of the BKZ-75 boiler (a),
its breakdown into control volumes for numerical simulation (b) and the design of burners:
c) direct-flow method of supplying air mixture; d) vortex method of supplying air mixture

The design of the burners of the combustion chamber of the BKZ-75 boiler during emergency mode (off burners are marked in red) are shown in figure 1 c, d. Two modes of fuel supply were investigated: a direct-flow method of supplying air mixture, when only two direct-flow burners operate from four burners, and two are in emergency mode (figure 1c) and the vortex method of supplying the mixture, when two vortex burners operate with four torches with a swirl angle of the mixture flow and tilting them to the center of symmetry of the boiler by 30 degrees, and two are in emergency mode (figure 1d).

**Results and discussion**

Figures 2-8 show the results of computational experiments: aerodynamics of the flow, temperature and concentration fields of carbon monoxide, nitrogen dioxide for two cases of fuel supply to the combustion chamber of the BKZ-75 boiler (direct-flow and vortex).

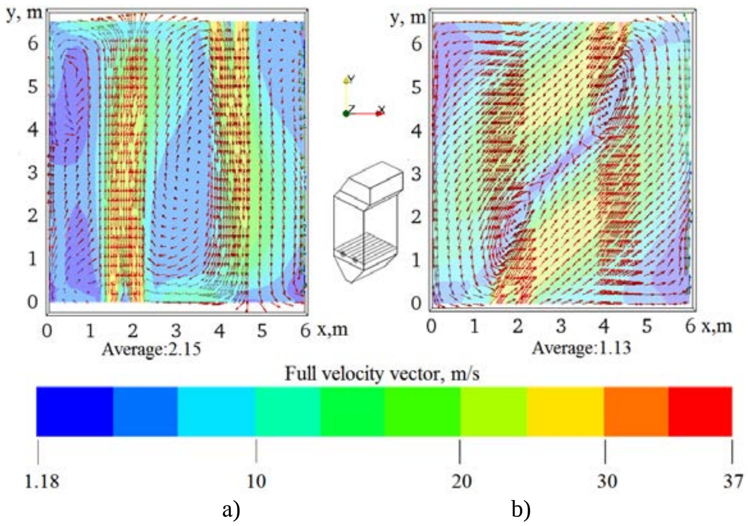


Figure 2 - Distribution of the full velocity vector in the region of the burner (z = 4.0 m) combustion chamber of the boiler BKZ-75 in **emergency mode**:
a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture

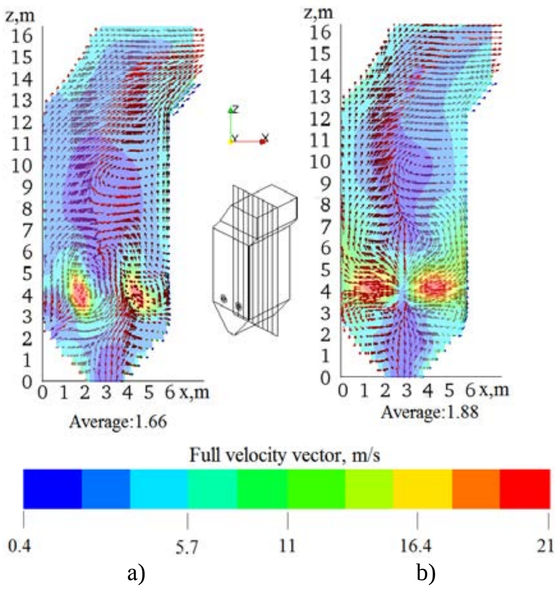


Figure 3 - Distribution of the full velocity vector in the central longitudinal section (y=3.3) of the combustion chamber of the boiler BKZ-75 in **emergency mode**:
a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture

The distribution of the field of the full velocity vector in various sections of the combustion chamber of the BKZ-75 boiler during emergency mode for the two studied modes of supply of air mixture (direct-flow and vortex) are shown in figures 2-3. An analysis of the figures shows that in the region of the burners with the direct-flow method of supplying the mixture, the flows colliding in the center at a right angle (figure 2a) are cut and, combined into two main flows, are directed to the exit from the combustion chamber (figure 3a).With the vortex method of supplying air mixtures (figure 2b), it is seen that flows counter-directed at an angle of 30° form a “vortex” flow in the center of the combustion chamber.

After the collision, the flows are additionally dissected into two vertical vortices above the burner installation zone, closer to the center of the combustion chamber, which favorably affects the mixing of fuel and oxidizer and, accordingly, the completeness of combustion of pulverized coal dust (figure 3 b). The main advantage of the vortex air mixture supply method is the provision of intensive heat and mass transfer in the reacting two-phase mixture due to the stable highly turbulent vortex flow.

A comparative analysis of the distribution of the average temperature in the cross section along the height of the combustion chamber of the BKZ-75 boiler during emergency mode for the two studied modes of supply of air mixture (direct-flow and vortex) is shown in figure 4. It can be noted that during the vortex flow of the air mixture, an increase in the length of the zone of maximum temperatures and a decrease in it at the exit from the combustion chamber are observed. The minimum in the curves are related to the low temperature of the air mixture entering the combustion chamber through the burners (z = 4 m). The temperature at the exit of the combustion chamber is confirmed by experimental data at TPPs [26] and to theoretical values obtained by the method of thermal calculation the CBTI [27].

An increase in temperature in the core of the torch and a decrease in it at the exit from the combustion chamber affect the chemical processes of combustion and the formation of harmful substances, such as carbon monoxide CO and nitrogen dioxide $NO_2$. An analysis of this effect can be done by considering figures 5-6, which show the distribution of the concentration of carbon monoxide CO and nitrogen dioxide. $NO_2$ the height of the combustion chamber, operating in emergency mode, for two methods of supplying air mixtures (direct-flow and vortex).

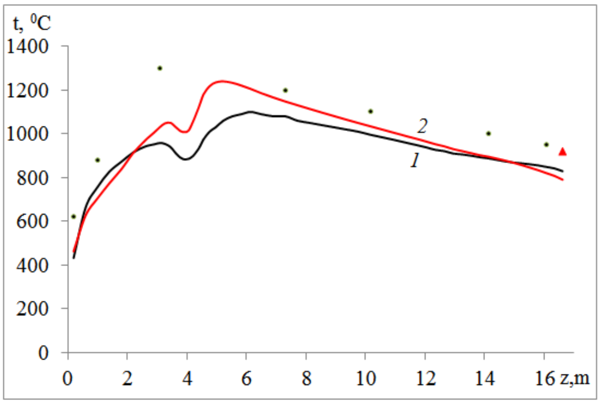


Figure 4 - Distribution of the temperature t along the height of the combustion chamber of the BKZ-75 boiler in **emergency mode**: 1- direct-flow method of supplying air mixture; 2 - vortex method of supplying air mixture; ● - experimental data at TPPs [26]; ▲ - is theoretical values obtained by the method of thermal calculation (CBTI – Central Boiler-and-Turbine Institute) [27]

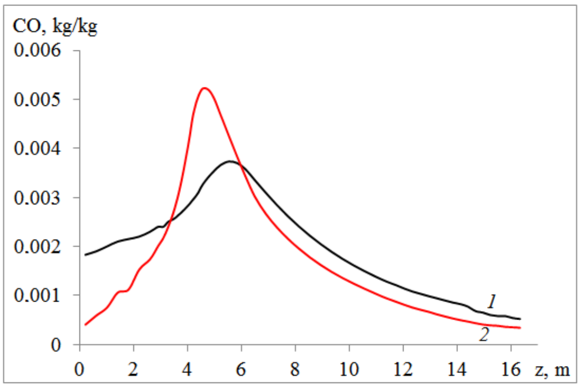


Figure 5 - Distribution the concentration of carbon monoxide CO at the exit of the combustion chamber the boiler BKZ-75 **emergency mode**:
1 - direct-flow method of supplying air mixture carbon monoxide CO; 2 - vortex method of supplying air mixture carbon monoxide CO

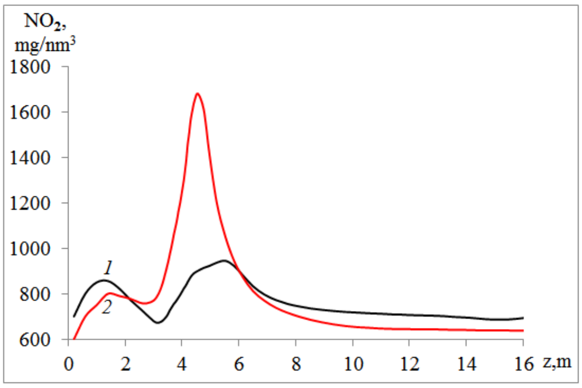


Figure 6 - Distribution the concentration of nitrogen dioxide $NO_2$ at the exit of the combustion chamber the boiler BKZ-75 **emergency mode**:
1 - direct-flow method of supplying air mixture nitrogen dioxide $NO_2$; 2 - vortex method of supplying air mixture nitrogen dioxide $NO_2$

Figure 5 shows the distribution curves of the average concentration of carbon monoxide over the height of the combustion chamber of the BKZ-75 boiler during emergency mode for the two cases studied. An analysis of the figure shows that, at the exit, the concentrations of carbon monoxide CO are - $5.2 \cdot 10^{-4}$ kg/kg, for the vortex method of supplying air mixture - $3.4 \cdot 10^{-4}$ kg/kg. The distribution of the average concentration of nitrogen dioxide $NO_2$ over the height of the combustion chamber of the BKZ-75 boiler during emergency mode for the two studied methods of supplying air mixture is shown in figure 6. As can be seen from Figure 6, a uniform decrease in $NO_2$ concentration is observed towards the exit from the combustion chamber, since this region contains less oxygen and a fuel component. In the case of using burners with swirling of the mixture flow, the temperature along the height of the combustion chamber monotonously decreases, as a result of which the rate of formation of nitrogen dioxide $NO_2$ decreases. At the exit of the combustion chamber, the average value of the concentration of nitrogen dioxide $NO_2$ with the direct-flow method of supplying the mixture is 688 mg/nm$^3$ (figure 6 curve 1), and with vortex burners - 636 mg/nm$^3$ (figure 6 curve 2), which is 52 mg/nm$^3$ less.

An analysis of the results shows that if the combustion chamber operates in emergency mode, then at the exit from it, the average concentration of carbon monoxide CO and nitrogen dioxide $NO_2$ decreases when using vortex burners. The calculated values of the concentrations of harmful substances (CO, $NO_2$,) at the exit from the combustion chamber comply with the MPC standards adopted in the power system of the Republic of Kazakhstan.

**Conclusion**

Based on the results of studies of emergency mode of the BKZ-75 combustion chamber, the following conclusions can be formulated:

- The characteristics of the combustion processes during emergency mode are compared for two cases: a direct-flow method for supplying air mixture, when only two direct-flow burners operate from four burners, and two are in emergency mode and **vortex method** of supplying the mixture, when two vortex burners operate with four torches with a swirl angle of the mixture flow and tilting them to the center of symmetry of the boiler by 30 degrees, and two are in emergency mode.
- During the vortex flow of air mixture, an increase in the length of the zone of maximum temperatures and a decrease in it at the exit of the combustion chamber are observed. The minimum in the curves are related to the low temperature of the air mixture entering the combustion chamber through the burners (z=4 m).
- If the combustion chamber is operating in emergency mode, then at the exit from it, the average concentration of carbon monoxide CO and nitrogen dioxide $NO_2$ decreases when using vortex burners. The calculated values of the concentrations of harmful substances (CO, $NO_2$,) at the exit from the combustion chamber comply with the MPC standards adopted in the power system of Kazakhstan.

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## ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ ТЕПЛОМАССОПЕРЕНОСА ПРИ ЧАСТИЧНОЙ ОСТАНОВКЕ ПОДАЧИ ТОПЛИВА В КАМЕРУ СГОРАНИЯ ТЭЦ

**Аннотация.** Казахстан является одним из государств, обладающих огромными запасами углеводородов, которые оказывают существенное влияние на формирование и состояние мирового энергетического рынка.

На территории республики имеются залежи около 33,600 млн. тонн угля (3,8% мировых запасов угля), 30 000 млн. баррелей нефти (1,8% мировых запасов) и 1,5 трлн кубометров природного газа (0,8% мировых запасов). Вследствие этого в нашей стране до 85% всей выработки электроэнергии производится путем сжигания ископаемого топлива, главным образом местного низкосортного угля. Энергоснабжение обеспечивается за счет производства электроэнергии на 69 электростанциях, основным источником которых является высокозольный казахстанский уголь Экибастузского, Карагандинского, Тургайского угольных бассейнов. Уголь в Казахстане обладает рядом преимуществ: малое содержание серы, высокий выход летучих веществ на сухую золу меньше массы и низкая цена, потому что уголь добывается в основном открытым способом. Тем не менее, он характеризуется низким качеством из-за высокого содержания золы в его составе (более 40%). Как следствие, использование такого топлива в теплоэнергетике приводит к проблемам в стабилизации пламени и горении в целом, в шлаковании конвективных поверхностей нагрева (экранов печей) и в загрязнении воздуха летучей золой, оксидами углерода, азота и серы ($CO_x$, $NO_x$, $SO_x$), углеводородами ($C_nH_m$) и другими продуктами сгорания. Кроме того, при использовании низкосортных углей увеличивается расход мазута или природного газа, используемых для растопки котла, подхвата и стабилизации горения пылеугольного факела, и ухудшается экологическая обстановка. Еще одной проблемой в энергоснабжении является то, что любое энергетическое предприятие нуждается в периодической остановке котельной, при этом возможны: аварийное отключение, плановое отключение и кратковременная остановка.

В данной статье с применением численных методов проведены исследования, позволяющие определить влияние вынужденной частичной остановки подачи угольной пыли (аварийный режим) через горелочные устройства на основные характеристики топочной камеры котла БКЗ-75 Шахтинской ТЭЦ. Паровой котел заводской марки БКЗ-75 – вертикально-водотрубный, производительностью 75 т/час (51,45 Гкал/ч). Котельный агрегат блочной конструкции является однобарабанным, с естественной циркуляцией и выполнен по П-образной схеме. Котел БКЗ-75 оборудован четырьмя пылеугольными горелками, установленными по две горелки с фронта и с тыла в один ярус. В котле сжигается пыль Карагандинского рядового (КР-200) угля, зольностью 35,1%, выходом летучих 22%, влажностью 10,6% и теплотой сгорания 18,55 MJ/kg. Методами компьютерного моделирования были исследованы различные режимы подачи пылеугольного топлива в камеру сгорания. Прямоточный способ подачи аэросмеси, когда из четырех горелок работают только две прямоточные горелки, а две находятся в аварийном режиме. Вихревой способ подачи аэросмеси, когда из четырех горелок работают две вихревые горелки с углом закрутки потока аэросмеси и наклоном их к центру симметрии котла на 30 градусов, а две находятся в аварийном режиме. Для исследования процессов тепломассопереноса в высокотемпературных средах использованы физико-математическая и химическая модели, включающие в себя систему трехмерных уравнений Навье–Стокса и уравнений тепломассопереноса с учетом источниковых членов, которые определяются химической кинетикой процесса, нелинейными эффектами теплового излучения, межфазного взаимодействия, а также многостадийностью химических реакций.

Выполненные вычислительные эксперименты позволили получить основные характеристики процесса тепломассопереноса в камере сгорания: аэродинамику течения, поля температуры и концентраций вредных веществ (оксиды углерода и азота) в объеме топочной камеры и на выходе из нее. Проведен сравнительный анализ для двух исследуемых аварийных режимов (прямоточный и вихревой), на основании которого был сделан вывод о том, что вихревой способ подачи аэросмеси при вынужденной остановке горелочных устройств топочной камеры приводит к уменьшению температуры, концентрации оксидов углерода CO и диоксидов азота $NO_2$ на выходе, но имеет их высокий уровень внутри топочного пространства. Вихревой способ подачи аэросмеси позволяет в значительной степени оптимизировать процесс сжигания низкосортных высокозольных углей в топочных камерах ТЭС и существенно снизить выбросы вредных веществ ($NO_2$ и CO) в окружающую среду. Полученные результаты 3D моделирования процессов тепломассопереноса в топочной камере котла БКЗ-75, работающей в аварийном режиме, подтверждает перспективность использования вихревого способа подачи аэросмеси с целью достижения требований энергоэффективного и экологически безопасного сжигания твердых топлив.

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## ЖЭО ЖАНУ КАМЕРАСЫНА ОТЫН БЕРУДІ ІШІНАРА ТОҚТАТУ КЕЗІНДЕГІ ЖЫЛУМАССАТАСМАЛДАУДЫ САНДЫҚ МОДЕЛДЕУ

**Аннотация.** Қазақстан әлемдік энергетикалық нарықтың қалыптасуы мен жай-күйіне елеулі әсер ететін көмірсутектердің орасан қоры бар мемлекеттердің бірі болып табылады. Республика аумағында шамамен 33,600 млн.тонна көмір (әлемдік көмір қорының 3,8%), 30 000 млн. баррель мұнай (әлемдік қорлардың 1,8%) және 1,5 трлн. текше метр табиғи газ (әлемдік қорлардың 0,8%) шоғырлары бар. Осының салдарынан біздің елімізде барлық электр энергиясын өндірудің 85% - ына дейін қазбалы отынды, негізінен жергілікті төмен сұрыпты көмірді жағу жолымен жүргізіледі. Энергиямен жабдықтау негізгі көзі Екібастұз, Қарағанды, Торғай көмір бассейндерінің жоғары көмірі болып табылатын 69 электр станцияларында электр энергиясын өндіру есебінен қамтамасыз етіледі. Қазақстандағы көмір бірқатар артықшылықтарға ие: күкірттің аз мөлшері, құрғақ күлге ұшатын заттардың жоғары шығуы массадан аз және төмен баға, себебі көмір негізінен ашық тәсілмен өндіріледі. Дегенмен, ол құрамында күлдің жоғары болуына байланысты төмен сапамен сипатталады (40%-дан астам). Жылуэнергетикада мұндай отынды пайдалану жалын тұрақтану мен жалпы жанудың, жылудың конвективті беттерін (пеш экрандарын) қождауда және ауаны ¦шпа күлмен, көміртегі, азот және күкірт тотықтарымен (COx, NOx, SOx), көмірсутектермен (CnHm) және басқа да жану өнімдерімен ластауда проблемаларға алып келеді. Сонымен қатар, төмен сортты көмірді пайдалану кезінде қазандықты жағу, ұстап қалу және шаңкөмір алауының жануын тұрақтандыру үшін пайдаланылатын мазут немесе табиғи газдың шығыны артады және экологиялық жағдай нашарлайды. Энергиямен жабдықтаудағы тағы бір проблема кез келген энергетикалық кәсіпорын қазандықты мерзімді тоқтатуға мұқтаж, бұл ретте: авариялық ажырату, жоспарлы ажырату және қысқа мерзімді тоқтату болуы мүмкін. Бұл мақалада сандық әдістерді қолдану арқылы Шахтинск ЖЭО БКЗ-75 қазандығының оттық камерасының негізгі сипаттамаларына жанарғы құрылғылары арқылы көмір тозаңын беруді мәжбүрлі ішінара тоқтату әсерін анықтауға мүмкіндік беретін зерттеулер жүргізілді. БКЗ-75 маркалы зауыт бу қазандығы-тік-суқұбыры, өнімділігі 75 т/сағ (51,45 Гкал/сағ). Блокты конструкцияның қазандық агрегаты бірбарабанды, табиғи айналыммен және П-тәрізді схема бойынша орындалған. БКЗ-75 қазандығы майданнан және тылдан бір қабатқа екі жанарғы орнатылған төрт шаң бұрышымен жабдықталған. Қазандықта Қарағанды қатардағы көмірдің (КР-200) шаңы жағылады, күлдігі 35,1%, ұшқыштың шығымы 22%, ылғалдылығы 10,6% және жану жылуы 18.55 MJ/kg. Компьютерлік модельдеу әдістерімен жану камерасына шаңкөмір отынын берудің әртүрлі режимдері зерттелді. Төрт жанарғылардан тек екі тікелей ағатын жанарғылар ғана жұмыс істейтін, ал екеуі авариялық режимде болатын аэроқоспаларды берудің тура ағынды тәсілі. Төрт жанарғылардан екі құйынды жанарғылар жұмыс істеп тұрған кезде аэрос қоспа ағынының бұралу бұрышы және оларды қазан симметриясының ортасына 30 градусқа еңкейтілген, ал екеуі авариялық режимде болады. Жоғары температуралы орталардағы жылу масса алмасу процестерін зерттеу үшін физика-математикалық және үш өлшемді Навье–Стокс теңдеулері мен көз мүшелерін есепке ала отырып, жылу масса алмасу теңдеулер жүйесін қамтитын химиялық модельдер, химиялық кинетикамен, жылулық сәулеленудің, фазааралық өзара әрекеттесудің сызықты емес әсерлерімен, сондай-ақ химиялық реакциялардың көп сатылы болуымен анықталатын.

Орындалған есептеу эксперименттері жану камерасында жылумасстасмалдау процесінің негізгі сипаттамасын алуға мүмкіндік берді: ағыс аэродинамикасы, температура өрісі және зиянды заттардың концентрациясы (көміртегі мен азот оксидтері) оттық камера көлемінде және одан шығуда. Екі зерттелетін апаттық режимдер үшін салыстырмалы талдау жүргізілді (тура ағынды және құйынды), оның негізінде оттық камераның жанарғы құрылғылары мәжбүрлі тоқтаған кезде аэро қоспаны берудің құйынды тәсілі температураның азаюына, CO көміртегі оксидтерінің және $NO_2$ азот диоксидтерінің шығуында шоғырлануына алып келеді, бірақ оттық кеңістіктің ішінде олардың жоғары деңгейі болады. Ауа қоспасын берудің құйынды тәсілі ЖЭС оттық камераларында төмен сортты жоғары көмірді жағу процесін айтарлықтай оңтайландыруға және қоршаған ортаға зиянды заттардың ($NO_2$ және CO) шығарындыларын айтарлықтай төмендетуге мүмкіндік береді. Апаттық режимде жұмыс істейтін БКЗ-75 қазандығының оттық

камерасындағы жылу масса алмасу процестерін 3D үлгілеудің алынған нәтижелері қатты отындарды энерготиімді және экологиялық қауіпсіз жағу талаптарына қол жеткізу мақсатында аэро қоспаны берудің құйынды тәсілін пайдаланудың перспективалылығын растайды.

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