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RESEARCH OF MULTIPERIODIC SOLUTIONS OF PERTURBED LINEAR AUTONOMOUS SYSTEMS WITH DIFFERENTIATION OPERATOR ON THE VECTOR FIELD

Abstract. A linear system with a differentiation operator *D* in the directions of vector fields of the form of the Lyapunov's system with respect to space independent variables and a multiperiodic toroidal form with respect to time variables is considered. All input data of the system multiperiodic depend on time variables or don't depend on them. In this case, some input data received perturbations depending on time variables. We study the question of representing the required motion described by the system in the form of a superposition of individual periodic motions of rationally incommensurable frequencies. The initial problems and the problems of multiperiodicity of motions are studied. It is known that when determining solutions to problems, the system integrates along the characteristics outgoing from the initial points, and then, the initial data are replaced by the first integrals of characteristic systems. Thus, the required solution consists of the following components: characteristics and first integrals of the characteristic systems of operator *D*, matricant and free term of the system itself. These components, in turn, have periodic and non-periodic structural components, which are essential in revealing the multiperiodic nature of the movements described by the system under study. The representation of a solution with the selected multiperiodic components is called the multiperiodic structure of the solution. It is realized on the basis of the well-known Bohr's theorem on the connection of a periodic function of many variables and a quasiperiodic function of one variable. Thus, more specifically, the multiperiodic structures of general and multiperiodic solutions of homogeneous and inhomogeneous systems with perturbed input data are investigated. In this spirit, the zeros of the operator D and the matricant of the system are studied. The conditions for the absence and existence of multiperiodic solutions of both homogeneous and inhomogeneous systems are established.

Keywords: multiperiodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field, perturbation.

1. Introduction. The foundations of the method used in this note were laid in [1, 2], which were further developed in [3–10] and applied to the study of solutions different problems in the partial differential equations [11, 12]. These methods with simple modifications extend to the study solutions of problems of the differential and integro-differential equations of different types [1-12], in particular, problems on multi-frequency solutions of equations from control theory [13]. The methods of research for multiperiodic solutions are successfully combined by methods for studying solutions of boundary value problems for equations of mathematical physics. Elements of the methods of [1, 2] can easily be found in [14,15], where time-oscillating solutions of boundary value problems are studied by the parameterization method.

As noted above, the considered system of partial differential equations along with multidimensional time contains space independent variables, according to which differentiation is carried out to the directions of the different vector fields. The autonomous case of this system was considered in [11, 12], where differentiation with respect to time variables was carried out in the direction of the main diagonal of space, and the free term of the system was independent of time variables. In this case, these parameters of the systems received perturbations depending on time variables. In the note, the method for studying multiperiodic structures of general and multiperiodic solutions is developed, the conditions for the existence of a multiperiodic solution are established, and its integral representation is given.

We consider the system of linear equations

$$Dx = Ax + f(\tau, t, \zeta) \tag{1.1}$$

with differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle a, \frac{\partial}{\partial t} \right\rangle + \left\langle \nu I \zeta + g, \frac{\partial}{\partial \zeta} \right\rangle, \tag{1.2}$$

where $\tau \in R$, $t = (t_1, ..., t_m) \in R^m$, $\zeta = (\zeta_1, ..., \zeta_l) \in R_\delta^{2l}$, $\zeta_j = (\xi_j, \eta_j) \in R_\delta^2$, $j = \overline{1, j}$, $R_\delta^2 = \{\zeta_j \in R_\delta^2 : |\zeta_j| = \sqrt{\xi_j^2 + \eta_j^2} < \delta, j = \overline{1,l}\}$, $\delta = const > 0$ are independent variables with areas of change; $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m} \right)$ и $\frac{\partial}{\partial \zeta} = \left(\frac{\partial}{\partial \zeta_1}, ..., \frac{\partial}{\partial \zeta_l} \right)$, $\frac{\partial}{\partial \zeta_j} = \left(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \eta_j} \right)$, $j = \overline{1,l}$ are vector differentiation operators; $I = diag\,(I_2, ..., I_2)$ is a matrix with l-blocks, I_2 is symplectic unit of the second order, $\nu = (\nu_1, ..., \nu_l)$ is a constant vector, $\nu I = diag\,(\nu_1 I_2, ..., \nu_l I_2)$, $a = (a_1(\tau, t), ..., a_m(\tau, t)) = a(\tau, t)$, $g = (g_1(\tau), ..., g_l(\tau)) = g(\tau)$ are vector functions, $\langle\ ,\ \rangle$ is the sign of the scalar product of vectors; A is a constant $n \times n$-matrix, $f = f(\tau, t, \zeta)$ is n-vector-function of variables $(\tau, t, \zeta) \in R \times R^m \times R_\delta^{2l}$.

The main objective of this note is to determine the multiperiodic structures of solutions of the problems (1.1) - (1.2).

2. Multiperiodic structure of zeros of the differentiation operator D. We introduce the equation

$$Du = 0 \tag{2.1}$$

with the required scalar function $u = u(\tau, t, \zeta)$ and the initial condition

$$u\big|_{\tau = \tau^0} = v(t) \in C_t^{(e)}(R^m), \tag{2.1'}$$

where D is the differentiation operator with respect to (τ, t, ζ) of the form (1.2). The solutions of equation (2.1) are called the zeros of the operator D.

Suppose that 1) the vector function $a(\tau, t)$ has the property of smoothness with respect to $(\tau, t) \in R \times R^m$ of order $(0, e) = (0, 1, ..., 1)$:

$$a(\tau + \theta, t + q\omega) = a(\tau, t) \in C_{\tau,t}^{(0,e)}(R \times R^m),\ q \in Z^m, \tag{2.2}$$

2) positive constants $\nu_1, ..., \nu_l$ are rationally incommensurable:

$$q_i \nu_i + q_j \nu_j \neq 0,\ q_i^2 + q_j^2 \neq 0,\quad q_i, q_j \in Z,\ (i,\ j = \overline{0, l}), \tag{2.3}$$

therefore, numbers $\alpha_j = 2\pi \nu_j^{-1}$, $j = \overline{1, l}$ are also incommensurable,

3) vector-functions $g_j(\tau) = (\varphi_j(\tau), \psi_j(\tau))$, $j = \overline{1, l}$ are continuous and β_j - periodic:

$$g_j(\tau + \beta_j) = g_j(\tau) \in C_\tau^{(0)}(R),\ j = \overline{1, l}, \tag{2.4}$$

where α_k, $k = \overline{1, l}$ and β_j, $j = \overline{1, l}$ are incommensurable positive constants.

It follows from condition (2.2) that the vector field $\dot{t} = a(\tau, t)$ determines the characteristic $t = \lambda(\tau, \tau^0, t^0)$, emanating from any initial point $(\tau^0, t^0) \in R \times R^m$, and moreover, it has the properties that are known from [2].

Lemma 2.1. *Let condition (2.2) be satisfied. Then under the condition* $v(t+q\omega)=v(t)\in C_t^{(e)}(R^m),\ q\in Z^m$ *the zeros* $u(\tau^0,\tau,t)=v(\lambda(\tau^0,\tau,t))$ *of the operator* D *with the initial data (2.1') have the multiperiodicity property of the form*

$$u(\tau^0+\theta,\tau+\theta,t+q\omega)=u(\tau^0,\tau,t),\ q\in Z^m.$$

The vector fields

$$\dot{\zeta}=\nu I\zeta+g(\tau) \tag{2.5}$$

determines the characteristic

$$\zeta=Z(\tau-\tau^0)[\zeta^0-z(\tau^0)]+z(\tau), \tag{2.6}$$

where $Z(\tau)=diag\ [Z_1(\tau),...,Z_l(\tau)],\ z(\tau)=(z_1(\tau),...,z_l(\tau)),\ \zeta^0=(\zeta_1^0,...,\zeta_l^0)$. Then we have the first integral of equation (2.5)

$$\zeta^0=Z(\tau^0-\tau)[\zeta-z(\tau)]+z(\tau^0)\equiv\mu(\tau^0,\tau,\zeta) \tag{2.7}$$

By virtue to the connection between $\sigma_j=\sigma_j(\tau)$ and $h_j=h_j(s_j,\sigma_j)$ of the form $\sigma_j(\tau)=h_j(\tau,\tau),\ \frac{d\sigma_j}{d\tau}=\frac{dh_j(\tau,\tau)}{d\tau}=\frac{\partial h_j(s_j,\sigma_j)}{\partial s_j}+\frac{\partial h_j(s_j,\sigma_j)}{\partial\sigma_j}$ with $\sigma_j=s_j=\tau$ leads to a transition from the differentiation operator D to the differentiation operator

$$\overline{D}=\frac{\partial}{\partial\tau}+\left\langle a(\tau,t),\frac{\partial}{\partial t}\right\rangle+\left\langle e,\frac{\partial}{\partial s}\right\rangle+\left\langle e,\frac{\partial}{\partial\sigma}\right\rangle+\left\langle \nu I h+g(\sigma),\frac{\partial}{\partial h}\right\rangle+\left\langle\frac{\partial h}{\partial s}+\frac{\partial h}{\partial\sigma},\frac{\partial}{\partial h}\right\rangle, \tag{2.8}$$

where $s=(s_1,...,s_l),\ \sigma=(\sigma_1,...,\sigma_l),\ g(\sigma)=(g_1(\sigma_1),...,g_l(\sigma_l)),\ e=(1,...,1)$ – l-vector, $h=(h_1,...,h_l),\ h_j=h_j(s_j,\sigma_j),\ j=\overline{1,l},\ \frac{\partial h}{\partial s}=\left(\frac{\partial h_1}{\partial s_1},...,\frac{\partial h_l}{\partial s_l}\right),\ \frac{\partial h}{\partial\sigma}=\left(\frac{\partial h_1}{\partial\sigma_1},...,\frac{\partial h_l}{\partial\sigma_l}\right).$

Lemma 2.2. *Let conditions (2.3) and (2.4) be satisfied. Then the zeros* $u(\tau^0,\tau,\zeta)=w(\mu(\tau^0,\tau,\zeta))$ *of the operator* D *with the initial condition* $u|_{\tau=\tau^0}=w(\zeta)\in C_\zeta^{(e)}(R^l)$ *have a multiperiodic structure of the form* $\overline{u}(s^0,s,\sigma,\zeta)=w(h(s^0-s,z(s^0),\zeta-z(\sigma)))$ *with the vector function* $h(s-s^0,z(\sigma),\ \zeta^0-z^0)=Z(s-s^0)[\zeta^0-z(s^0)]+z(\sigma)$*, at that*

$$\begin{aligned}&\overline{u}(s^0,s,\sigma,\zeta)\Big|_{\sigma=s=\tau\tilde{e},\ s^0=\tau^0\tilde{e}}=u(\tau^0,\tau,\zeta),\\&h(\tilde{e}\tau^0-\tilde{e}\tau,z(\tilde{e}\tau^0),\zeta-z(\tilde{e}\tau))=\mu(\tau^0,\tau,\zeta).\end{aligned} \tag{2.9}$$

The following theorem is proved on the bases of these Lemmas 2.1 and 2.2.

Theorem 2.1. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $u(\tau^0,\tau,t,\zeta)$ *of equation (2.1) with the initial condition* $u|_{\tau=\tau^0}=u^0(t,\zeta)\in C_{t,\zeta}^{(e,\tilde{e})}(R^m\times R^l)$ *is determined by the relation* $u(\tau^0,\tau,t,\zeta)=u^0(\lambda(\tau^0,\tau,t),\mu(\tau^0,\tau,\zeta))$*, which under the conditions* $\lambda(\tau^0,\tau+\theta,t)=\lambda(\tau^0,\tau,t)$ *and* $u^0(t+q\omega,\zeta)=u^0(t,\zeta),\ q\in Z^m$ *has a multiperiodic structure with respect to* (τ,t,s,σ) *with period* $(\theta,\omega,\alpha,\beta)$ *of the form*

$$\overline{u}(\tau^0,\tau,t;s^0,s,\sigma,\zeta)=u^0(\lambda(\tau^0,\tau,t),h(s^0-s,z(s^0),\zeta-z(\sigma))),$$

where the vector-function $h(s,z,\zeta)$ *has the form* $h(s-s^0, z(\sigma), \zeta^0 - z^0) = Z(s-s^0)[\zeta^0 - z(s^0)] + z(\sigma)$, $\hat{e} = (1,\ldots,1)$ *is* m*-vector,* $\tilde{e} = (1,\ldots,1)$ *is* l*-vector, moreover* $\bar{u}\big|_{\sigma=s=\tilde{e}\tau,\ s^0=\tilde{e}\tau^0} = u(\tau^0,\tau,t,\zeta)$.

3. The multiperiodic structure of the solution of a homogeneous linear D-system with constant coefficients. We consider a homogeneous linear system

$$Dx = Ax \tag{3.1}$$

with a differentiation operator D of the form (1.2) and a constant $n\times n$-matrix A.

We will put the problem of determining the multiperiodic structure of the solution x of the system (3.1) with the initial condition

$$x\big|_{\tau=\tau^0} = u(t,\zeta) \in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m \times R^l\right). \tag{3.1°}$$

To this end, we begin the solution of the problem by studying the multiperiodic structure of the matricant

$$X(\tau) = \exp[A\tau] \tag{3.2}$$

of the system (3.1). We need the following lemmas, to do this, which are given without proof.

Lemma 3.1. *If* $f_j(\tau+\theta_j) = f_j(\tau),\ j=\overline{1,r}$ *is some collection of the periodic functions with rationally commensurate periods:* $\theta_j\theta_k^{-1} = r_{jk}$ *is a rational number for* $j,k=\overline{1,r}$, *then for these functions exist a common period* θ*:* $f_j(\tau+\theta) = f_j(\tau),\ j=\overline{1,r}.$

Lemma 3.2. *If the real parts of all eigenvalues equal to zero and all the elementary divisors are simple of the constant matricant* $Y(\tau) = \exp[I\tau]$, *then all the elements of the matrix* I *are periodic functions*.

We consider the multiperiodic matrix $T(\hat{\tau}) = T(\tau_1,\ldots,\tau_\rho)$ with period $\gamma = (\gamma_1,\ldots,\gamma_\rho)$, where $\gamma_1,\ldots,\gamma_\rho$ are rationally incommensurable constants. Since $(\partial/\partial\tau_k)Y_{j_k}(\tau_k) = J_j Y_{j_k}(\tau_k)$, the matrix $T(\hat{\tau})$ satisfies the equation

$$\hat{D}T(\hat{\tau}) = I\,T(\hat{\tau}), \tag{3.3}$$

where the operator $\hat{D}$ is determined by

$$\hat{D} = \left\langle \hat{e}, \frac{\partial}{\partial\hat{\tau}} \right\rangle = \frac{\partial}{\partial\tau_1} + \ldots + \frac{\partial}{\partial\tau_\rho}, \tag{3.4}$$

$\hat{e} = (1,\ldots,1)$ is a ρ-vector. Obviously, under $\hat{\tau} = \hat{e}\tau$ we have $T(\hat{e}\tau) = Y(\tau)$ and

$$\dot{Y}(\tau) = \dot{T}(\hat{e}\tau) = I\,T(\hat{e}\tau) = I\,Y(\tau). \tag{3.5}$$

Thus, the multiperiodic matrix $T(\hat{\tau})$ defines the multiperiodic structure of the matricant $Y(\tau)$

$$Y(\tau) = T(\tau_1,\ldots,\tau_\rho)\big|_{\tau_1=\ldots=\tau_\rho=\tau}. \tag{3.6}$$

Lemma 3.3. *The matricant* $Y(\tau)$ *of the system (3.5) under the conditions of Lemma 3.2 has a multi-periodic structure in the form of a matrix* $T(\hat{\tau}) = T(\tau_1,\ldots,\tau_\rho)$ *which satisfies the system (3.3) with the differentiation operator (3.4) and along the characteristics* $\hat{\tau} = \hat{e}\tau$ *of the operator* $\hat{D}$ *turns into* $Y(\tau)$, *in other words, these matrices are related by the relation (3.6)*

Indeed, we making the replacement $X = Y(\tau)Z$ in the equation

$$\dot{X} = AX \tag{3.7}$$

obtain the equation $\dot{Z} = Y^{-1}(\tau)[AY(\tau) - \dot{Y}(\tau)]Z$. Therefore, according to Lemma 3.3, the multiperiodic structure of the matricant (3.2), by virtue of equality $X(\tau) = Y(\tau)\cdot Z(\tau)$, is determined by a matrix $\hat{X}(\tau,\hat{\tau})$ of the form

$$\hat{X}(\tau,\hat{\tau}) = X(\tau,\tau_1,...,\tau_\rho) = T(\tau_1,...,\tau_\rho)e^{R\tau}, \tag{3.8}$$

which is connected by the matricant $X(\tau)$, by relation

$$\hat{X}(\tau,\hat{\tau})\Big|_{\hat{\tau}=\hat{e}\tau} = X(\tau). \tag{3.9}$$

Theorem 3.1. *In the presence of complex eigenvalues of the matrix* A, *the matricant (3.2) of the system (3.7) has a multiperiodic structure defined by the matrix (3.8) and relations (3.3) - (3.6), and it along the characteristics* $\hat{\tau} = \hat{e}\tau$ *of the operator* $\hat{D}$ *satisfies condition (3.9). The matrix* $T(\hat{\tau})$ *turns into a constant matrix in the absence of complex eigenvalues.*

Now the solution of the objectives set can be formulated as Theorem 3.2.

Theorem 3.2. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $x(\tau^0,\tau,t,\zeta)$ *of the problem (3.1) - (3.1°) defined by relation*

$$x(\tau^0,\tau,t,\zeta) = X(\tau)u(\lambda(\tau^0,\tau,t),\mu(\tau^0,\tau,\zeta)) \tag{3.10}$$

has a multi-periodic structure in the form of a vector-function

$$\hat{x}(\tau^0,\tau,\hat{\tau},t,\ s^0 s,\sigma,\zeta) = \hat{X}(\tau,\hat{\tau})\bar{u}(\lambda(\tau^0,\tau,t),h(s^0 - s,z(s^0),\zeta - z(\sigma))), \tag{3.11}$$

that satisfies equation

$$\overline{\overline{D}} = A\hat{x} \tag{3.12}$$

with the differentiation operator

$$\overline{\overline{D}} = \overline{D} + \hat{D}, \tag{3.13}$$

defined by relations (2.8) and (3.4).

Proof. The representation (3.10) is known from [2], and (3.11) follows from the proved Theorems 2.1 and 3.1. The identity (3.12) can be verified by a simple check.

Theorem 3.3. *Under the conditions of the Theorem 3.2, the system (3.1) allowed nonzero multiperiodic solutions enough for the matrix* A *to have at least one eigenvalue* $\lambda = \lambda(A)$ *with the real part* $\operatorname{Re}\lambda(A) = 0$ *equal to zero.*

The theorem could be proved on the basis of a similar theorem from the theory of the systems of ordinary differential equations.

We have the following theorem from the theorem 3.3, as a corollary.

Theorem 3.4. *Under the conditions of the Theorem 3.3, the system (3.1) did not admit the multiperiodic solution other than trivial, it is sufficient that all eigenvalues of the matrix* A *have nonzero real parts.*

The general solution x of the system (3.1) can be represented in the form

$$x(\tau,t,\zeta) = X(\tau)u(\tau,t,\zeta), \tag{3.14}$$

where $u = u(\tau,t,\zeta)$ is the zero of the operator D with the general initial condition for $\tau = 0$: $x(0,t,\zeta) = u(0,t,\zeta) = u_0(t,\zeta)$, $X(\tau) = \exp[A\tau]$ is the matricant of the system.

Theorem 3.5. . *Under the conditions (2.2) - (2.4), the system (3.1) had* (θ,ω) *-periodic with respect to* (τ,t) *solutions of the form (3.14) corresponding to the multiperiodic zero of the operator* D *with the same periods, it is necessary and sufficient that the monodromy matrix* $X(\theta)$ *satisfies condition*

$$\det[X(\theta) - E] = 0. \tag{3.15}$$

Proof. Under the conditions of the theorem, its justice is equivalent to the solvability of equation $X(\tau+\theta)u = X(\tau)u$ in the space of (θ,ω)-periodic with respect to (τ,t) zeros $u = u(\tau,t,\zeta)$ of the operator D.

We arrive at the solvability of the system of equations $[X(\theta)-E]u = 0$, which is equivalent to the condition (3.15) taking into account the properties of the matricant $X(\tau+\theta) = X(\tau)X(\theta)$ from the system $X(\tau+\theta)u = X(\tau)u$.

In conclusion, we note that the fulfillment of condition

$$\det[X(\theta)-E] \neq 0 \tag{3.16}$$

guarantees the absence of such solutions.

Theorem 3.6. *Let conditions (2.2) - (2.4) and (3.16) be satisfied. Then the system (3.1) allowed nonzero (θ,ω)-periodic solutions of the form (3.14) necessary and sufficient for the functional-difference equations*

$$u(\tau+\theta, t+q\omega, \zeta) = [X(\theta)-E]^{-1} X(\theta)[u(\tau+\theta, t+q\omega, \zeta) - u(\tau,t,\zeta)],\ q \in Z^m \tag{3.17}$$

to be solvable in the space of zeros of the operator D.

Proof. Under the condition (3.16) from the definition of (θ,ω)-periodicity with respect to (τ,t) of solution (2.7), we have the equation (3.17). We must be to take into account that $u(\tau,t,\zeta)$ is the zero of the operator D to complete the proof. If the equation (3.17) has only zero solutions, then, under the condition (3.16), the system (3.1) does not have a nontrivial multiperiodic solution.

4. The multiperiodic structure of an inhomogeneous linear system with operator *D*. Consider the inhomogeneous linear equation (1.1) corresponding to the homogeneous equation (3.1), where the n-vector function $f(\tau,t,\zeta)$ satisfies condition

$$f(\tau+\theta, t+q\omega, \zeta) = f(\tau,t,\zeta) \in C^{(0,\hat{e},\tilde{e})}_{\tau,t,\zeta}(R\times R^m \times R^l). \tag{4.1}$$

Assume that the condition (3.16) is fulfilled and we search for the (θ,ω)-periodic with respect to (τ,t) solution $x(\tau,t,\zeta)$ of the system (1.1) that corresponds to zero $u(\tau,t,\zeta)$ of the operator D possessing the property of multiperiodicity with the same periods (θ,ω) for (τ,t).

Therefore, we have the solution

$$x(\tau,t,\zeta) = X(\tau)u(\tau,t,\zeta) + X(\tau)\int_0^\tau X^{-1}(s) f(s, \lambda(s,\tau,t), \mu(s,\tau,\zeta))ds \tag{4.2}$$

with zero $u(\tau+\theta, t+q\omega, \zeta) = u(\tau,t,\zeta),\ q\in Z^m$ of the operator D having the property $x(\tau+\theta, t+q\omega, \zeta) = x(\tau,t,\zeta),\ q \in Z^m$.

By accepting the notation based on (4.2)

$$f_\theta(s, \lambda(s,\tau,t), \mu(s,\tau,\zeta)) = \begin{cases} f(s, \lambda(s,\tau,t), \mu(s,\tau,\zeta)),\ \tau \xrightarrow{s} 0, \\ f(s, \lambda(s,\tau+\theta,t), \mu(s,\tau+\theta,\zeta)),\ 0 \xrightarrow{s} \tau+\theta, \end{cases}$$

where $\gamma \xrightarrow{s} \delta$ means changes in the variable s from γ to δ, the multiperiodic solutions can be presented in compact form

$$x(\tau,t,\zeta) = [X^{-1}(\tau+\theta) - X^{-1}(\tau)]^{-1} \int_\tau^{\tau+\theta} X^{-1}(s) f_\theta(s, \lambda(s,\tau,t), \mu(s,\tau,\zeta))ds. \tag{4.3}$$

Obviously, if the system (3.1) does not have multiperiodic solutions, except for zero, then the solution (4.3) of the system (1.1) is a unique multiperiodic solution.

Further, we have solutions

$$\hat{x}(s,\sigma,\hat{\tau},\tau,t,\zeta)=\left[\hat{X}^{-1}(\tau+\theta,\hat{\tau}+\hat{e}\theta)-\hat{X}^{-1}(\tau,\hat{\tau})\right]^{-1}\int_{\tau}^{\tau+\theta}X^{-1}(\varepsilon)f_{\theta}(\varepsilon,\lambda(\varepsilon,\tau,t),h(\varepsilon-s,z(\varepsilon),\zeta-z(\sigma)))d\varepsilon \qquad (4.4)$$

of the equation

$$\overline{\overline{D}}\hat{x}=A\hat{x}+f(\tau,t,\zeta) \qquad (4.5)$$

with the differentiation operator (3.13) from representation (4.3) on the basis of multiperiodic structures (2.9) and (3.8) of the quantity $\mu(s,\tau,\zeta)$ and $X(\tau)$.

Теорема 4.1. *Assume that conditions (2.2) - (2.4), (3.16) and (4.1) are satisfied, and the homogeneous system (3.1) does not have multiperiodic solutions except zero. Then the system (1.1) has a unique* (θ,ω)*-periodic solution (4.3) for which the* $(\alpha,\beta,\gamma,\theta,\omega)$*-periodic with respect to* $(s,\sigma,\hat{\tau},\tau,t)$ *structure (4.4) satisfies equation (4.5) with the differentiation operator (3.13).*

In conclusion, note that we can derive the multiperiodic structure of the general solution (4.2) of the system (1.1) similarly to formula (4.4).

Conclusion. A method for studying the multiperiodic structure of oscillatory solutions of perturbed linear autonomous systems of the form (1.1) - (1.2) was developed. The main essence of the method for studying the multiperiodic structures of solution of the system under consideration is a combination of the known methods [1-3] with the methods used in [11, 12] for the autonomous systems. In conclusion, the sufficient conditions for the existence of the multiperiodic solutions of linear systems (1.1) - (1.2) with the differentiation operator *D* in the directions of a toroidal vector field with respect to time variables and of the form of Lyapunov's systems with respect to space variables were established. Moreover, relation (4.3) is an integral representation of the multiperiodic solution of the system, and (4.4) determines its multiperiodic structure.

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ВЕКТОРЛЫҚ ӨРІС БОЙЫНША ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛЫ ҚОЗДЫРЫЛҒАН СЫЗЫҚТЫ АВТОНОМДЫҚ ЖҮЙЕЛЕРДІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІН ЗЕРТТЕУ

Аннотация. Тәуелсіз кеңістік айнымалысына қатысты Ляпунов жүйесі түріндегі және уақыт айнымалысына қатысты, көппериодты тороидалды түрдегі векторлық өрістер бағыты бойынша *D* дифференциалдау операторлы сызықты жүйе қарастырылады. Жүйені анықтайтын барлық берілген өлшемдер уақыт айнымалысынан көппериодты тәуелді, не олардан тәуелсіз болады. Бұл жағдайда жүйені анықтайтын кейбір берілгендерге уақыт айнымалысынан тәуелді қоздыртқы берілген. Рационалды өлшенбейтін жиіліктердің жекеленген периодты қозғалыстарының суперпозициясы түріндегі жүйе арқылы сипатталған ізделінді қозғалыс туралы сұрақ зерттеледі. Бастапқы есептер және қозғалыстардың көппериодтылығы туралы есептер зерттеледі. Есептің шешімін анықтау кезінде жүйенің бастапқы нүктеден шығатын характеристика маңайында интегралданатыны, одан кейін бастапқы берілгендер характеристикалық жүйенің бірінші интегралдарымен ауыстырылатыны белгілі. Сонымен, ізделінді шешім келесі компоненттерден тұрады: *D* операторының характеристикалық жүйесінің характеристикасы мен бірінші интегралдары, жүйенің бос мүшесі мен матрицанты. Бұл компоненттердің зерттелуші жүйемен сипатталған қозғалыстың көппериодтылық табиғатын ашу кезінде маңызды мағынасы бар болатын периодты және периодты емес құрылымдық құраушылары болады. Шешімді ерекшеленген көппериодты құраушылар арқылы сипаттауды шешімнің көппериодтылық құрылымы деп атайды. Ол көп айнымалы периодты функциялар мен бір айнымалы квазипериодты функцияларының байланысы туралы Бордың танымал теоремасы негізінде жүзеге асады. Сонымен, жүйелерді анықтайтын берілгендері қоздырылған жағдайды біртекті және біртексіз жүйелердің жалпы және көппериодты шешімдерінің көппериодты құрылымы нақты зерттелген. Осылайша *D* операторының нөлдері мен жүйенің матрицанты зерттелген. Біртекті және біртексіз жүйелердің көппериодты шешімдерінің бар болу және болмау шарттары тағайындалған.

Түйін сөздер: көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі, қоздыртқы.

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ИССЛЕДОВАНИЕ МНОГОПЕРИОДИЧЕСКИХ РЕШЕНИЙ ВОЗМУЩЕННЫХ ЛИНЕЙНЫХ АВТОНОМНЫХ СИСТЕМ С ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ ПО ВЕКТОРНОМУ ПОЛЮ

Аннотация. Рассматривается линейная система с оператором дифференцирования D по направлениям векторных полей вида системы Ляпунова относительно пространственных независимых переменных и многопериодического тороидального вида относительно временных переменных. Все входные данные системы либо многопериодично зависят от временных переменных, либо от них не зависят. В данном случае некоторые входные данные получили возмущения, зависящие от временных переменных. Исследуется вопрос о представлении искомого движения, описанного системой в виде суперпозиции отдельных периодических движений рационально несоизмеримых частот. Изучаются начальные задачи и задачи о многопериодичности движений. Известно, что при определении решений задач система интегрируется вдоль характеристик, исходящих из начальных точек, а затем начальные данные заменяются первыми интегралами характеристических систем. Таким образом, искомое решение состоит из следующих компонентов: характеристик и первых интегралов характеристических систем оператора D, матрицанта и свободного члена самой системы. Эти компоненты, в свою очередь, имеют периодические и непериодические структурные составляющие, которые имеют существенное значение при раскрытии многопериодической природы движений, описанных исследуемой системой. Представление решения с выделенными многопериодическими составляющими названо многопериодической структурой решения. Оно реализуется на основе известной теоремы Бора о связи периодической функции от многих переменных и квазипериодической функции одной переменной. Таким образом, более конкретно исследуются многопериодические структуры общих и многопериодических решений однородных и неоднородных систем с возмущенными входными данными. В таком духе изучаются нули оператора D и матрицант системы. Устанавливаются условия отсутствия и существования многопериодических решений как однородных, так и неоднородных систем.

Ключевые слова: многопериодическое решение, автономная система, оператор дифференцирования, Ляпунова векторное поле, возмущение.

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THE TECHNOLOGICAL CONSTRUCTS OF MATHEMATICAL TRAINING FOUNDING IN HIGHER EDUCATION

Abstract. The article presents the development of innovative technological constructs of professional training of future specialists in higher education on the basis of the deployment of spirals and clusters of founding of mathematical knowledge in the course of synergetic effects actualization in teaching mathematics. The founding procedures are based on the study of complex knowledge, fractal interactions and integrative relationships in modern mathematics and their transfer into the technological field of training. The components and content of teaching mathematics technologies and innovative activity of students to master the complex content of mathematics as a pedagogical task in various conditions of information technologies support are presented. Effective development of professional competences, communicative and personal qualities of students, increase of motivation of future specialists are predicted.

Keywords: founding principle, technological constructs, professional training, visual modeling, fractal geometry.

The basis of an innovative approach to the selection of the content of mathematical education of the future specialist in higher education in the conditions of modernization is the mastery of complex knowledge with the manifestation of synergetic effects and cognitive style of future professional activity on the basis of the processes of personal experience founding and visual modeling of mathematical objects and processes [20]. Pedagogical technology of mathematical education synergy is the essence of the joint activity of the teacher and the student, leading to self-organization and achievement of the planned and probabilistically guaranteed results of the development of complex mathematical knowledge. However, in real life, university developments are usually reduced to the traditional content of vocational training, a proven set of subjects and their volume in the structure of curricula (unfortunately, this trend applies not only to higher but also to secondary school). Technological constructs of founding concept developed by V. D. Shadrikov and E. I. Smirnov, presented in this article, provide real opportunities for a particular university to build innovative educational constructs, a new structure and content of vocational training , based on already tested theories and technologies of vocational education [12]. A unique experience of the Yaroslavl State Pedagogical University named after K. D. Ushinsky on implementation of innovative content of professional training of the future teacher of mathematics within the experimental educational standard in the specialty "mathematics" (order No. 2046 of 14.05.2001, MO of the Russian Federation). in the period from 2001 to 2006, and then in a pilot experiment on the manifestation of synergetic effects [2] showed the high efficiency of the developed technology and the possibility of its transfer to new conditions of vocational education [2]. Control sections on mathematical preparedness of control and experimental groups of students showed significant positive changes of students of the experimental group studying on the experimental program on the basis of the founding conception. As a result, a third of the students of the experimental group graduated from the university with honors (which exceeds the average indicators of graduates of the faculty of physics and mathematics over the past decade in YSPU by 20%). Moreover, 25% of graduates continued professional pedagogical education (postgraduate studies, job seekers, real work in universities, management in the education system) with elements of research activities. Significant effects in the development of motivation, creativity and communication are revealed [3].

Purpose of research

As the analysis of the state higher pedagogical education conducted above shows, the lack or complete absence of a sound methodological basis in determining the vocational education content leads to the fact that the new content is not much different from the previous special training. At the same time, the reduced volume of subject training, weak professionalization, practical destruction of professional selection system will lead to deepening of crises and contradictions in professional training. Therefore, the purpose of this study involves the identification of technological constructs for the effective development of complex mathematical knowledge on the basis of founding conception and actualization of modern achievements in science , which determine the trends of implementation. The basic educational program of the university should be formalized and materialized in the form of specific academic disciplines, tools and forms of educational activity not only justified methodologically didactic (cognitive) processes that form the goal-setting, acquisition, application and transformation of personal experience. It should be improved the adaptation processes of characterizing technological professional tests of acceptance by the student the profession and the personal processes directed on manifestation of self-organization, features and development of motivations and emotions, reflection and self-regulation, self-assessment and a choice, intelligence and creativity of the personality in the course of development of difficult knowledge. The student should already get acquainted with the nonlinear style of thinking in post-non-classical sciences , know and find associations in real life of such phenomena of collective ordering as the effect of Zhabotinsky-Belousov, Binar Cells ("the road of giants" in Ireland), the Ginzburg-Landau theory of superconductivity in the quantum system, the Tray - Volterra equations in the predator-prey system, the Koch snowflake and the Schwartz cylinder, the Ferhulst scenario and the "butterfly effect" of the strange attractor Lorenz, etc. The leading idea is that a key aspect of synergy manifestation effects in mathematics teaching of complex knowledge-based adaptation of modern achievements in science. It state the possibility of the phases actualization and study the characteristics of essence development of sophisticated mathematical knowledge, phenomena and procedures, creating the conditions for communication and dialogue of cultures, identify attributes of self-organization of the content, processes and interactions (attractors, bifurcation points, basins of attraction, iterative procedures) during the development of "problem areas" of mathematics. Thus, the present study is an attempt to develop a technology of modern achievements adaptation in science to teaching of mathematics based on computer modeling and design, visual and mathematical modeling of complex knowledge in the "problem areas" of mathematical education with the manifestation of synergetic effects and the identification of new research by-products based on self-organization of cognitive activity.

Methodology, methods and technological constructs

The implementation of the announced technology is associated with the development of students ' complex knowledge by means of mathematical and computer modeling in a rich information and educational environment. An effective tool for the development of complex mathematical knowledge and the development of intellectual operations of students thinking can be a study and adaptation to school or university mathematics of modern achievements in science, clearly and significantly presented in applications to real life, the development of other sciences, high technology and industry. Mathematics education as a complex and open social system carries with it a huge potential of self-organization and positive manifestation of synergetic effects in different directions: the development and education of the individual in the project activities, ordering the content and structure of cognitive experiences of communication and social interaction of the subjects on the basis of cultures dialogue. Synergy of mathematical education will be considered by us as a symbiosis and qualitative change of nonlinear effects of self-organization and self-development of the individual during the development of mathematical activity in the control of complex stochastic processes based on the coordination of different factors and began in three contexts: content (semiotic), procedural (simulation) and social adaptation. Synergy of mathematical education is characterized by the presence of internal attributes (mechanisms) of self-organization and order parameters, which form the success of the educational system at increasingly complex levels. At the same time, didactic processes acquire a new quality: natural science knowledge is enriched with a humanitarian aspect, humanitarian knowledge acquires a scientific basis for substantiating the essence using natural science and mathematical apparatus and methods.

An important context is the external factors formulation of influence in the form of a plurality of goal-setting, building stages and hierarchies of symbolic and figurative-geometric activities in the direction of essence founding of mathematical objects and procedures [3], search and analysis of side solutions using information technology , identify bifurcation transitions and basins of attraction in the processes under study on the basis of variability and parameterization, ensuring the coherence of information flows in the emergence of a new product based on the cultures dialogue (including, in the conditions of network interaction). In [13] identified and characterized in all stages of mathematical education synergy manifestation: preparatory, informative, technological, assessment, corrective and generate transforming. In recent decades, the post-non-classical scientific picture of the world as a paradigm effect has become the most important concept, which is based on the priority of the processes of self-organization of dynamic nonlinear systems ((G. Haken [9]), T. Kuhn, E. N. Knyazeva, B. Mandelbrot [5], I. R. Prigozhin [8], S. P. Kurdyumov, G. G. Malinetsky [7], K. Mainzer, etc. That is, there were opportunities and the principles of technique of self-organization and dialogue of cultures of the students through the development of synergetic paradigm of development of mathematics in the context of a reasoned, coherent and level of opening-up and creative to overcome the "problem areas" of mathematical activities.

Scientists, philosophers, educators and psychologists (I. Kant, G. Hegel, I. Prigogine, S. P. Kurdyumov, G. Haken, K. Mainzer, V. V. Orlov, A. N. Polyakov [8], V. S. Stepin, I. S. Utrobin, H..The alfvén, T. S., Vasilyev, etc.) convincingly demonstrated that the effective development of the individual occurs during the development of complex knowledge (different levels of difficulty depending on student's personal development, including inclusive education), creating situations of overcoming difficulties in the process of mastering knowledge and a unified picture of the world on the basis of a high degree of deployment of students educational and professional motivation in a single network of interactions, independence and coherence. In cognition of the complex, the process of cognition itself "becomes a communication, a loop between cognition (phenomenon, object) and cognition of this cognition" (E. Moren).

We highlight the following system - genetic contexts of mathematical education synergy at the University (also A. A. Verbitsky [1]).

1. Procedural contexts. The basic concept of presented concept of modern achievements adaptation in science is the principle and technology of founding of personal experience (E. Husserl, S. L. Rubinstein, V. D. Shadrikov, E. I. Smirnov [10],[14], etc.). Therefore, the concept of founding of the personality formation process acts as an effective mechanism for overcoming professional crises of becoming a specialist and actualization of integrative links between science, vocational education and school. Adaptation processes are considered by scientists psychologists and teachers as a dynamic complex of integral interaction of internal results (system of knowledge, skills, attitudes, competencies, values) and adequate mechanisms of adaptation of the individual to changes in the environment and the activities results with developing effect (A. A. Rean [11], Yu. I. Tolstykh [19], S. I. Soroko [18], etc.). In accordance with S. N. Dvoryadkina and E. I. Smirnov [4] such can be as the synergetic effects of adaptation processes realization: cognitive, motivational, professional, creative, socio - economic and spiritual-moral. The processes of creating a motivational field for the study of complex mathematical constructs require the computer design and visual modeling of modern achievements in science (strange Lorentz attractor, fuzzy sets and fuzzy-logic, Menger's sponge, Ferhulst's script, etc.). Building hierarchies in the deployment of the essence of the generalized construct of "problem zone" in mathematics education on the basis of parameterization and abstraction, search for bifurcation points and basins of attraction by means of construction of iterative processes on the basis of information technology support and create the mechanisms of complex knowledge adaptation to school and university mathematics.

2. Meaningful context of the synergy in mathematical activity is the sensitive mechanism that will allow to actualize the factors of success in solving creative problems on the basis of research activity and self-organization of students. Therefore, the primary means of manifestation of synergy, mathematics education, and the mechanism of formation of exploratory behavior of students in learning mathematics we consider the development and introduction in educational process of research practice-oriented complex problems in the "problem areas" of mathematics education in the form of a complex multi-step mathematical-informational tasks (M. Klakla, V. S. Sekovanov, E. I. Smirnov, etc.). Research activity of the students is realized in a specially organized environment (for example, resource classes [4]) against the background of growth of motives of self-actualization and self-organization, identification of priority of value orientations in mathematical activity. An important factor in the context of the meaningful manifestation of the synergy of mathematical education is the productive work on the study of new mathematical properties and characteristics of generalized constructs of self-organization: fractal objects, mathematical models of instability of solutions of nonlinear dynamic systems, means of coding and encryption, cellular automata, fuzzy sets and fuzzy logic, computer simulation of multi-faceted surfaces of the cylinder Schwartz, stochastic structures on strange attractors, etc. (V. S. Sekovanov, E. I. Smirnov, S. N. Dvoryadkina [4] E. I. Smirnov, A. D. Uvarov [15].

3. Personality - adaptation and social context of the synergy of mathematical education. Human interaction with the world and people activates with its internal potentials, which is the basis of his self-knowledge, self-regulation and self-actualization, thus ensuring his personal self-development. In this regard, special attention is paid to the problems of group interaction organization of students, which is the most important source of their self-actualization and development, an incentive for creative activity and further personal growth. Ponderous procedure of transition from cash of an entity to a generalized potential development in the form of a perfect object (process or phenomenon, status, personal qualities) are multi-stage, multifunctional, integrative and aimed at actualization within and cross-curricular links. The personal adaptation component is associated with the expression of the characteristics and qualities of personal development and adaptation of the students in the process of mastering in modern scientific knowledge in the direction of self-actualization ("I'm interested"), self-determination ("what I can do"), self-organization ("I'm able to manage the process"), self-development ("I can do something new")[17].

Technology of synergy in the study of "problem areas" of mathematical education

The technology of identification and research "zones of modern achievements in science (problem areas)" in the relation to teaching mathematics allows you to design and implement the stages of adaptation of modern achievements in science to the current state of the experience of students mathematical activity , allows you to integrate knowledge from different fields of the science in the context of the development of complex knowledge. We highlight a number of technological stages of the founding procedures in the process of modern scientific knowledge adaptation to school mathematics with the manifestation of synergetic effects and the reflection of phenomenological type of an essence modeling of generalized construct:

1. **Development of standards and samples of phenomenology of visual modeling of generalized construct and results of diagnostic procedures** of specific manifestations of the essence of the generalized construct.
2. **Creating a motivational field in the development of generalized construct**: visual modeling (lessons, lectures, video clips, project activities, presentations, business games) motivational and applied situations of different interpretations of standards and examples of synergy.
3. **Practice-oriented and research complexes of tasks** for updating the deployment of individual educational trajectories for small groups of students (determining the composition and orientation of small groups , the distribution of roles, selection and updating of practice-oriented research activities on the stages of founding and adaptation of the generalized construct).
4. **Multiple goal-setting of the research processes** of generalized construct of "problem zone".
5. **The willingness to debate and multiplicity of solutions to the problem**; identifying selection criteria for making a diagnosis and finding solutions to the research practice-oriented tasks based on diagnostic information, systematized in the form of thunderous complexes.
6. **Creation of the creative environment** in the process of the essence mastering of generalized construct (stimulation of success situation; work in small groups and dialogue of cultures; tolerance to uncertainty and development of divergent thinking; identification and popularization of patterns of creative behavior and its results); collection and variety of forms and methods of information presentation; development of statistical packages and office editors, computer algebra systems and Web-support.

7. **Ability to adapt and develop in social communications** based on the dialogue of mathematical, information, natural science and humanitarian cultures. **An effective dialogue of mathematical, informational, natural science and humanitarian cultures** on the basis of components computer and mathematical modeling and stages of generalized construct adaptation in "zone of modern achievements in science" of university mathematics.

8**. Updating the attributes of synergy (bifurcation, attractors, fluctuations, basins of attraction)** in the research process, the generalized construct of founding; identify patterns, analogies , associations, the dynamics of the investigated processes, phenomena and facts; forecast and "by-products" of the research.

A synergistic effect of the study of polyhedral surfaces of Schwartz cylinder.

Example 1. Mathematics education as a complex and open social system carries a huge potential of self-organization and positive manifestation of synergetic effects in different directions: the development and education of the personal, the orderliness of the content and structure of cognitive experiences of communication and social interaction of the subjects on the basis of cultures dialogue. It is necessary to design techniques and methods of reflection and study of technological parameters of the generalized construct against the background of the functioning of the adaptation system and obtain new results: in our case, the generalized construct of scientific knowledge – the concept of surface area is indirectly updated through computer and mathematical modeling of the research processes of the "area" of the side surface of Schwartz cylinder [16-17].

Multiple goal-setting of the actualization processes of surface area concept by methods of investigation of the "area" of Schwartz cylinder (content aspect)): pathological properties of the "area" of lateral surface of the cylinder are well studied in the so-called "regular" case (see for example [2]). This occurs when its height H is divided into m equal parts (respectively – layers of the cylinder), and the circle lying at the base is divided into n equal parts, followed by a shift φ on each layer by $\frac{\pi}{n}$. With such a triangulation of lateral surface of the cylinder, the formula for calculating its "area", by means of the resulting polyhedral with $m, n \to \infty$ is:

$$S_q = 2\pi R\sqrt{R^2 \frac{\pi^4}{4} q^2 + H^2}, \quad \text{where:} \quad q = \lim_{m,n\to\infty} \frac{m}{n^2} \qquad (1)$$

Thus, the "area" of lateral surface Sq of regular Schwarz cylinder of height H and radius R (if this limit exists – finite or infinite) is completely determined by the limit (1). At the same time, due to the independent nature of the aspiration $m, n \to \infty$, the result of the limiting process becomes weakly predictable, multivalued, with the absence of regularities in the chaotic deployment of fractal structures of polyhedral. B. Mandelbrot [5] showed that $m = n^k$ the area of a polyhedral surface grows as n^k $(k \neq 2)$. There are the hierarchies of issues related to the study of multi-faceted surfaces of Schwartz cylinder and solved by means of computer and mathematical modeling of research activities in small groups of students in a remote environment or in the form of research of multi-stage mathematical information tasks. Such studies conducted by students in resource or laboratory-calculation classes, in the performance of multi-stage mathematical and information tasks, in the course of project activities or network interaction develop intellectual operations of thinking, increase educational motivation and the quality of mathematical actions development.

Consider a circle centered at point A and a radius $g_1 = 1$. A regular hexagon is inscribed in the circle and a radius is drawn so that it crosses the side of the hexagon at the point U. Suppose that the point T moves along the circle. In this case, we put in accordance with the central angle $c_1 = \alpha$ of the length of the segment *UT*, we get the function *f*(*α*). The introduced function is limited and periodic, namely $0 \leq |f(\alpha)| \leq 1 - \frac{\sqrt{3}}{2}$ and period $T = \frac{\pi}{6}$. The function *f*(*α*) can be defined explicitly:

$$f(\alpha) = 1 - \frac{\frac{\sqrt{3}}{2}}{\sin(120^0 - (\alpha - [\frac{\alpha}{60^0}] \cdot 60^0))} \quad (2)$$

It is easy to determine a function $f_n(\alpha)$ similar to the function in formula (2) in the case where an arbitrary regular n-gon is inscribed in the circle. Indeed, denote by $\varphi = \frac{360^0}{n}$ the central angle of the inscribed n-gon, and then $f_n(\alpha)$ take the form:

$$f_n(\alpha) = 1 - \frac{\sin(90^0 - \frac{\varphi}{2})}{\sin(90^0 + \frac{\varphi}{2} - (\alpha - [\frac{\alpha}{\varphi}] \cdot \varphi))} \quad (3)$$

We define the following function $g(\alpha)$ as a functional series:

$$g(\alpha) = \sum_{n=1}^{\infty} f_{k \cdot n}(\alpha) \quad (4)$$

where functions $f_{k \cdot n}(\alpha)$ are defined by formula (3). It is easy to see that the graph of the function $g(\alpha)$ has a fractal structure, like the graph of the van Der Warden function [21]. Now consider the layer of the cylinder Schwartz, crossed by the plane of its orthogonal axis. There is a natural problem. Let Schwartz cylinder with height $H = 1$ and radius $R = 1$ be given. In this case, its upper base is divided into n equal parts, and the height into m equal parts. Draw a section perpendicular to the axis of the cylinder through an arbitrary point x on it. If it tends to infinity, and fixed, what kind of function $g(\alpha)$ will be defined in formula (4)?

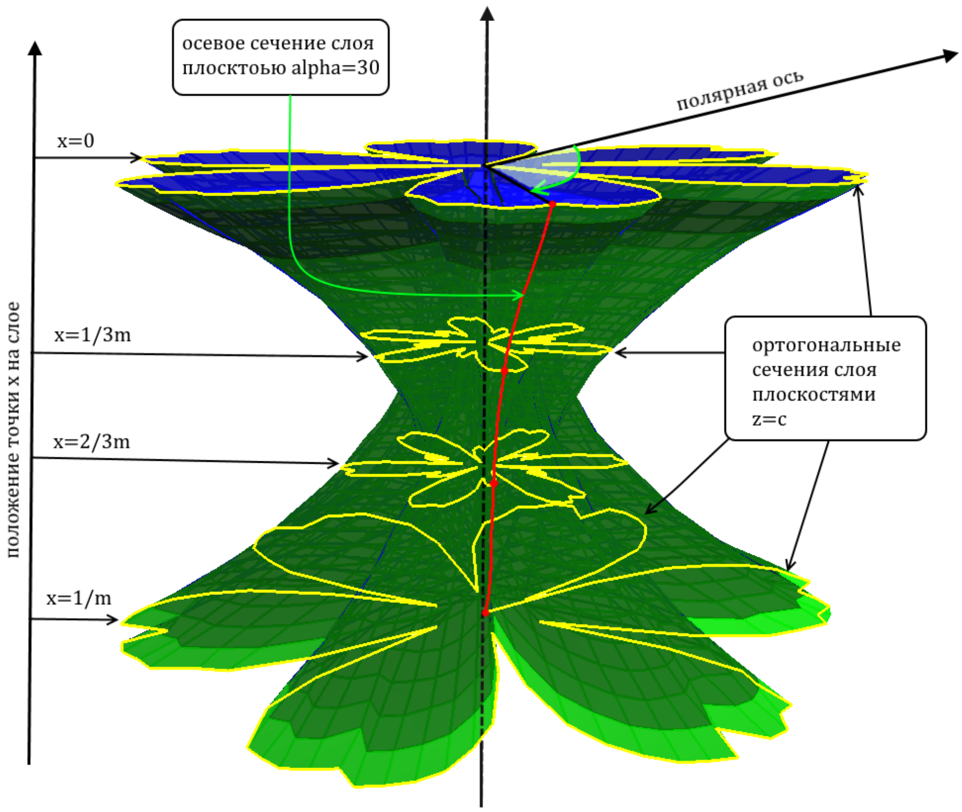

Figure 1 - A portion of the surface $z = s(\alpha, x)$

If we assume that in formula (4) variables α and ε are independent, then the series defined by this formula can be viewed as a function of two variables $s(\alpha, x)$. The graph of this function is the surface. The following figure shows a portion of the surface $z = s(\alpha, x)$, with $0^0 \le \alpha \le 360^0$ and $0 \le x \le \frac{1}{30}$. We believe that the height of one layer of Schwartz cylinder is equal to $\frac{1}{30}$ (for clarity, the surface is depicted in a cylindrical coordinate system). In the last figure, the level lines shown in yellow correspond to the graphs of the function $g(\alpha)$ in the polar coordinate system at $k = 0, k = \frac{1}{3}, k = \frac{2}{3}, k = 1$.

Similarly, other "zones of modern achievements in science" can be studied: elements of fractal geometry, cellular automata, coding and encryption of information, the theory of chaos and catastrophes. As the example shows, the longitudinal study of "zones of modern achievements in science" imposes increased requirements for their selection and number, at the same time developing the effect of the development of students ' complex knowledge in the context of modern achievements in science and dialogue of mathematical, information, natural science and humanitarian cultures is difficult to overestimate.

Results. Thus, the technological constructs of complex knowledge development on the basis of the concept of personal experience founding, as well as computer design and technology for the study of generalized constructs to identify the essence of "problem areas" of university mathematics are identified and characterized. Fractal characteristics of the surface area in detail of nonlinear dynamics of growth of polyhedral complexes areas at crushing of triangulations of lateral surface of Schwartz cylinder or "boot" by means of computer and mathematical modeling are investigated. Bifurcation points, attraction pools, computational procedures and fluctuations of state parameters, computer design and side results of the study of the "area" of lateral surface of regular and irregular Schwartz cylinder are identified and characterized. Hierarchy forms and means of students ' research activity: resource and laboratory and design classes, complex multi-step mathematical and information jobs, design methods and networking.

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ЖОҒАРЫ БІЛІМ БЕРУДЕ МАТЕМАТИКАЛЫҚ ДАЙЫНДЫҚТЫ ЖҮЗЕГЕ АСЫРУДЫҢ ТЕХНОЛОГИЯЛЫҚ КОНСТРУКТІЛЕРІ

Аннотация. Мақалада математиканы оқытудағы синергетикалық әсерлерді өзектендіру барысында математикалық білімді фундаментациялау кластерлері мен спиральдарды өрістету негізінде педагогикалық ЖОО-да болашақ мамандарды кәсіби даярлаудың инновациялық технологиялық конструкторларын әзірлеу ұсынылған. Фундаменталды процедуралар негізінде күрделі білімді, фракталды өзара іс-қимылдарды және қазіргі математикада интегративтік байланыстарды зерттеу және оларды кәсіби дайындықтың технологиялық алаңына көшіру жатыр. Ақпараттық технологияларды қолдаудың әр түрлі жағдайларында педагогикалық міндет ретінде, математиканың күрделі мазмұнын меңгеру бойынша студенттердің инновациялық қызметі және математиканы оқыту технологияларының компоненттері мен мазмұны ұсынылған. Студенттердің кәсіби құзыреттілігін, коммуникативтік және жеке қасиеттерін тиімді дамыту, болашақ мамандардың уәждемесін арттыру болжанады. Модернизациялау жағдайында болашақ маманның математикалық білім мазмұнын іріктеудің инновациялық тәсілі – синергетикалық әсерлер мен болашақ кәсіби іс-әрекеттің танымдық стилін, математикалық нысандар мен процестердің тұлғалық қалыптасуы мен визуалды модельдеу негізінде көрінетін кешенді білімді игеру. Математикалық білім берудің синергетикалық педагогикалық технологиясы – бұл өзін-өзі ұйымдастыруға және күрделі математикалық білімді игерудің жоспарланған және ықтимал кепілдендірілген нәтижелеріне қол жеткізуге әкелетін мұғалім мен оқушының бірлескен іс-әрекетінің мәні. Алайда, нақты өмірде университеттің дамуы, әдетте кәсіптік білім беру мазмұнының дәстүрлі мазмұнына, бекітілген оқу пәндерінің жиынтығына және оқу жоспарлары құрылымындағы олардың көлеміне азаяды. Өкінішке орай, бұл үрдістің көрінісі тек жоғары деңгейге ғана емес, сонымен бірге орта мектепке де қатысты. Осы мақалада келтірілген В.Д. Шадриков пен Е.И. Смирнов

әзірлеген іргетас тұжырымдамасының технологиялық конструкциялары белгілі бір университеттің бұрыннан бекітілген кәсіптік білім беру теориялары мен технологияларына негізделген кәсіби білім берудің жаңа құрылымы мен мазмұны бар инновациялық білім беру құрылыстарын салуға нақты мүмкіндіктер береді. К.Д. Ушинский атындағы Ярославль мемлекеттік педагогикалық университетінің «математика» мамандығы бойынша эксперименттік білім беру стандартының бөлігі ретінде болашақ математика мұғалімін кәсіби даярлаудың инновациялық мазмұнын жүзеге асырудағы бірегей тәжірибесі ұсынылады.

Түйін сөздер: қорландыру принципі, технологиялық конструкторлар, кәсіби дайындық, көрнекі модельдеу, фракталды геометрия.

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ТЕХНОЛОГИЧЕСКИЕ КОНСТРУКТЫ ФУНДИРОВАНИЯ МАТЕМАТИЧЕСКОЙ ПОДГОТОВКИ В ВЫСШЕМ ОБРАЗОВАНИИ

Аннотация. В статье представлена разработка инновационных технологических конструктов профессиональной подготовки будущих специалистов в педагогическом вузе на основе развертывания спиралей и кластеров фундирования математических знаний в ходе актуализации синергетических эффектов в обучении математике. В основе фундирующих процедур лежит исследование сложного знания, фрактальных взаимодействий и интегративных связей в современной математике и их переноса в технологическое поле профессиональной подготовки. Представлены компоненты и содержание технологий обучения математике и инновационной деятельности студентов по освоению сложного содержания математики как педагогической задачи в различных условиях поддержки информационных технологий. Прогнозируется эффективное развитие профессиональных компетенций, коммуникативных и личностных качеств студентов, повышение мотивации будущих специалистов. В основе инновационного подхода к отбору содержания математического образования будущего специалиста в высшей школе в условиях модернизации лежит овладение сложным знанием с проявлением синергетических эффектов и когнитивным стилем будущей профессиональной деятельности на основе процессов фундирования опыта личности и наглядного моделирования математических объектов и процессов. Педагогическая технология синергии математического образования представляет собой существо совместной деятельности преподавателя и студента, ведущее к самоорганизации и достижению планируемых и вероятностно гарантированных результатов освоения сложного математического знания. Однако в реальной жизни вузовские разработки сводятся, как правило, к традиционному наполнению содержания профессиональной подготовки, апробированному набору учебных предметов и их объёму в структуре учебных планов. К сожалению, проявление этой тенденции относится не только к высшей, но и к средней школе. Технологические конструкты концепции фундирования, разработанной В.Д. Шадриковым и Е.И. Смирновым, представленные в настоящей статье, дают реальные возможности для конкретного вуза выстраивать инновационные образовательные конструкты, новой структурой и содержанием профессиональной подготовки, опирающиеся на уже апробированные теории и технологии профессионального образования. Уникальный опыт Ярославского государственного педагогического университета им. К.Д. Ушинского по реализации инновационного содержания профессиональной подготовки будущего учителя математики в рамках экспериментального образовательного стандарта по специальности «математика».

Ключевые слова: принцип фундирования, технологические конструкты, профессиональная подготовка, наглядное моделирование, фрактальная геометрия.

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MULTIPERIODIC SOLUTIONS OF LINEAR SYSTEMS INTEGRO-DIFFERENTIAL EQUATIONS WITH D_c-OPERATOR AND ε-PERIOD OF HEREDITARY

Abstract. The article explores the questions of the initial problem and the problem of multiperiodicity solutions of linear systems integro-differential equations with an operator of the form $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ and with finite hereditary period $\varepsilon = const > 0$ by variable τ that describe hereditary phenomena. Along with the equation of zeros of the special differentiation operator D_c are considered linear systems of homogeneous and inhomogeneous integro-differential equations, sufficient conditions are established for the unique solvability of the initial problems for them, both necessary and sufficient conditions of multiperiodic existence are obtained by (τ, t) with periods (θ, ω) of the solutions. The integral representations of multiperiodic solutions of linear inhomogeneous systems with the uniqueness property are determined 1) in the particular case when the corresponding homogeneous systems have exponential dichotomy and 2) in the general case when the homogeneous systems do not have multiperiodic solutions, except for the trivial one. The article proposes a research technique for solving problems that satisfy initial conditions and have the property of multiperiodicity with a given ε heredity period for linear systems of integro-differential equations with a special partial differential operator D_c. Multiperiodic solutions obtained along characteristics $t = t^0 + c\tau - c\tau^0$ with fixed (τ^0, t^0) are used as an application in the theory of quasiperiodic solutions of systems of integro-differential equations.

Key words: integro-differential equation, hereditary, fluctuation, multiperiodic solution.

1. Problem statement.

In this paper, we've researched the problem of the existence of (θ, ω)-periodic solutions $u(\tau, t)$ by $(\tau, t) = (\tau, t_1, \ldots, t_m) \in R \times R^m$ systems of

$$D_c u(\tau, t) = A(\tau, t)u(\tau, t) + \int_{\tau-\varepsilon}^{\tau} K(\tau, t, s, t - c\tau + cs)u(s, t - c\tau + cs)ds + f(\tau, t) \qquad (1.1)$$

with a differentiation operator D_c of the form

$$D_c = \partial/\partial\tau + \langle c, \partial/\partial t\rangle, \qquad (1.2)$$

that turns into the operator of the total derivative $d/d\tau$ along the characteristics $t = c\tau - cs + \sigma$ with initial data $(s, \sigma) \in R \times R^m$, where $R = (-\infty, +\infty)$, c is constant vector, $\partial/\partial t = (\partial/\partial t_1, \ldots, \partial/\partial t_m)$ is vector, $\langle c, \partial/\partial t\rangle$ is the scalar product of vectors, $A(\tau, t)$ and $K(\tau, t, s, \sigma)$ are given $n \times n$-matrices, $f(\tau, t)$ is n-vector-function, $(\theta, \omega) = (\theta, \omega_1, \ldots, \omega_m)$ is vector-period with rationally incommensurable coordinates, ε is positive constant.

The problem of this kind involves the research problems of hereditary vibrations in mechanics and electromagnetism. For example, if the oscillation phenomenon is hereditary in nature, then the equation of motion of the string at a known moment $m(\tau)$ is set by changing the angle of string torsion $\omega(\tau)$, subordinated to the ratio $m(\tau) - \mu \frac{d^2\omega(\tau)}{d\tau^2} = h\omega(\tau) + \int_{\tau-\varepsilon}^{\tau} \varphi(\tau, s)\omega(s)ds$, where μ and h are constants and ε is the hereditary period of the vibrational phenomenon. It is also known that the hereditary biological phenomenon "predator-prey" is related by the law of oscillations described by the system of integro-differential equations.

The given integro-differential equations are the mathematical model hereditary phenomena described by system

$$\frac{dx}{dt} = P(\tau)x(\tau) + \int_{\tau-\varepsilon}^{\tau} Q(\tau, s)x(s)ds + \psi(\tau). \tag{1.3}$$

Where $x(\tau)$ is unknown n-vector-function; $P(\tau)$, $Q(\tau, s)$ are $n \times n$-matrices; $\psi(\tau)$ is n-vector-function; $\varepsilon > 0$ is a constant. Since the process is oscillatory, as a rule, $P(\tau)$ and $\psi(\tau)$ are almost periodic by τ, and the kernel $Q(\tau, s)$ has the property of diagonal periodicity by $(\tau, s) \in R \times R$. In particular, the indicated input data of system (1.3) are quasiperiodic by $\tau \in R$ with a frequency basis $\nu_0 = \theta^{-1}, \nu_1 = \omega_1^{-1}, \ldots, \nu_m = \omega_m^{-1}$, then in the theory of fluctuations, the question of the existence of quasiperiodic solutions $x(\tau)$ of system (1.3), with a changed frequency basis $\tilde{\nu}_0 = \theta^{-1}, \tilde{\nu}_1 = c_1\omega_1^{-1}, \ldots, \tilde{\nu}_m = c_m\omega_m^{-1}$, is of great importance and we set $\varepsilon < \theta = \omega_0 < \omega_1 < \ldots < \omega_m$. An important role in solving this problem is played by the well-known theorem of G. Bohr on the deep connection between quasiperiodic functions and periodic functions of many variables. According to this theorem, matrix-vector functions are defined $A = A(\tau, t)$, $K = K(\tau, t, s, \sigma)$, $\sigma = t - c\tau + cs$, $f = f(\tau, t)$, $u = u(\tau, t)$ with properties of $A\big|_{t=c\tau} = P(\tau)$, $K\big|_{t=c\tau} = Q(\tau, s)$, $f\big|_{t=c\tau} = \psi(\tau)$, $u\big|_{t=c\tau} = x(\tau)$ and the operator $d/d\tau$ is replaced by a differentiation operator D_c of the form (1.2).

Thus, the problem of quasiperiodic fluctuations in systems (1.5) becomes equivalent to the problem on the existence of (θ, ω)-periodic by (τ, t) solutions $u(\tau, t)$ of the system partial integro-differential equations of the form (1.1) with differentiation operator (1.2).

The above problems on string vibrations and fluctuations in the numbers of two species living together associated with the task indicate the relevance of the latter, in terms of its applicability in life. Along with this, it is worth paying special attention to the fact that the methods of researching multiperiodic solutions of integro-differential equations and systems of such partial differential equations belong to a poorly studied section of mathematics. Therefore, the development of methods of the theory of multiperiodic solutions of partial differential integro-differential equations is of special scientific interest.

In the present work are investigated to obtain conditions for the existence of multiperiodic solutions of linear systems integro-differential equations with a given differentiation operator D_c. To achieve this goal, the initial problems for the considered systems of equations are solved from the beginning, the necessary and sufficient conditions for the existence of multiperiodic solutions of linear systems are established, integral structures of solutions linear systems are determined.

The theoretical basis of this research is based on the work of several authors. As noted above, taking into account the hereditary nature of various processes of physics, mechanics, and biology leads to the consideration of integro-differential equations [1–3], especially to the research of problems for them related to the theory of periodic fluctuations [2]. If the heredity of the phenomenon is limited to a finite period ε of time τ, then the hereditary effect is specified by the integral operator with variable limits from $\tau - \varepsilon$ to τ. Integro-differential equations describing phenomena with such hereditary effects are considered in [3]. The various processes of hereditary continuum mechanics are described by partial integro-differential equations, the study of which began with the works [1]. The work of many authors is devoted to finding effective signs of solvability and the construction of constructive methods for researching problems for systems of differential equations, we note only [2,4,5]. The research of multi-frequency oscillations led to the concept of multidimensional time. In this connection, of the theory solutions of partial differential equations that are periodic in multidimensional [6–13]. In [6], an approach is implemented where quasiperiodic solutions of ordinary differential equations are studied with a transition to the study of multiperiodic solutions of partial differential equations. This method was developed in [7–11] with its extension to the solution of a number of oscillation problems in systems of integro-differential equations.

In this research, it is examined for the first time that the problem of the existence multiperiodic solutions of systems integro-differential equations with a special differentiation operator D_c, describing hereditary processes with a finite period ε of hereditary time τ. In solving this problem, we encountered the problems associated with the multidimensionality of time; not developed general theory of such systems; determination of structures and integral representations of solutions of linear systems equations; extending the results of the linear case to the nonlinear case; the smoothness of the solutions integral equations equivalent to the problems under consideration, etc. These barriers to solving problems have been overcome due to the spread and development of the methods of works [12-13] used to solve similar problems for systems of differential equations.

2. Zeros of the differentiation operator and its multiperiodicity

By the zero of the operator D_c we mean a smooth function $u = u(\tau, t)$ satisfying the equation of $D_c u = 0$. The linear function y is a general solution of the characteristic equation with the initial data (τ^0, t^0), its integral is the zero of the operator D_c satisfying condition $h(\tau^0, \tau, t)\big|_{\tau=\tau^0} = t$. Note that if $\psi(t)$ is an any smoothness function $e = (1, \dots, 1)$, by $t \in R^m$, then

$$u(\tau^0, \tau, t) = \psi(h(\tau^0, \tau, t)) \qquad (2.1)$$

is the zero of the operator D_c satisfying condition $u\big|_{\tau=\tau^0} = \psi(t)$. Since the $\psi(t)$ is arbitrary in the class $C_t^{(e)}(R^m)$, relation (2.1) is a general formula of the zeros.

We give the properties of the characteristics of the operator D_c:

$$h(s + \theta, \tau + \theta, t) = h(s, \tau, t), \qquad (2.2)$$

$$h(s, \tau + \theta, t) = h(s, \tau, t) - c\theta, \qquad (2.3)$$

$$h(s, \tau, t + q\omega) = h(s, \tau, t) + q\omega,\ q\omega = (q_1\omega_1, \dots, q_m\omega_m),\ q \in Z^m. \qquad (2.4)$$

If $u(\tau, t)$ is the zero (θ, ω)-periodic, then the $u\big|_{\tau=\tau^0} = u^0(t)$ is ω-periodic by t:

$$u^0(t + q\omega) = u^0(t) \in C_t^{(e)}(R^m), q \in Z^m. \qquad (2.5)$$

Therefore, (2.5) is a necessary condition for the (θ, ω)-periodicity of zero $u(\tau, t) \in C_{\tau,t}^{(1,e)}(R \times R^m)$. Suppose that for zero $u(\tau, t)$ is satisfied (2.5). From (2.1):

$$u(\tau, t) = u^0(h(\tau^0, \tau, t)). \qquad (2.6)$$

Based on (2.3), zero $u(\tau, t)$ is θ-periodic by τ if $u^0(h(\tau^0, \tau + \theta, t)) = \ = u^0(h(\tau^0, \tau, t) - c\theta)$. This takes place if there is a vector $q^0 \in Z^m$ and

$$c\theta + q^0\omega = 0. \qquad (2.7)$$

By virtue of (2.2), the zeros $u(\tau^0, \tau, t)$ form (2.1) have the property of diagonal θ-periodicity by (τ^0, τ). The proof follows from (2.2) and (2.1).

Theorem 2.1. *1) If condition (2.7) is not satisfied, then only constants are the* (θ,ω)*-periodic zeros and it does not have multiperiodic variables zeros. 2) If condition (2.7) is satisfied, then any zero of the operator* D_c *with an initial function of the form (2.5) is* (θ,ω)*-periodic, in particular, it can be any constant. 3) Zero of the form (2.1) has the property of diagonal* θ*-periodicity by* (τ^0,τ)*, and from its* θ*-periodicity zeros by* τ *follows its* θ*-periodicity by* τ^0.

3. Linear homogeneous equations and its multiperiodic solutions.

We consider the initial problem for a linear homogeneous system

$$D_c u(\tau,t)=A(\tau,t)u(\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds \tag{3.1}$$

$$u(\tau,t)\big|_{\tau=\tau^0}=u^0(t)\in C_t^{(e)}(R^m) \tag{3.1^0}$$

under assumptions of

$$A(\tau+\theta,t+q\omega)=A(\tau,t)\in C_{(\tau,t)}^{(0,2e)}(R\times R^m),\ q\in Z^m, \tag{3.2}$$

$$K(\tau+\theta,t+q\omega,s,\sigma)=K(\tau,t,s+\theta,\sigma+q\omega)=K(\tau,t,s,\sigma)\in$$
$$\in C_{\tau,t,s,\sigma}^{(0,2e,0,2e)}(R\times R^m\times R\times R^m),\ q\in Z^m. \tag{3.3}$$

From [17-19] with (3.2), using the method of successive approximations, we can construct a matricant $W(\tau^0,\tau,t)$ of the system $D_c w(\tau,t)=A(\tau,t)w(\tau,t)$, and

$$D_c W(\tau^0,\tau,t)=A(\tau,t)W(\tau^0,\tau,t),\ W(\tau^0,\tau^0,t)=E, \tag{3.4}$$

$$D_c W^{-1}(\tau^0,\tau,t)=-W^{-1}(\tau^0,\tau,t)A(\tau,t), \tag{3.5}$$

$$W(\tau^0+\theta,\tau+\theta,t+q\omega)=W(\tau^0,\tau,t),\ q\in Z^m. \tag{3.6}$$

Then, using the replacement of $u(\tau,t)=W(\tau^0,\tau,t)v(\tau,t)$ system (3.1) is reduced to the form of integro-differential equation

$$D_c v(\tau,t)=\int_{\tau-\varepsilon}^{\tau}Q(\tau^0,\tau,t,s,h(s,\tau,t))v(s,h(s,\tau,t))ds \tag{3.7}$$

with the kernel $Q(\tau^0,\tau,t,s,\sigma)=W^{-1}(\tau^0,\tau,t)K(\tau,t,s,\sigma)W(\tau^0,s,\sigma)$. By virtue of (2.2)-(2.4), (3.3) and (3.4)-(3.6), $Q(\tau^0,\tau,t,s,\sigma)$ has the properties:

$$Q(\tau^0+\theta,\tau+\theta,t+q\omega,s+\theta,h(s+\theta,\tau+\theta,t+q\omega))=Q(\tau^0,\tau,t,s,h(s,\tau,t))=$$
$$=Q(\tau^0,\tau,t,s,\sigma)\in C_{\tau^0,\tau,t,s,\sigma}^{(1,1,e,1,e)}(R\times R\times R^m\times R\times R^m),q\in Z^m$$

Further, under condition (3.3), integrating along the characteristics: $\tau=\eta$, $t=h(\eta,\tau,t)$, using the group property of the characteristic, from equation (3.7):

$$V(s,\tau,t)=E+\int_s^{\tau}d\eta\int_{\eta-\varepsilon}^{\eta}Q(s,\eta,h(\eta,\tau,t),\xi,h(\xi,\tau,t))V(s,\xi,h(\xi,\tau,t))d\xi, \tag{3.8}$$

Obviously, by virtue of (3.8) and multiperiodicity, we have

$$D_c V(s,\tau,t)=\int_{\tau-\varepsilon}^{\tau}Q(s,\tau,t,\xi,h(\xi,\tau,t))V(\xi,h(\xi,\tau,t))d\xi, \tag{3.9}$$

$$V(s,s,t)=E. \tag{3.9^0}$$

The A, K, ε are such that matrix $V(s,\tau,t)$ is invertible. The matrix $U(s,\tau,t)=$ $=W(s,\tau,t)V(s,\tau,t)$: $D_c U(s,\tau,t)=A(\tau,t)U(s,\tau,t)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi$,

$$U(s,s,t)=E, \tag{3.10}$$

$$U(s+\theta,\tau+\theta,t+q\omega)=U(s,\tau,t)\in C_{s,\tau,t}^{(1,1,e)}(R\times R\times R^m),\ q\in Z^m. \tag{3.11}$$

Properties (3.10)-(3.11) are consequences of (3.4)-(3.6), (3.9)-(3.9^0). The matrix $U(s,\tau,t)$ is called the resolving operator of system (3.1).

Theorem 3.1. *Let conditions (3.2)-(3.3) are satisfied. Then the solution $u(\tau^0,\tau,t)$ of the problem (3.1)-(3.1^0) is uniquely determined by the relation*

$$u(\tau^0,\tau,t)=U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t)). \tag{3.12}$$

Proof.◄By condition (3.1^0), (2.5), (2.6) $u^0(h(\tau^0,\tau,t))$, is the zero of the operator D_c. Taking into account the group property of the characteristic, (3.10) it is shown that (3.12) satisfies the system (3.1).►

Theorem 3.2. *Let the conditions of theorem 3.1 are satisfied. The solution $u(\tau,t)$ of system (3.1) is (θ,ω)-periodic, it is necessary, that its initial function $u(0,t)=u^0(t)$ at $\tau=0$ should be ω-periodic continuously differentiable:*

$$u^0(t+q\omega)=u^0(t)\in C_t^{e/\omega}(R^m),\ q\in Z^m. \tag{3.13}$$

Proof.◄Indeed, for $\tau^0=0$, from the solution (3.12) we have

$$u(\tau,t)=U(0,\tau,t)u^0(h(0,\tau,t)), \tag{3.14}$$

and it is (θ,ω)-periodic by (τ,t), in particular

$$u(\tau,t+q\omega)=u(\tau,t), q\in Z^m. \tag{3.15}$$

Using (3.14), (3.15), (2.4), (3.11) and (3.10), we find. $u^0(t+q\omega)=u^0(t)$. The smoothness of $u^0(t)$ follows from smoothness of solution $u(\tau,t)$ of system (3.1).

Theorem 3.3. *In order for the solution $u(\tau,t)$ of system (3.1) for being ω-periodic by $t\in R^m$ under the conditions of theorem 3.2, it is necessary and sufficient for condition (3.13) be satisfied by the initial function $u^0(t)$ for $\tau=0$.*

Proof.◄ Necessity follows from Theorem 3.2. For sufficiency, to show relation (3.15) follows from condition (3.13).►

Theorem 3.4. *In order for the solution $u(\tau,t)$ to be θ-periodic by $\tau\in R$ under the conditions of theorem 3.3, it is necessary and sufficient that the initial function $u^0(t)$ a ω-periodic solution of the linear ω-periodic by t functional difference system with difference $\rho=c\theta$ by t*

$$U(0,\theta,t)u^0(t-c\theta)=u^0(t) \tag{3.16}$$

◄*The necessary and sufficient condition (3.16) follows from (3.14), (2.3), (3.10)* .►

Theorem 3.5. *In order for the solution $u(\tau,t)$ to be (θ,ω)-periodic solution of (3.1) generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the operator D_c under the conditions of theorem 3.4, it is necessary and sufficient that the $u_0(\tau,t)=v(t)$ be an eigenvector of the monodromy matrix $U(0,\theta,t)=V(t)$: $[V(t)-E]v(t)=0$.*

The necessary and sufficient condition (3.16) follows from Theorems 2.1 and 3.4.

We assume that the operator $U\left(\tau^{0},\tau,t\right)$ of system (3.1) satisfies condition

$$\left|U\left(s,\tau,t\right)\right|\leq a\,e^{-\alpha(\tau-s)},\ a\geq 1,\ \alpha>0,\ \tau\geq s. \tag{3.17}$$

Theorem 3.6. *In order for the system of integro-differential equations (3.1) has no multiperiodic solutions, except for the zero one under the conditions of theorem 3.4, the fulfillment of condition (3.17) is sufficient.*

Note that theorem 3.6 is valid if condition (3.17) is replaced by condition $\left|U\left(s,\tau,t\right)\right|\leq a\,e^{\alpha(\tau-s)}$, $a\geq 1,\ \alpha>0$, $\tau\leq s$. The resolving operator $U(s,\tau,t)$ is represented as:

$$U(s,\tau,t)=U_{-}(s,\tau,t)+U_{+}(s,\tau,t), \tag{3.18}$$

$$D_c U_{\mp}(s,\tau,t)=A(\tau,t)U_{\mp}(s,\tau,t)+\int_{\tau-\varepsilon}^{\tau}K\left(\tau,t,\xi,h(\xi,\tau,t)\right)U_{\mp}\left(s,\xi,h(\xi,\tau,t)\right)d\xi, \tag{3.19}$$

and satisfying conditions

$$\left|U_{-}\left(s,\tau,t\right)\right|\leq a\,e^{-\alpha(\tau-s)},\ \tau\geq s,\ a\geq 1 \text{ and } \alpha>0, \tag{3.20}$$

$$\left|U_{+}\left(s,\tau,t\right)\right|\leq a\,e^{\alpha(\tau-s)},\ \tau\leq s,\ a\geq 1 \text{ and } \alpha>0. \tag{3.21}$$

Under conditions (3.18)-(3.21), they say that the resolving operator $U(s,\tau,t)$ has the property of exponential dichotomy.

Theorem 3.7. *Let conditions (3.2), (3.3), and (3.18)-(3.21) be satisfied. Then system (3.1) has no multiperiodic solutions, except for the trivial one.*

4. Linear inhomogeneous equations and its multiperiodic solutions

We consider the system of integro-differential equations

$$D_c u(\tau,t)=A(\tau,t)u(\tau,t)+\int_{\tau-\varepsilon}^{\tau}K\left(\tau,t,\xi,h(\xi,\tau,t)\right)u\left(\xi,h(\xi,\tau,t)\right)d\xi+f(\tau,t), \tag{4.1}$$

where the $f(\tau,t)$ is given n-vector-function possessing property

$$f(\tau+\theta,t+q\omega)=f(\tau,t)\in C_{\tau,t}^{(0,e)}\left(R\times R^{m}\right),\ q\in Z^{m}. \tag{4.2}$$

Find a solution to system (4.1) satisfying the initial condition

$$u\big|_{\tau=\tau^{0}}=u^{0}(t)\in C_{t}^{(e)}\left(R^{m}\right). \tag{4.1^{0}}$$

We seek a particular solution $u^{*}\left(\tau^{0},\tau,t\right)$ of system (4.1) with zero initial condition

$$u^{*}\left(\tau^{0},\tau,t\right)\big|_{\tau=\tau_{0}}=0. \tag{4.1*}$$

with an unknown n-vector function $v(\tau,t)\in C_{\tau,t}^{(0,e)}\left(R\times R^{m}\right)$ in the form

$$u^{*}\left(\tau^{0},\tau,t\right)=\int_{\tau^{0}}^{\tau}U(s,\tau,t)v\left(s,h(s,\tau,t)\right)ds. \tag{4.3}$$

Acting by the operator D_c on vector-function (4.3), considering $v(\tau,t)$, we have

$$D_c u^{*}\left(\tau^{0},\tau,t\right)=A(\tau,t)u^{*}\left(\tau^{0},\tau,t\right)+\int_{\tau-\varepsilon}^{\tau}K\left(\tau,t,\xi,h(\xi,\tau,t)\right)u^{*}\left(\tau^{0},\xi,h(\xi,\tau,t)\right)d\xi+v(\tau,t). \tag{4.5}$$

Substituting (4.3) and (4.5) into (4.1) we obtain that $v(\tau,t)=f(\tau,t)$. Then

$$u^*\left(\tau^0,\tau,t\right)=\int_{\tau^0}^{\tau}U(s,\tau,t)f\left(s,h(s,\tau,t)\right)ds. \qquad (4.6)$$

The solution (4.6) satisfies condition (4.1^*). The general Cauchy solution of system (4.1) with initial condition (4.1^0) has the form

$$u\left(\tau^0,\tau,t\right)=U\left(\tau^0,\tau,t\right)u^0\left(h\left(\tau^0,\tau,t\right)\right)+u^*\left(\tau^0,\tau,t\right). \qquad (4.7)$$

Theorem 4.1. *Under conditions (3.2), (3.3) and (4.2), the initial problem (4.1) - (4.1^0) has the unique solution in the form (4.6)-(4.7).*

◄ For the proof we use (4.6) and (4.7).►

Under the conditions of Theorem 3.7, the existence of multiperiodic solutions of system (4.1) is investigated by the method of Green's functions, exponential dichotomy of $G(s,\tau,t)=\begin{cases}U_-(s,\tau,t), & \tau\ge s\\ -U_+(s,\tau,t), & \tau<s\end{cases}$.

Theorem 4.2. *Suppose that the conditions of theorem 4.1 are satisfied and the matrix $A(\tau,t)$ with kernel $K(\tau,t,s,\sigma)$ are such that the system (3.1) has the property of exponential dichotomy, expressed by the relation (3.18)-(3.21). Then system (4.1) has the unique (θ,ω)-periodic solution $u^*(\tau,t)=\int_{-\infty}^{+\infty}G(s,\tau,t)f\left(s,h(s,\tau,t)\right)ds$, satisfying estimate $\left\|u^*\right\|\le a/\alpha\left\|f\right\|$.*

◄ Under the conditions of Theorem 3.1, the properties of the Green's function, the proof of the theorem is carried out. Exponential dichotomy (3.1) ensures the uniqueness of a multiperiodic solution to system (4.1).►

Lemma 4.1. *Let the homogeneous linear system (3.1) under conditions (3.2), (3.3) and (4.2) have no (θ,ω)-periodic solutions except zero. Then the corresponding inhomogeneous linear system (4.1) can have at most one (θ,ω)-periodic solution.*

Finding the (θ,ω)-periodic solution $u(\tau,t)$ of system (4.1) among the solutions with initial conditions, it is shown that it is defined as

$$u(\tau,t)=\left[U^{-1}(0,\tau+\theta,t)-U^{-1}(0,\tau,t)\right]^{-1}\int_{\tau}^{\tau+\theta}\tilde{U}_\theta(s,\tau,t)f_\theta\left(s,h(s,\tau,t)\right)ds. \qquad (4.8)$$

Theorem 4.3. *Suppose that conditions (3.2), (3.3), (4.2) are satisfied and the linear homogeneous system (3.1) has no (θ,ω)-periodic solutions, except for the trivial one. Then the system of inhomogeneous linear integro-differential equations (4.1) has the unique (θ,ω)-periodic solution $u(\tau,t)$ of the form (4.8).*

Note that uniqueness follows from Lemma 4.1.

We note that the research problems of the considered systems can be studied along characteristics with fixed initial data. From the proved theorems as a corollary, we have statements about the existence of solutions to the initial problems for ordinary integro-differential equations and about the existence of their quasiperiodic solutions in the sense of Bohr generated by multiperiodic solutions of the original systems.

Conclusion.

In this article proposes the method for (research) researching solutions of problems that satisfy the initial conditions and have the property of multiperiodicity with given periods for systems of integro-differential equations with a special D_c operator in partial differential, ε hereditary effect and the linear integral operator. This technique is a generalization of methods and solutions of similar problems for systems of partial differential equations with the operator D_c. The problems under consideration in this formulation are researched for the first time. The relevance of the main problem is substantiated. The solutions of all the subtasks analyzed to achieve the goal are formulated as theorems with proofs. Scientific novelties include the multi-periodicity theorems of zeros of the operator D_c; about solutions to initial problems for all considered of systems; about necessary as well as sufficient conditions for the

existence of multiperiodic solutions of both homogeneous and inhomogeneous systems, the integral representations of solutions systems in cases: exponential dichotomy and the absence of non-trivial multiperiodic solutions. We note that the consequences deduced by examining the results obtained along the characteristics refer to their applications in the theory of quasiperiodic solutions of systems ordinary integro-differential equations. The technique that developed here is quite applicable to the research of problems of hereditary-string vibrations and the "predator-prey" given in delivered part of work, which can be attributed to examples of applied aspect.

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D_c-ОПЕРАТОРЛЫ ЖӘНЕ ε -ЭРЕДИТАРЛЫҚ ПЕРИОДТЫ СЫЗЫҚТЫ ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІ

Аннотация. Мақалада $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$ операторлы, $c - const$ және тұқым қуалаушылық сипаттағы құбылыстарды сипаттайтын τ уақыты бойынша ε ақырлы эредитарлық периодты сызықты интегралды-дифференциалдық теңдеулер жүйесінің көппериодты шешімдері жөніндегі есептер мен бастапқы есеп мәселелері зерттеледі. Арнайы D_c дифференциалдау операторының нөлдерінің теңдеуімен қатар, сызықты біртекті және біртекті емес интегралды-дифференциалдық теңдеулер жүйесі қарастырылды, олар үшін бастапқы есептердің бірмәнді шешілімділігінің жеткілікті шарттары анықталған, (τ, t) бойынша (θ, ω) периодты, көппериодты шешімдердің бар болуының қажетті де, жеткілікті де шарттары алынған. Жалғыздық шартына ие сызықты біртекті емес жүйенің көппериодты шешімдерінің интегралдық өрнектері, 1) дербес жағдайда, яки теңдеуге сәйкес біртекті жүйелер экспоненциалды дихотомиялық қасиетке ие болғанда және 2) жалпы жағдайда, біртекті жүйелердің нөлден басқа көппериодты шешімдері болмағанда айқындалды. Мақалада дербес туындылы арнайы D_c дифференциалдау операторлы сызықты интегралды-дифференциалдық теңдеулер жүйесі үшін берілген ε эредитарлық периодты көппериодтылық қасиетіне ие және бастапқы шарттарға қанағаттандыратын есептерді шешудің зерттеу әдістемесі ұсынылған. (τ^0, t^0) бекітілген $t = t^0 + c\tau - c\tau^0$ характеристикалар бойында алынған көппериодты шешімдер қарапайым интегралды-дифференциалдық теңдеулер жүйесінің квазипериодты шешімдер теориясында қолданбалы түрде пайдаланылады.

Түйін сөздер: интегралды-дифференциалдық теңдеу, эредитарлық, флуктуация, көппериодты шешім.

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МНОГОПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ЛИНЕЙНЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С D_c- ОПЕРАТОРОМ И ε -ПЕРИОДОМ ЭРЕДИТАРНОСТИ

Аннотация. В статье исследуются вопросы начальной задачи и задачи о многопериодичности решений линейных систем интегро-дифференциальных уравнений с оператором вида $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c - const$ и конечным периодом эредитарности $\varepsilon = const > 0$ по времени τ, которые описывают явления наследственного характера. Наряду с уравнением нулей специального оператора дифференцирования D_c рассмотрены линейные системы однородных и неоднородных интегро-дифференциальных уравнений, для них установлены достаточные условия однозначной разрешимости начальных задач, получены как необходимые, так и достаточные условия существования многопериодических по (τ, t) с периодами (θ, ω) решений. Определены интегральные представления многопериодических решений линейных неоднородных систем, обладающих свойством единственности, 1) в частном случае, когда соответствующие однородные системы обладают экспоненциальной дихотомичностью и 2) в общем случае, когда однородные системы не имеют многопериодических решений, кроме тривиального. В статье предложена методика исследования решения задач, удовлетворяющих начальным условиям и обладающих свойством многопериодичности с заданным $\varepsilon-$ периодом эредитарности для линейных систем интегро-дифференциальных уравнений со специальным оператором дифференцирования D_c в частных производных. Многопериодические решения, полученные вдоль характеристик $t=t^0+c\tau-c\tau^0$ с фиксированной (τ^0, t^0), применяются в виде приложения в теории квазипериодических решений систем обыкновенных интегро-дифференциальных уравнений.

Ключевые слова: интегро-дифференциальное уравнение, эредитарность, флуктуация, многопериодическое решение.

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ON BASIS PROPERTY OF SYSTEMS ROOT VECTORS OF A LOADED MULTIPLE DIFFERENTIATION OPERATOR

Abstract. In the case of non-self-adjoint ordinary differential operators, the basis property of systems of eigenfunctions and associated functions (E&AF), in addition to the boundary value conditions, can be affected by values of coefficients of the differential operator. Moreover, it is known that the basic properties of E&AF can be changed at a small change of values of the coefficients. This fact was first noted in V.A. Il'in. Ideas of V.A. Il'in for the case of non-self-adjoint perturbations of the self-adjoint periodic problem were developed in A.S. Makin where operator was changed due to perturbation of one of the boundary value conditions.

In Sadybekov M.A., Imanbaev N.S., we studied another version of the non-self-adjoint perturbation of the self-adjoint periodic problem. In contrast to A.S. Makin, in Sadybekov M.A. and Imanbaev N.S. perturbation occurs due to the change in the equation, which belongs to the class of so-called loaded differential equations, where the basic properties of root functions are investigated.

In this paper we consider perturbations of a second order differential equation of the spectral problem with a loaded term, containing a value of the unknown function at the point zero, with regular, but not strongly regular boundary value conditions. Question about basis property of eigenfunctions and associated functions (E&AF) systems of a loaded multiple differentiation operator is studied.

Keywords: Eigenvalues, eigenfunctions, associated functions, adjoint operator, multiple differentiation, loaded operaor, Riesz basis, root vectors.

Mathematics subject classification: 34B09, 34L15, 34L05

1. Introduction

In the case of non-self-adjoint ordinary differential operators, the basis property of systems of eigenfunctions and associated functions (E&AF), in addition to the boundary value conditions, can be affected by values of coefficients ofthe differential operator. Moreover, it is known that the basic properties of E&AF can be changedat a small change of values of the coefficients. This fact was first noted in [1]. Ideas of [1] for the case of non-self-adjoint perturbations of the self-adjoint periodic problem were developed in [2], [3], where operator was changed due to perturbation of one of the boundary value conditions.

In [4], we studied another version of the non-self-adjoint perturbation of the self-adjoint periodic problem. In contrast to [2], [3], in [4] perturbation occurs due to the change in the equation, which belongs to the class of so-called loaded differential equations, where the basic properties of root functions are investigated. Problems about the basis properties of root functions of loaded differential operators are thoroughly studied in [5], [6]. In these papers, it was possible to extend the spectral decomposition method of V.A.Il'in [1] to the case of loaded differential operators. By the other method questions about the basis property of functional-differential equations were investigated in [7].

The Riesz basis property of the E&AFsystem of periodic and anti-periodic Sturm-Liouville problems was studied in [8].

In [9], consisting of the Sturm-Liouville equation, together with eigenparameter that depended on boundary conditions and two supplementary transmission conditions; we constructed the resolvent operator and prove theorems on expansions in terms of eigenfunctions in modified Hilbert Space $L_2(a,b)$.

The basis properties in $Lp(-1,1)$ of root functions of the nonlocal problems for the equations with involution have been studied in [10]. Moreover, using these asymptotic formulas, we proved that the root functions of these operators form a Riesz basis in the space $L_2(0,1)$ [11].

In the case when the potential is zero, the system of eigenfunctions of the periodic problem is usual trigonometric system, which forms a complete orthonormal system in $L_2(0,1)$. And if the potential is non-zero, then additional research is required, which answer is the results of [4].

2. Problem Statement and Main Result

Inthispaperweconsiderperturbations of equation of the following spectral problem with a loaded term containing value of the unknown function at the point zero:

$$L_1(u) \equiv -u''(x) + \overline{q(x)}u(0) = \lambda u(x), \quad q(x) \in L_2(0,1), \quad 0 < x < 1, \tag{1}$$

$$U_1(u) = u(0) - u(1) = 0, \quad U_2(u) = u'(1) = 0 \tag{2}$$

Firstweconstruct characteristic determinant of the spectral problem

(1)-(2). Assumingthat $u(0)$ isasomeindependentconstant, it is easy to prove that,when $\lambda \neq 0$, general solution of (1)can be represented as follows:

$$u(x) = C_1 \cdot \cos\sqrt{\lambda}x + C_2 \cdot \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + u(0)\int_0^x \overline{q(\xi)}\frac{\sin\sqrt{\lambda}(x-\xi)}{\sqrt{\lambda}}d\xi \tag{3}$$

Hence, supposingfirst $x = 0$, andthensatisfying (3) bytheboundaryvaluecondition (2), wegetthe system of three equations, which can be represented in the following vector-matrix form:

$$\begin{bmatrix} -1 & 0 & 1 \\ -\cos\sqrt{\lambda} & -\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} & 1-\int_0^1 \overline{q(\xi)}\frac{\sin\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}}d\xi \\ \sqrt{\lambda}\sin\sqrt{\lambda} & -\cos\sqrt{\lambda} & -\int_0^1 \overline{q(\xi)}\cos\sqrt{\lambda}(1-\xi)d\xi \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Byusing simple calculations, we obtain

$$\Delta_1(\lambda) = -1 \cdot \left(\frac{1}{\sqrt{\lambda}} \int_0^1 \overline{q(\xi)} \cos\sqrt{\lambda}(1-\xi)\sin\sqrt{\lambda}d\xi + \cos\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\int_0^1 \overline{q(\xi)}\sin\sqrt{\lambda}(1-\xi)\cos\sqrt{\lambda}d\xi \right) +$$
$$+ 0 + \left(\cos^2\sqrt{\lambda} + \sin^2\sqrt{\lambda}\right) \tag{4}$$

After the standard transformation of (4), we find that the characteristic determinant $\Delta(\lambda)$ of the spectral problem (1) - (2) is represented as

$$\Delta_1(\lambda) = 1 - \cos\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\int_0^1 \overline{q(\xi)}\sin\sqrt{\lambda}\xi d\xi \tag{5}$$

Now we define the operator L_1^*. Using the Lagrange formula for all functions $u \in D(L_1)$ and $v \in D(L_1^*)$, due to boundary value conditions (2), we find:

$$\int_0^1 L_1(u)\overline{v(x)}dx - \int_0^1 u(x)\overline{L_1^*(v)}dx = \overline{v(0)} \cdot u'(0) + u(0) \cdot \left(\overline{v'(1)} - \overline{v'(0)} + \int_0^1 q(x)\overline{v(x)}dx \right) - \int_0^1 \overline{v''(x)}u(x)dx$$

Consequently, theoperator L_1^* isaconjugateoperator to the operator L_1, which is given by the differential expression:

$$L_1^*(v) \equiv -v''(x) = \overline{\lambda}v(x), \quad 0 < x < 1 \tag{1a}$$

and the boundary-value conditions

$$V_1(v)=v'(0)-v'(1)=\int_0^1 q(x)v(x)dx, \quad q(x)\in L_2(0,1), \quad V_2(v)=v(0)=0\,. \tag{2b}$$

If $q(x)\equiv 0$, then this problem is called Samarskii - Ionkin problem [13]. Inthiscase, boundaryvalueconditions (2) and (2b) are regular, but not strongly regular boundary value conditions [13]. Characteristic determinant of the Samarskii-Ionkin problem will be $\Delta_0(\lambda)=\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right)$. The number $\lambda_0^0=0$ isasimpleroot, thatis single eigenvalue, and $v_0(x)=\sqrt{3}x$ is a corresponding eigenfunction of the Samarskii-Ionkin problem. O ther eigen values of the Samarskii-Ionkin problem are double: $\lambda_k^0=(2k\pi)^2, \quad k=1,2,3,\ldots$.

Moreover, $v_{k0}^0=\sqrt{2}\sin(2k\pi x)$ are the ir corresponding eigen functions, and $v_{k1}^0=\frac{\sqrt{2}}{2}x\cos(2k\pi x)$ are associated functions.

Due to the biorthogonality conditions $\left(v_{k1}^0,u_{k1}^0\right)=1$ we have that $u_{k1}^0=4\sqrt{2}\cos(2k\pi x)$ is an eigen function and $u_{k0}^0=2\sqrt{2}(1-x)\sin(2k\pi x)$ is an associated function of the conjugate problem to the Samarskii-Ionkin problem. The system $\left\{u_{k0}^0,u_{k1}^0\right\}$ forms Riesz basis in $L_2(0,1)$ [13].

Function $q(x)$ can bep resented in the form of biorthogonal expansion in a Fourier series by the system $\left\{u_{k0}^0,u_{k1}^0\right\}$:

$$q(x)=\sum_{k=1}^{\infty}a_{k0}u_{k0}^0+\sum_{k=0}^{\infty}a_{k1}u_{k1}^0=\sum_{k=1}^{\infty}a_{k0}\cdot 2\sqrt{2}(1-x)\sin(2k\pi x)+\sum_{k=0}^{\infty}a_{k1}\cdot 4\sqrt{2}\cos(2k\pi x). \tag{6}$$

Using (6), wefind more convenient representation of the determinant $\Delta_1(\lambda)$. To do it first we calculate the integral in (5). Simple calculations show that

$$\int_0^1\overline{q(\xi)}\sin\sqrt{\lambda}\xi d\xi=2\sqrt{2\lambda}\left(1-\cos\sqrt{\lambda}\right)\left(\sum_{k=0}^{\infty}\left[\frac{2k\pi\overline{a_{k0}}}{\left(\lambda-(2k\pi)^2\right)^2}+\frac{2\overline{a_{k1}}}{\lambda-(2k\pi)^2}\right]\right).$$

By usingthisresult, determinant (5) after conversion is converted to

$$\Delta_1(\lambda)=\Delta_0(\lambda)\cdot A(\lambda), \tag{7}$$

Where $$A(\lambda)=1-4\sqrt{2}\sum_{k=1}^{\infty}\left(\pi\overline{a_{k0}}\frac{k}{\left(\lambda-(2k\pi)^2\right)^2}+\overline{a_{k1}}\frac{k}{\lambda-(2k\pi)^2}\right).$$

Therefore, it is proved

Theorem2.1. Characteristic determinant of the spectral problem (1) - (2) when $q(x)\neq 0$ can be represented in the form (7), where $\Delta_0(\lambda)$is the characteristic determinant of the Samarskii - Ionkin problem, a_{k0}, a_{k1} are Fourier coefficients of the biorthogonal expansion (6) of the function $q(x)$ by the E&AF system of the conjugate Samarskii-Ionkin spectral problem.

Thefunction $A(\lambda)$in (7) has poles of the second and first orders at points $\lambda=\lambda_k^0$, but the function $\Delta_0(\lambda)$has zeros of the second order at these points. Thus, thefunction $\Delta_1(\lambda)$, representedbytheformula(7), is an entire analytic function of the variable λ.

Ifatsomeindex j coefficientsof(6) $a_{j0}=0, a_{j1}=0$, then $\lambda_j^1=\lambda_j^0$is a double eigenvalue of the spectral problem (1) – (2).

If $a_{j0} = 0, a_{j1} \neq 0$, then $\lambda_j^1 = \lambda_j^0$ isasimpleeigenvalue of the spectral problem (1) – (2).

The characteristic determinant (7) looks more simply, when $q(x)$ is represented as (6) with a finite first sum. That is, when there exists a number N such that $\overline{a_{k0}} = 0,\ \overline{a_{k1}} = 0$ for all $k > N$. Inthiscase, the (7) -thformulatakesthefollowingform

$$\Delta_1(\lambda) = \Delta_0(\lambda)\left[1 - 4\sqrt{2}\sum_{k=1}^{N}\left(\pi\overline{a_{k0}}\frac{k}{\left[\lambda-(2k\pi)^2\right]^2} + \overline{a_{k1}}\frac{k}{\lambda-(2k\pi)^2}\right)\right] \quad (8)$$

From this particular case of the formula (8) we justify the following

Corollary 2.1. For any pre-assigned numbers: complex $\tilde{\lambda}$ and natural $\tilde{m}$, there always exists a function $q(x)$ such that $\tilde{\lambda}$ is an eigenvalue of the problem (1) - (2) of the multiplicity $\tilde{m}$.

From the analysis of (8), we note, that $\Delta_1(\lambda_k^0) = 0$ for all $k > N$. That is all eigen values $\lambda_k^0,\ k > N$, of the Samarskii-Ionkin problem are eigenvalues of the spectral problem (1) - (2). Moreover, multiplicity of the eigen values $\lambda_k^0,\ k > N$, is also preserved.

From the orthogonality condition $q(x) \perp v_{j0}^0,\quad q(x) \perp v_{j1}^0$ for all $j > N$ it follows, that in this case

$$\int_0^1 \overline{q(x)} v_{j0}^0(x)dx = \int_0^1 \overline{q(x)} v_{j1}^0(x)dx = 0.$$

Therefore, eigenfunctions $v_{j0}^0(x)$ andassociatedfunctions $v_{j1}^0(x)$ oftheSamarskii-Ionkinproblemforall $j > N$ satisfythespectralproblem(1)–(2) and, consequently, they are eigenfunctions and associated functions of the spectral problem (1) – (2). Thus, inthiscaseE&AFsystemofthespectralproblem (1) – (2) and E&AFof the Samarskii-Ionkin problem (forming Riesz basis) differ from each other only in a finite number of first members. Consequently, the E&AF system of the spectral problem (1) - (2) also forms the Riesz basis in $L_2(0,1)$.

By*B*wedenoteasetoffunctions $q(x) \in L_2(0,1)$, representable in the form of a finite series (6), which is everywhere dense in $L_2(0,1)$. Thus, we formulate the main result of our paper:

Theorem 2.2. Let $q(x) \in L_2(0,1)$. Then E&AF system of the spectral problem (1) – (2) forms Riesz basis in $L_2(0,1)$ and the set B is everywhere dense in $L_2(0,1)$.

Since the adjoint operators simultaneously possess the Riesz basis property of root functions, consequently, we get

Corollary 2.2. The set Bof functions $q(x) \in L_2(0,1)$, for which the E&AF system of the conjugate problem (1a) - (2b), that is, of the multiple differentiation operator L_1^* with integral perturbation of the first boundary value condition of the Samarskii-Ionkin problem, forms a Riesz basis in $L_2(0,1)$, is every where dense $L_2(0,1)$.

Previously, other approaches to the study of similar problems (1a) - (2b) with integral perturbation of the second boundary value condition were published in our papers [14], [15], [16].

The work paper [17], we prove uniqueness theorem, by one spectrum, for a Sturm-Liouville operator with non-separated boundary value conditions and a real continuous and symmetric potential.

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ЕСЕЛІ ДИФФЕРЕНЦИАЛДАНАТЫН ЖҮКТЕЛГЕН ОПЕРАТОРДЫҢ ТҮБІРЛІК ВЕКТОРЛАР ЖҮЙЕСІНІҢ БАЗИСТІЛІГІ ЖАЙЛЫ

Аннотация. Бұл мақалада регулярлы, бірақ күшейтілген регулярлы емес шеттік шарттармен берілген жүктелген екінші ретті дифференциалдық оператордың спектралдық есебі қарастырылады. Еселі дифференциалданатын жүктелген оператордың түбірлік векторлар жүйесінің базистілігі зерттеледі. Кез келген өзіне-өзі түйіндес шеттік шарттармен және өзіне-өзі түйіндес формальді дифференциалдық амалмен берілген, спектрі дискретті болатын оператордың меншікті функциялары жүйесінің ортонормаланған базис құрайтындығы белгілі жай. Сонымен бірге өзіне-өзі түйіндес емес жай дифференциалдық операторлардың түбірлік функциялары жүйесінің базистілігіне шеттік шарттардан бөлек, дифференциалдық оператордың коэффициентерінің мәндері де әсер ететіндігі белгілі. Мұндай жағдайда, коэффициентердің мәндері шамалы ғана өзгергеннің өзінде, түбірлік функциялардың базистілік қасиеттеріне бірден әсер етеді. Тұңғыш рет мұндай факт В.А. Ильиннің жұмысында келтірілді. Өзіне-өзі түйіндес периодты есеп үшін В.А. Ильиннің идеясы өзіне-өзі түйіндес емес толқыту жағдайында А.С. Макиннің еңбегінде дамытылды. Бұл жұмыста, шеттік шарттардың біреуін толқытқан кезде оператор өзгерген болатын. Ал, М.А. Садыбеков пен Н.С. Иманбаевтың мақаласында периодты шарттармен берілген жүктелген екінші ретті дифференциалдық оператордың меншікті функциялары жүйесінің базистілік қасиеттері зерттелген. Бұл мақалада қарастырылған есепте өзіне-өзі түйіндес периодты есеп үшін өзіне-өзі түйіндес емес толқыту жағдайы болып табылады, бірақ А.С. Макиннің еңбегінде зерттелген есептен М.А. Садыбеков пен Н.С. Иманбаевтың мақаласында қарастырылған есептің ерекшелігі шеттік шарттардың емес, теңдеудің толқытылуында болып тұр.

Жүктелген дифференциалдық операторлардың түбірлік функцияларының базистілік қасиеттерін зерттеу мәселелері И.С. Ломовтың жұмыстарында зерттелді. В.А. Ильиннің спектралдық жіктеу әдісі И.С. Ломовтың мақалаларында жүктелген дифференциалдық операторлар үшін сәтті қолданылып дамытылды. А.М. Гомилко мен Г.В. Радзиевскийдің жұмыстарында функционалдық-дифференциалдық теңдеулердің түбірлік векторларының базистілік мәселелері басқа әдістермен зерттелген.

Ал, аталмыш мақалада екінші ретті, нөл нүктесінде жүктелінген дифференциалдық теңдеу үшін периодты емес шеттік шарттармен берілген есептің характеристикалық анықтауышы жазылып, оның спектралдық параметр бойынша бүтін аналитикалық функция болатындығы көрсетіліп, меншікті мәндері анықталған. Осыған сәйкес, түбірлік функцияларының базистілік қасиеттерінің орнықтылығы жайлы теорема дәлелденеді. Бұл есепке түйіндес есеп - толқытылған Самарский-Ионкин есебі болатындығы көрсетілген.

Түйін сөздер: меншікті мәндер, меншікті функциялар, тіркелген функциялар, түйіндес оператор, еселі дифференциалдау, жүктелген оператор, Рисс базистілігі, түбірлік векторлар.

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О БАЗИСНОСТИ СИСТЕМ КОРНЕВЫХ ВЕКТОРОВ НАГРУЖЕННОГО ОПЕРАТОРА КРАТНОГО ДИФФЕРЕНЦИРОВАНИЯ

Аннотация. В настоящей статье рассматривается возмущения дифференциального уравнения второго порядка спектральной задачи с нагруженным слагаемым, содержащий значение искомой функции в точке нуль, с регулярными, но неусиленно регулярными краевыми условиями. Исследуется вопрос базисности систем собственных и присоединенных функций (СиПФ) нагруженного оператора кратного дифференцирования. Хорошо известно, что система собственных функций оператора, заданного формально самосопряженным дифференциальным выражением, с произвольными самосопряженными краевыми условиями, обеспечивающими дискретный спектр, образует ортонормированный базис. Наряду с этим, известно, что в случае несамосопряженных обыкновенных дифференциальных операторов на базисность систем корневых функций, помимо краевых условий, могут влиять также значения коэффициентов дифференциального оператора. При этом базисные свойства корневых функций могут изменяться даже при сколь угодном малом изменении значений коэффициентов. Впервые этот факт был отмечен в работе В.А. Ильина. Идеи В.А. Ильина были развиты А.С. Макиным на случай несамосопряженного возмущения самосопряженной периодической

задачи. Оператор в работе А.С. Макина изменялся за счет возмущения одного из краевых условий. В статье М.А. Садыбекова, Н.С. Иманбаева исследованы базисные свойства корневых функций нагруженного дифференциального оператора второго порядка с периодическими краевыми условиями, который также является несамосопряженным возмущением самосопряженной периодической задачи. В отличие от работы А.С. Макина, в статье М.А. Садыбекова и Н.С. Иманбаева возмущение происходит за счет изменения уравнения. Вопросы базисности корневых функций нагруженных дифференциальных операторов были изучены в работах И.С. Ломова. Ему удалось распространить метод спектральных разложений В.А. Ильина на случай нагруженных дифференциальных операторов. Другим методом вопросы базисности функционально-дифференциальных уравнений были исследованы в работе А.М. Гомилко и Г.В. Радзиевского.

А в настоящей работе исследуется вопросы базисности корневых векторов дифференциального оператора второго порядка с нагруженным слагаемым в точке ноль с непериодическими краевыми условиями. Построен характеристический определитель, который является целой аналитической функцией. Доказана теорема об устойчивости свойства базисности корневых векторов и построен сопряженный оператор, который оказался возмущенной задачей Самарского-Ионкина.

Ключевые слова: собственные значение, собственные функций, присоединенные функции, сопряженный оператор, кратное дифференцирование, нагруженный оператор, базис Рисса, корневые вектора.

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ANALYTICAL SOLUTION OF PARTIAL TASKS OF SHEAR WAVE IN A CYLINDRICAL LAYER (in the case of the constant values γ-α + 2 = 0 and α = β)

Abstract. The concept of phase velocity is introduced as the rate of change of the phase medium in studies of shear wave processes of circular elements in deformable bodies. In the case of harmonic oscillations of a cylindrical shell, the phase velocity is expressed in terms of the frequency of natural vibrations freely supported along the edges of the shell, and therefore, the study of waves in a cylindrical layer is most directly related to the problem of determining the natural forms and vibration frequencies of shells of finite length. The results of this work on one-dimensional cylindrical waves in elastic and viscoelastic media and rods allow us to study the influence of the characteristics of the material of the media on the wave fields in the material. The problems of the theory of viscoelasticity have recently attracted the special attention of many researchers and engineers in connection with the use of polymer materials in various industries.

Key words: deformable bodies, shear wave, vibrations, cylindrical shell, rod, viscoelastic medium.

FORMULATION OF THE PROBLEM

Let a shear cylindrical wave propagate in an elastic inhomogeneous transversally isotropic cylindrical layer. At the moment t = 0, the tangential stress pulseσ_{zr}or the displacementu_z, but changing in coordinate θ.

We solve the problem in dimensionless variables

$$\tau = \frac{bt}{r_0}; \;\; r = \frac{R}{r_0}; \;\;\; u = \frac{u_z}{r_0} \tag{1}$$

where r_0 - is the inner radius of the layer; b - is the shear wave velocity.

Hooke's law in an elastic inhomogeneous medium has the form

$$\sigma_{zr} = \mu_1(r)\frac{\partial u}{\partial r};$$
$$\sigma_{z\theta} = \mu_2(r)\frac{1}{r}\frac{\partial u}{\partial r}; \tag{2}$$

The equation of motion reduces to the following

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{1}{r}\sigma_{zr} = \rho(r)b^2\frac{\partial^2 u}{\partial \tau^2} \tag{3}$$

Substituting (2) into equation (3) we obtain the basic equation, which has the form

$$\frac{\partial^2 u}{\partial r^2}+\left[\frac{1}{r}+\frac{\mu_1'(r)}{\mu_1(r)}\right]\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\mu_2(r)\partial^2 u}{\mu_1(r)\partial\theta^2}=\frac{b^2\rho(r)}{\mu_1(r)}\frac{\partial^2 u}{\partial\tau^2} \quad (4)$$

Let the boundary conditions for this problem have the form

$$\sigma_{zr}=f_m(\tau)\cos(m\theta)\text{for} r=1 \quad (5)$$

$$\sigma_{zr}=0\text{for} r=r_1 \qquad (\text{and} r_1>1) \quad (6)$$

In addition to the boundary conditions, it is necessary to specify initial conditions that are zero in our problem, i.e.

$$\left.\frac{\partial r}{\partial\tau}\right|_{\tau=0}=0 \qquad u|_{\tau=0}=0 \quad (7)$$

Since a linear problem is considered, it is advisable to use the one-sided Laplace transform over dimensionless time to solve it.

We apply the Laplace transform with respect to τ to equation (4) and obtain

$$\frac{\partial^2 u_0}{\partial r^2}+\left[\frac{1}{r}+\frac{\mu_1'(r)}{\mu_1(r)}\right]\frac{\partial u_0}{\partial r}+\frac{\mu_2(r)\partial^2 u_0}{r^2\mu_1(r)\partial\theta^2}=\frac{\rho(r)b^2}{\mu_1(r)}p^2u_0 \quad (8)$$

The solution to equation (8) is sought in the form

$$u_0=T(r)\cos(m\theta) \quad (9)$$

Flat

$$u_0=\int_0^\infty u(r,\theta,t)\,e^{-p\tau}d\tau \quad (10)$$

Then equation (8) takes the form

$$\frac{\partial^2 T(r)}{\partial r^2}+\left[\frac{1}{r}+\frac{\mu_1'(r)}{\mu_1(r)}\right]\frac{\partial T(r)}{\partial r}-\left[\frac{m^2\mu_2(r)}{r^2\mu_1(r)}+\frac{\rho(r)b^2}{\mu_1(r)}p^2\right]T(r)=0 \quad (11)$$

In the future, we will assume that the in homogeneity of the medium has the form

$$\mu_1(r)=\mu_{10}r^\alpha;\quad \mu_2(r)=\mu_{10}r^\beta;\quad \rho(r)=\rho_0r^\gamma \quad (12)$$

Moreover, $b^2=\frac{\mu_{10}}{\rho_1}$; $\alpha,\beta,\gamma-$ constants.

Then equation (11) takes the form

$$\mathrm{r^2T''(r)+r(1+\alpha)T'(\alpha)-\left(m^2r^{\beta-\alpha}\gamma_1^2+p^2r^{\gamma-\alpha+2}\right)T(r)=0} \quad (13)$$

Here $\gamma_1^2=\frac{\mu_{20}}{\mu_{10}}$

Suppose that $\gamma-\alpha+2=0$and$\alpha=\beta$.

Then equation (13) takes the form

$$r^2T''(r)+r(1+\alpha)T'(r)-[\gamma_1^2m^2+m^2]T(r)=0 \quad (14)$$

The general solution of equation (14) is equal to

$$T=C_1r^{-(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}+C_2r^{-(\frac{\alpha}{2}-\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}} \quad (15)$$

The boundary conditions of the problem in the images takes the form

$$\frac{dT}{dr}=-\frac{f_{m0}(P)}{\rho_0b^2} \quad \text{при} \quad r=1 \quad (16)$$

$$\frac{dT}{dr}=0\text{при}r=r_1 \quad (17)$$

The constants C_1 and C_2 are determined from the boundary conditions (16) - (17) and have the form

$$C_1=\frac{f_{m0}(p)}{\rho_0 b^2(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}[1-r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}]}$$

$$C_2=-\frac{f_{m0}(p)r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}}{\rho_0 b^2(\frac{\alpha}{2}-\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}[1-r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}]} \qquad (18)$$

Substituting (18) into (15), we obtain

$$T=\frac{f_{m0}(p)}{\rho_0 b^2}\left\{\frac{r^{-(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}[1-r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}]}-\frac{r^{-(\frac{\alpha}{2}-\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}[1-r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}]}\right\} (19\ )$$

or

$$T=\frac{f_{m0}(p)}{\rho_0 b^2}\sum_{n=0}^{\infty}\left\{\frac{r^{-(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}r_1^{-2\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}\right.$$

$$\left.-\frac{r^{-(\frac{\alpha}{2}-\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}}{\left(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}\right)}r_1^{-2(n-1)\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}\right\}$$

$$=\frac{f_{m0}(p)}{\rho_0 b^2}r^{-\frac{\alpha}{2}}\sum_{n=0}^{\infty}\left\{\frac{e^{-[lnr+2nlnr_1]\sqrt{p^2+\gamma_1^2m^2+\frac{\alpha^2}{4}}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}-\frac{e^{-[lnr+2(n+1)lnr_1]\sqrt{p^2+\gamma_1^2m^2+\frac{\alpha^2}{4}}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}}\right\} \quad (20)$$

We introduce the notation

$$\varphi_1=lnr+2nlnr_1\ ;$$
$$\varphi_1=-lnr+2(n+1)lnr_1 \qquad (21)$$

Then (20) takes the form

$$T=\frac{f_{m0}(p)}{\rho_0 b^2}r^{-\frac{\alpha}{2}}\sum_{n=0}^{\infty}\left\{\frac{e^{-\varphi_1(r)\sqrt{p^2+\gamma_1^2m^2+\frac{\alpha^2}{4}}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+(\frac{\alpha^2}{4})}}-\frac{e^{-\varphi_2(r)\sqrt{p^2+\gamma_1^2m^2+\frac{\alpha^2}{4}}}}{(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+(\frac{\alpha^2}{4})}}\right\}$$

Considertheexpression

$$T_1(r)=\frac{e^{\left[-\varphi_1(r)\sqrt{p^2+\gamma_1^2m^2+\frac{\alpha^2}{4}}\right]}}{\left(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2m^2+\left(\frac{\alpha^2}{4}\right)}\right)}=$$

$$= \frac{e^{[-\varphi_1(r)\sqrt{p^2+\gamma_1^2 m^2+\frac{\alpha^2}{4}}]}}{\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}} \sum_{k=0}^{\infty} (-\frac{\alpha}{2})^k \frac{1}{(\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)})^k} \qquad (22)$$

Wedenote

$$F_{10} = \frac{e^{[-\varphi_1(r)\sqrt{p^2+\gamma_1^2 m^2+\frac{\alpha^2}{4}}]}}{\sqrt{p^2+\gamma_1^2 m^2+(\frac{\alpha^2}{4})}}; \dots$$

$$F_{1k} = \frac{e^{[-\varphi_1(r)\sqrt{p^2+\gamma_1^2 m^2+\frac{\alpha^2}{4}}]}}{(\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)})^{k+1}} \qquad (23)$$

There is a relation

$$F_{12} = \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]}{1!} F_{10} d\xi; F_{11} = \int_{\varphi_1(r)}^{\infty} F_{10} d\xi$$

$$F_{13} = \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]^2}{2!} F_{10} d\xi; \dots F_{1k} = \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]^{k-1}}{(k-1)!} F_{10} d\xi \qquad (24)$$

Then expression (22) takes the form

$$T_1(r) = F_{10} - \frac{\alpha}{2} \int_{\varphi_1(r)}^{\infty} F_{10} d\xi + \left(\frac{\alpha}{2}\right)^2 \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]}{1!} F_{10} d\xi$$
$$- \left(\frac{\alpha}{2}\right)^3 \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]^2}{2!} F_{10} d\xi + \dots + \left(\frac{\alpha}{2}\right)^k \int_{\varphi_1(r)}^{\infty} \frac{[\xi-\varphi_1(r)]^{k-1}}{(k-1)!} F_{10} d\xi + \dots$$
$$= F_{10} - \frac{\alpha}{2} \int_{\varphi_1(r)}^{\infty} \Big\{ \{1 - \frac{\alpha}{2} \frac{[\xi-\varphi_1(r)]}{1!} + \left(\frac{\alpha}{2}\right)^2 \frac{[\xi-\varphi_1(r)]^2}{2!} - \dots$$
$$+ \left(\frac{\alpha}{2}\right)^{k-1} \frac{[\xi-\varphi_1(r)]^{k-1}}{(k-1)!} + \dots \} F_{10} d\xi \qquad (23)$$

or

$$\mathrm{T}_1 = F_{10} - \frac{\alpha}{2} \int_{\varphi_{1(r)}}^{\infty} e^{-\left(\frac{\alpha}{2}\right)(\xi-\varphi(r))} F_{10} d\xi \qquad (24)$$

Let

$$T_2 = \frac{e^{\left[\varphi_2(r)\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}\right]}}{\left(\frac{\alpha}{2}+\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}\right)} =$$

$$= -\frac{e^{[-\varphi_2(r)\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}]}}{\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}} \cdot \sum_{n=0}^{\infty} (\frac{\alpha}{2})^k \frac{1}{(\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)})^k} \qquad (25)$$

We denote

$$F_{20} = \frac{e^{[-\varphi_2(r)\sqrt{p^2+\gamma_1^2 m^2+\frac{\alpha^2}{4}}]}}{\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)}};\ \ldots \tag{26}$$

$$F_{2k} = \frac{e^{[-\varphi_2(r)\sqrt{p^2+\gamma_1^2 m^2+\frac{\alpha^2}{4}}]}}{(\sqrt{p^2+\gamma_1^2 m^2+\left(\frac{\alpha^2}{4}\right)})^{k+1}}$$

As previously put

$$F_{22} = \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_2(r)]}{1!} F_{20} d\xi;\ F_{21} = \int_{\varphi_2(r)}^{\infty} F_{20} d\xi$$

$$F_{23} = \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_2(r)]^2}{2!} F_{20} d\xi;\ \ldots F_{2k} = \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_2(r)]^{k-1}}{(k-1)!} F_{20} d\xi \tag{27}$$

Consequently:

$$\begin{aligned} -T_2(r) = F_{20} - \frac{\alpha}{2}\int_{\varphi_2(r)}^{\infty} F_{20} d\xi + \left(\frac{\alpha}{2}\right)^2 \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_1(r)]}{1!} F_{20} d\xi \\ + \left(\frac{\alpha}{2}\right)^3 \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_2(r)]^2}{2!} F_{20} d\xi + \cdots + \left(\frac{\alpha}{2}\right)^k \int_{\varphi_2(r)}^{\infty} \frac{[\xi-\varphi_2(r)]^{k-1}}{(k-1)!} F_{20} d\xi + \cdots F_{20} \\ - \frac{\alpha}{2}\int_{\varphi_2(r)}^{\infty} \Big\{ \{1 + \frac{\alpha}{2}\frac{[\xi-\varphi_2(r)]}{1!} + \left(\frac{\alpha}{2}\right)^2 \frac{[\xi-\varphi_2(r)]^2}{2!} - \cdots \\ + \left(\frac{\alpha}{2}\right)^{k-1} \frac{[\xi-\varphi_2(r)]^{k-1}}{(k-1)!} + \cdots \} F_{20} d\xi = \quad F_{20} + F_{20} + \frac{\alpha}{2} \int_{\varphi_2(r)}^{\infty} e^{-\left(\frac{\alpha}{2}\right)(\xi-\varphi_2(r))} F_{20} d\xi \end{aligned} \tag{28}$$

Thus, the expression for T takes the form

$$\begin{aligned} -T = \frac{f_{m0}(p)}{\rho_0 b^2} r^{-\left(\frac{\alpha}{2}\right)} \sum_{n=0}^{\infty} \Big\{ F_{10} - \frac{\alpha}{2}\int_{\varphi_1(r)}^{\infty} e^{-\left(\frac{\alpha}{2}\right)(\xi-\varphi_2(r))} F_{10} d\xi + F_{20} + \\ + \frac{\alpha}{2}\int_{\varphi_2(r)}^{\infty} e^{-\left(\frac{\alpha}{2}\right)(\xi-\varphi_2(r))} F_{20} d\xi \Big\} \end{aligned} \tag{29}$$

Inverting the expression in p, for the displacement u (r, θ, τ) we obtain

$$\begin{aligned} u(r,\theta,\tau) = \frac{\cos(m\theta)}{\rho_0 b^2} r^{-\left(\frac{\alpha}{2}\right)} \Big\{ \sum_{n=0}^{\infty} \int_{\varphi_1(r)}^{\tau} [\hat{F}_{10} - \frac{\alpha}{2} \int_{\varphi_1(r)}^{\xi} e^{-\left(\frac{\alpha}{2}\right)(\xi-\varphi_1(r)} \hat{F}_{10} d\xi] f_m(\tau-\xi) d\xi \\ + \sum_{n=0}^{n_2} \int_{\varphi_2(r)}^{\tau} [\hat{F}_{20} + \frac{\alpha}{2} \int_{\varphi_2(r)}^{\xi} e^{\frac{\alpha}{2}[\xi-\varphi_2(r)]} \hat{F}_{20} d\xi] f_m(\tau-\xi) d\xi \Big\} \end{aligned} \tag{30}$$

Where

$$\hat{F}_{10} = J_0 \sqrt{\frac{\alpha^2}{4} + m^2\gamma_1^2} \cdot \sqrt{\tau^2 - \varphi_1^2(r)} H(\tau - \varphi_1(r))$$

$$\hat{F}_{20} = J_0\sqrt{\frac{\alpha^2}{4} + m^2\gamma_1^2} \cdot \sqrt{\tau^2 - \varphi_2^2(r)}H(\tau - \varphi_1(r))$$

$$n_1 = \left[\frac{\tau - lnr}{2lnr_1}\right]; n_2 = \left[\frac{\tau + lnr}{2lnr_1}\right] - 1 \quad (31)$$

Thenumbers$n_1$and$n_2$show the number of cylindrical waves diverging and descending, or incident and reflecting from the boundary $r_1 = r$affecting the point r in the perturbed region, depending on the dimensionless time τ.

Similarly, we can obtain expressions for the stresses $\sigma_{zr}$and $\sigma_{z\theta}$.

$$\sigma_{zr} = r^{\frac{\alpha}{2}}\left\{\sum_{n=0}^{n_1}[\int_{\varphi_1(r)}^{\tau} f(\tau - \xi)\frac{d\hat{F}_{10}}{dr}d\xi - \frac{1}{r}f[\tau - \varphi_1(r)] - \sum_{n=0}^{n_1}[\int_{\varphi_1(r)}^{\tau} f(\tau - \xi)\frac{d\hat{F}_{10}}{dr}d\xi - \frac{1}{r}f[\tau - \varphi_1(r)]\cos(m\theta)\right\} \quad (32)$$

$$\sigma_{z\theta} = m\mu_{20}r^{-\frac{\alpha+r}{2}}\frac{1}{\rho_0 b^2}r^{-\frac{\alpha}{2}}\left\{\sum_{n=0}^{n_1}\left[\hat{F}_{10} - \frac{\alpha}{2}e^{-\frac{\alpha}{2}[\xi - \varphi_1(r)]}\hat{F}_{10}d\xi\right]f_m(\tau - \xi)d\xi + + \sum_{n=0}^{n_1}\int_{\varphi_2(r)}^{\tau}[\hat{F}_{20} + \frac{\alpha}{2}\int_{\varphi_2(r)}^{\infty}e^{-\frac{\alpha}{2}[\xi - \varphi_2(r)]}\hat{F}_{20}d\xi]f_m(\tau - \xi)d\xi\right\}\cos(m\theta)\sin(m\theta) \quad (33)$$

Consideraparticularcase.

For$\gamma_1 = 1$theshearmodulusis$\mu_{10} = \mu_{20}$;hence the solution for an isotropic medium, i.e. the propagation of shear cylindrical waves occurs in an isotropic medium and forF_{10}, F_{20}we get:

$$\hat{F}_{10} = J_0\sqrt{\frac{\alpha^2}{4} + m^2} \cdot \sqrt{\tau^2 - \varphi_1^2(r)}H(\tau - \varphi_1(r))$$

$$\hat{F}_{20} = J_0\sqrt{\frac{\alpha^2}{4} + m^2} \cdot \sqrt{\tau^2 - \varphi_2^2(r)}H(\tau - \varphi_1(r)) \quad (34)$$

For$\alpha = 0$and$f(\tau, \theta) = \frac{2}{\pi}\sigma_0 S(\theta)H(\xi)$, expressions (31)

Take the form

$$\hat{F}_{10} = J_0(m\gamma_1\sqrt{r^2 - \varphi_1^2(r)}H(\tau - \varphi_1(r))$$

$$\hat{F}_{20} = J_0(m\gamma_1\sqrt{r^2 - \varphi_2^2(r)}H(\tau - \varphi_2(r)) \quad (35)$$

Formulas (30), (31) and (32) give an exact solution to the problem, taking into account the entire complex wave picture.

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ЦИЛИНДРЛІК ҚАБЫҚШАДАҒЫ ЫҒЫСУ ТОЛҚЫНДАРЫНЫҢ ДЕРБЕС ЕСЕБІНІҢ АНАЛИТИКАЛЫҚ ШЕШІМІ
(тұрақты мәні $\gamma - \alpha + 2 = 0$ және $\alpha = \beta$ шамасы жағдайында)

Аннотация. Деформацияланатын денелердегі дөңгелек элементтердің ығысу толқыны процестерін зерттеуде, фазалық жылдамдық ұғымы фазалық ортаның өзгеру жылдамдығы ретінде енгізіледі. Цилиндрлік қабықтың гармоникалық тербелісі жағдайында, фазалық жылдамдық қабықтың шеттерінде еркін тірелген тербеліс жиілімен сипатталады, сондықтан цилиндрлік қабаттағы толқындарды зерттеу арқылы ұзындықтағы қабықтардың табиғи формалары мен тербеліс жиілігіне тікелей байланысты. Жүргізілген жұмыстың нәтижесі бір өлшемді цилиндрлік толқындардың серпімді және жабысқақ ортадағы, шыбықтағы, материалдағы толқын өрісінің характеристикасын зерттеуге мүмкіндік береді.

Көптеген зерттеулерде толқындардың сипаттамаларын анықтау үшін әдетте екі әдіс қолданылады:

1) ортаның белгілі бір уақыт моментіне сәйкес келетін лездік күйі зерттеледі.

2) қарастырылып отырған нүктеде дененің күйі уақытының өзгеруін зерттеу.

Қарастырылып отырған зерттеулер жүйе материалының реологиялық қасиеттерін ескере отырып жүргізілсе немесе қоршаған ортаның айналасында болса, онда бұл реологиялық қасиеттерді көрсетеді, бұл әдістерді қолдану айтарлықтай қиынға соғады. Мұндай жағдайларда, комплексті фазалық жылдамдықтың реологиялық параметрлері, тербеліс жиілігінің нақты мәндері есептеледі. Бұл жұмыс жазықтық пен дөңгелек элементтердің толқындық процестерінің тұрақтылығының динамикасын зерттеуге арналған, сонымен қатар қабатты, серпімді жазықтық бетіне қозғалатын жүктемелердің әсері туралы жазықтық есептері, сызықты емес деформациялардан болатын кернеулердің заңы қарастырылған. Бұл есептің қолданбалылығы сол динамикалық есептерді шешудің әртүрлі сандық алгоритмдерін жасау үшін қолданылады.

Деформацияланатын ортадағы әртүрлі периодты және периодты емес қозғалысының басты мәні қарапайым гармоникалық типтегі жазықтық толқындары, олардың әсері осы бетке жақын орналасқан. Сондықтан Реле таралу толқынының есебін қарастыра аламыз. Жартылай жазықтықтағы материалдың қозғалыс теңдеуі потенциалда φ,ψ толқын теңдеулерімен сипатталады. Құрылымдарды немесе құрылымдарды жобалау кезінде маңызды шарттардың бірі – құрылымдардың тұрақтылық жағдайы мен элементтері ескеріледі.

Егер ұзындықтың дөңгелек серпімді өзегін қарастыратын болсақ, белгілі бір уақытта штамның ұштарына интенсивтіліктің осьтік сығылатын $P(t)$ күші қолданылады деп болжаймыз. Дөңгелек шыбықтың тұрақтылықты жоғалтуы математикалық теория негізінде және дөңгелек өзектің көлденең тербелісі негізінде зерттеледі [3]. Осы мәселелерге сүйене отырып, оған қалыпты немесе айналмалы ығысу кернеуі қолданылған кезде, қатаң немесе деформацияланатын шекаралармен шектелген серпімді қабаттың тербелісінің кейбір аксиметриялық мәселелерін қарастырамыз. Қарастырылып отырған мәселелердің шешімдері интегралдық түрлендірулердің көмегімен координата және уақыт бойынша алынған.

Түйін сөздер: деформацияланатын дене, ығысу толқыны, тербеліс, цилиндрлік тербеліс, сырық.

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АНАЛИТИЧЕСКОЕ РЕШЕНИЕ ЧАСТНЫХ ЗАДАЧ СДВИГОВЫХ ВОЛН В ЦИЛИНДРИЧЕСКОМ СЛОЕ
(при раскладе постоянных величин $\gamma - \alpha + 2 = 0$ и $\alpha = \beta$)

Аннотация. При исследованиях сдвиговых волновых процессов круговых элементов в деформируемых телах вводится понятие фазовой скорости как скорости изменения фазовой среды. В случае гармонических колебаний цилиндрической оболочки фазовая скорость выражается через частоту собственных колебаний, свободно опертой по краям оболочки, и поэтому исследование волн в цилиндрическом слое имеет самое прямое отношение к проблеме определения собственных форм и частот колебаний оболочек конечной длины. Проводимые в данной работе результаты по одномерным цилиндрическим волнам в упругих и вязкоупругих средах и стержнях позволяют исследовать влияние характеристик материала сред на волновое поле в материале.

Во многих исследованиях для определения характеристик волн обычно поступают двумя методами.

1.Исследуется мгновенное состояние среды, соответствующее некоторому фиксированному моменту времени.

2.Исследуется изменение во времени состояние рассматриваемого тела в некоторой фиксированной точке.

Если исследования проводятся с учетом реологических свойств материала рассматриваемой системы или имеется окружающая систему среда, также в общем случае проявляющая реологические свойства, использование этих способов значительно затруднено. В таких случаях изучается влияние реологических параметров на составляющие комплексной фазовой скорости при определенных значениях частот колебаний. Поэтому работа посвящена изучению динамики устойчивости волновых процессов плоских и круговых элементов, а также рассматривается класс плоских задач о воздействии подвижных нагрузок на поверхность слоистой упругой полуплоскости при нелинейном законе зависимости напряжений от деформаций. Задачи данного класса представляют большой прикладной интерес и, кроме того, могут служить эталоном для разработки тех или иных численных алгоритмов для решения динамических задач.

Среди различных периодических и непериодических движений деформируемых сред важное значение имеют плоские волны простого гармонического типа, распространяющиеся по поверхности тела или полуплоскости, влияние которых ограничивается окрестностью этой поверхности. Поэтому можно рассмотреть задачу о распространении волны Релея.

Если рассмотреть круглый упругий стержень длины, то можем предполагать, что к торцам стержня в какой-либо момент времени прикладывается осевая сжимающая сила интенсивности $P(t)$.Потеря устойчивости круглого стержня будет исследоваться на основе математической теории и поперечного колебания круглого стержня, изложенной в работе [3]. На основе этих задач можно рассмотреть некоторые осесимметричные задачи колебания упругого слоя, ограниченные жесткими или деформируемыми границами при воздействии на него нормального или вращательного касательного напряжения. Решения рассматриваемых задач получены с использованием интегральных преобразований по координате или по времени.

Ключевые слова: деформируемое тело, сдвиговая волна, колебания, цилиндрическая оболочка, стержень, вязкоупругая среда.

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M-FUNCTION NUMBERS: CYCLES AND OTHER EXPLORATIONS. PART 1

Abstract. This paper establishes the cyclic properties of the M-Function, which we define as a function, [M(n)], that takes a positive integer, adds to it the sum of its digits and the number produced by reversing its digits, and then divides the entire sum by three. Our definition of the M-Function is influenced by D. R. Kaprekar's work on a remarkable class of positive integers, called self- numbers, and his procedure, [K(n)], of adding to any positive integer the sum of its digits [1]. We analyze the distribution of numbers that make the defined M-Function behave like a cyclic function, and observe that many such "cycles" form arithmetic sequences. We examine the distribution of numbers that produce integer ratios between the outputs of Kaprekar's and the M-Function functions, [K(n)/M(n)]. We also prove that the set of numbers with equal outputs to both Kaprekar's and M-Function functions, [K(n)=M(n)], is infinite.

Key words: M-Function, D.R.Kaprekar, self-numbers.

1. Introduction. Indian mathematician D.R. Kaprekar is especially known for the discovery of the "Kaprekar Constant." His another prominent work, described by the famous American science writer Martin Gardner in his book "Time travel and other mathematical bewilderments," [1] is called self-numbers, discovered by Kaprekar in 1949. Choose any integer n and add to it the sum of its digits S (n). The resulting number K (n) = n + S (n) is called a *digitaddition*, and the initial number n is called its *generator*. A digitaddition can have several generators. A self-number is a positive integer that does not have a generator. In the "Columbian Numbers" article published by mathematicians Recaman and Bange in "The American Mathematical Monthly" [2] magazine in 1974, it was proven that there are infinitely many self-numbers.

The discoveries that D. R. Kaprekar made engaged not only serious mathematics scholars and researchers, but also astonished many high school students - Kaprekar's core discoveries do not require knowledge of concepts outside of a normal high-school curriculum to understand. Followed by Kaprekar's discoveries, many scientific articles, scientific projects in mathematics, and software products globally examined various new properties of the "Kaprekar Constant" and the sets of self-numbers and digitadditions. There were many math Olympiad problems based on Kaprekar's remarkable class of positive integers and their properties. Thus, Kaprekar's discoveries inspired and drew the attention of mathematicians of many levels.

This paper outlines the investigation of a function similar to Kaprekar's function, a function defined as the M-Function. M-Function takes a positive integer, adds to it the sum of its digits and the number produced by reversing its digits, and then divides the entire sum by three:

For positive integer n → M (n) = (n + S (n) + r (n)) / 3, where r (n) is a number with the digits of n in reverse order. For example, if n = 358, M (358) = (358 + (3+5+8) + 853) / 3 = 409.

In the case of Kaprekar's digitaddition procedure, the inequality n < K (n) holds true for all positive integers n. However, in the case of the M-Function, all three inequality cases are possible: n < M (n), n = M (n), n > M (n). As it turns out, there could be many interesting properties and consistencies that fascinatingly flow out of the defined M-Function. The procedure for obtaining new positive integers n →

M (n) is a simple mathematical operation, yet it can produce fascinating properties. The research also involved a study of relationships between Kaprekar's function and the M-Function, and raises general questions concerning a particular number set (distribution problems) or the mutual relationship of several number sets (such as the coincidence of elements of two sets, multiple ratios of elements of integer sets).

The research outlined in this paper focuses on discovering fundamental mathematical dependencies, properties and theories within Number Theory field. Hence, the openings of the research are valuable to the ever-growing field of Number Theory. Perhaps, its results would have no immediate practical applications, however, as the scientific development progresses forward, there might be a number of applications of M-Functions beyond the fundamental Number Theory in fields like computer science and computational biology.

2. Distribution of *m*-cycles and their properties. Let's give some definitions. Let N be the set of positive integers. For all $n \in N$, let $S(n)$ be the sum of digits of n, $\bar{n}$ be the number produced by reversing n's digits.

Let the number $d(n)$ be the "order" of the number, the quantity of digits of n. Then, for any n such that $n\epsilon N$, the condition $10^{d(n)-1} \leq n < 10^{d(n)}$ is true.

The definition of the M-Function is $M(n) = \frac{1}{3}(n + S(n) + \bar{n})$ and $K(n) = n + s(n)$.

If $n = M(n)$ then n is called a stationary number.

Let l be the least positive integer such that $M^l(n) = n$ for some $n\epsilon N$. Then, the number l is the length of the cycle:

$$n \to M(n) \to M^2(n) \to \ldots \to M^{l-1}(n) \to M^l(n) = n.$$

Table 1 Table of the distribution of *m* – cycles for numbers 1 to 10^{10}

	The length of the cycle (l)																		
$d(n)$	1	2	3	4	5	6	9	10	12	13	15	16	18	19	21	23	24	∑ (total)	%
1	9																	9	100
2	4		1															5	5.556
3	4		3		1													8	$8.89 * 10^{-1}$
4	12	3	2	2				1										20	$2.22 * 10^{-1}$
5	8	1	12			4	2											27	$3 * 10^{-2}$
6	8		9			6							1					24	$2.67 * 10^{-3}$
7	12	10	16			3		17	2			1			1	1	1	64	$7.11 * 10^{-4}$
8	8		12			2	1		1				1		2			27	$3.0 * 10^{-5}$
9	8	7	21			7	4		1		2		3		1		1	55	$6.11 * 10^{-6}$
10	12	2	17	2	1	3	5	3	3	1	2		3	2	1		1	58	$6.44 * 10^{-7}$
∑	85	23	93	4	2	25	12	21	7	1	4	1	8	2	5	1	3	297	

Using a C++ program, we can compute all $m-$ cycles for numbers from 1 to 10^{10}, and make a list of them. Based on this data, we can compose a table of distribution of m-cycles in the set N (Table 1). Their total quantity is 297. Also, we can observe from the table that the length l of m-cycles can be of any value from 1 to 24, except the numbers 7, 8, 11, 14, 17, 20 and 22. m-cycles with length $l = 10$ occur just among the numbers when $d(n) = \{4, 7, 10\}$. As seen from the table, the proportion of m-cycles out of all positive integers in the given order decreases rapidly as d(n) increases. In the table below, the percentage (%) represents the proportion of m-cycles out of all positive integers in the given order $d(n)$. So, among ten-digit numbers (we are considering $9*10^9$ integers), there are 58 m-cycles, which means that the percentage of m-cycles out of all 10-digit positive integers is $6.44*10^{-7}$%.

Conjecture 1. m-cycles with length greater than $l = 24$ don't exist. m-cycles with length $l = 10$ occur only in the numbers that have order $d(n) = 3k + 1$, where $k \epsilon N$.

Considering the table, we might be curious in looking at m – cycles with length $l = 10$: they might possess interesting properties.

Among the four-digit numbers, there is only one m −cycle with length $l = 10$:

$$1297 \to 3079 \to 4267 \to 3970 \to 1594 \to 2188 \to 3673 \to 2485 \to 2782 \to 1891 \to 1297$$

These numbers form an arithmetic progression, where $a_1 = 1297$ - is the initial term, $d = 297$- is the common difference.

Hence:

$1297 = a_1$, $3079 = a_1 + 6d = a_7$, $4267 = a_1 + 10d = a_{11}$,
$3970 = a_1 + 9d = a_{10}$, $1594 = a_1 + d = a_2$, $2188 = a_1 + 3d = a_4$,
$3673 = a_1 + 8d = a_9$, $2485 = a_1 + 4d = a_5$, $2782 = a_1 + 5d = a_6$,
$1891 = a_1 + 2d = a_3$.

If we write in terms of a(n) where a1 = 1297 and the nth term belongs to the arithmetic sequence, then we can write the sequence of the cycle as follows:

$$a_1 \to a_7 \to a_{11} \to a_{10} \to a_2 \to a_4 \to a_9 \to a_5 \to a_6 \to a_3 \to a_1$$

Note that the term a_8= 3376 is not in the sequence, which we can define as a stationary number ($a_8 = M(a_8)$).

We can draw a circle and label points 1-11 on the circle (in order), each equally spaced. We can construct a hendecagon, an eleven-sided polygon, by connecting points in the order of the cycle (that is, connect a1 to a7, then a7 to a11, etc.), excluding a8. We can draw a "mirror" AB with a length of the circle's diameter. We can observe an interesting picture: the mirror is symmetric with respect to the diameter AB, where the endpoint A is 8! (Figure 1).

We can draw another circle, but this time with the order of the cycle in terms of an arithmetic progression (points 1, 7, 11, … , 6, 3), also equally spaced. We can construct two pentagons, connecting every other point for one pentagon, and connecting the remaining points on the other. (Figure 2).

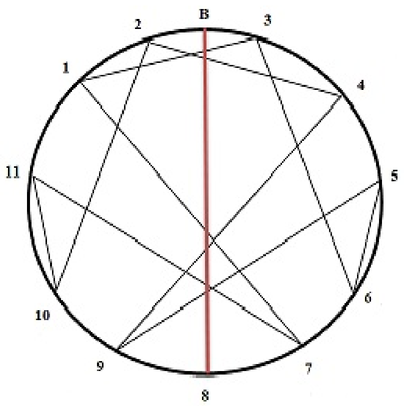

Figure 1

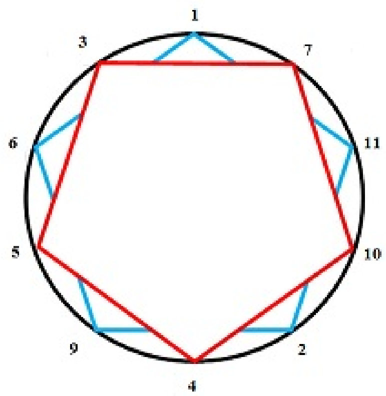

Figure 2

We can sum the vertices of each pentagon, and notice that the sums are equal:

$$1 + 11 + 2 + 9 + 6 = 7 + 10 + 4 + 5 + 3 = 29$$

In terms of the arithmetic sequence, this found property means the following:

$$a_1 + a_{11} + a_2 + a_9 + a_6 = a_7 + a_{10} + a_4 + a_5 + a_3.$$

If we designate numbers by the sequence of the cycle, it will have the following order:

$$u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_6 \to u_7 \to u_8 \to u_9 \to u_{10} \to u_1$$

and the given property will be written as:

$$u_1 + u_3 + u_5 + u_7 + u_9 = u_2 + u_4 + u_6 + u_8 + u_{10}.$$

Now, let's look at m −cycles with $l = 10$ for seven-digit numbers: there are 17 of them. There are only 8 $m -$ cycles where the numbers form an arithmetic progression in some sequence.

1) There are 4 m −cycles with a common difference of $d = 32670$ that are related to each other with a difference of 1000002

$a_1 = 1102982$, a_8, a_9, a_6, a_4, a_{10}, a_3, a_2, a_5, a_7.

+1000002

$b_1 = 2102984$, b_8, b_9, b_6, b_4, b_{10}, b_3, b_2, b_5, b_7.

+1000002

$c_1 = 3102986$, c_8, c_9, c_6, c_4, c_{10}, c_3, c_2, c_5, c_7.

+1000002

$e_1 = 4102988$, e_8, e_9, e_6, e_4, e_{10}, e_3, e_2, e_5, e_7.

Where $b_i = a_i + 1000002$, $c_i = a_i + 2000004$, $e_i = a_i + 3000006$, $i = \overline{1,10}$. Note that 1000002, 2000004, 3000006 are stationary numbers!

2) There are also 4 m −cycles with a common difference of $d = 27270$:

$f_1 = 1127282$, f_8, f_9, f_6, f_4, f_{10}, f_3, f_2, f_5, f_7.

+1000002

$g_1 = 2127284$, g_8, g_9, g_6, g_4, g_{10}, g_3, g_2, g_5, g_7.

+1000002

$h_1 = 3127286$, h_8, h_9, h_6, h_4, h_{10}, h_3, h_2, h_5, h_7.

+1000002

$j_1 = 4127288$, j_8, j_9, j_6, j_4, j_{10}, h_3, j_2, j_5, j_7.

Where $g_i = f_i + 1000002$, $h_i = f_i + 2000004$, $j_i = f_i + 3000006$, $i = \overline{1,10}$.

All 8 m-cycles of length 10 form a similar arithmetic sequence:

$$a_1 \to a_8 \to a_9 \to a_6 \to a_4 \to a_{10} \to a_3 \to a_2 \to a_5 \to a_7 \to a_1 .$$

Placing points 1-10 around the circle in order and connecting the points in the order of the arithmetic sequence, we get another symmetry with respect to diameter AB (Figure 3). The endpoints of the diameter, B and A, are centered in the midpoint of an arc between 5 and 6, and between 10 and 1, respectively. We can draw another circle, but this time putting the points in the order of the arithmetic sequence terms. We get the following circle with 10 "slices" (Figure 4).

We see that the sum of diametrically opposite 2 numbers is always equal to

$$1 + 10 = 8 + 3 = 9 + 2 = 6 + 5 = 4 + 7 = 11.$$

If we consider the sequence in terms of a cycle rather than in terms of an arithmetic progression, then we can describe it as follows:

$$u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_6 \to u_7 \to u_8 \to u_9 \to u_{10} .$$

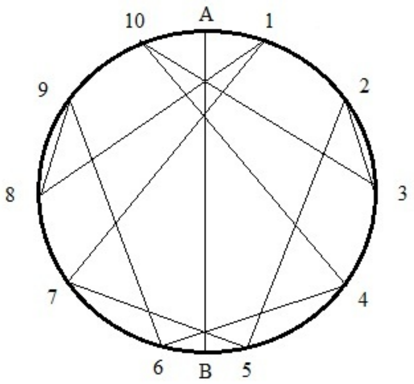

Figure 3

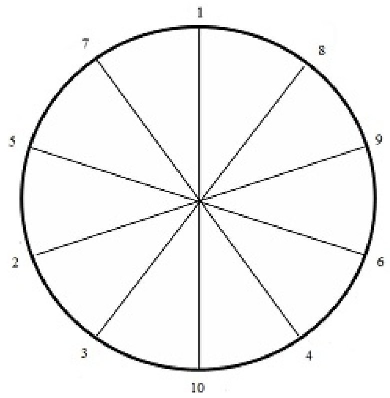

Figure 4

The observed pattern means that for all 8 m-cycles, the following equation holds true:

$$u_1 + u_6 = u_2 + u_7 = u_3 + u_8 = u_4 + u_9 = u_5 + u_{10}.$$

We verified the aforementioned statement algebraically.

3. Solutions to the equation K(n) = M(n). Let's investigate the following question: for what values of n, where $n \in N$, will the output of Kaprekar's function $K(n)$ be equal to the output of the M-Function $M(n)$? Let $K(n) = M(n)$ for some $n \in N$. Then, we can simplify the equation:

$$n + S(n) = \frac{1}{3}(n + S(n) + \bar{n}),$$

$$3n + 3S(n) = n + S(n) + \bar{n}\,.$$

Here we get the equation

$$2n = -2 * S(n) + \bar{n}\,. \qquad \text{(Equation 1)}$$

By solving Equation 1 using a C++ program, we obtain the following results for numbers up to 10^{10}:

27	24894	450009	24600294	450000009
459	45009	2460294	45000009	2460000294
4509	246294	4500009	246000294	4500000009

As evident in the list of solutions, starting with 6-digit numbers up until 10 digits, there are only 2 solutions for each order $d(n)$, having the following types:

$$\alpha_d = 246\underbrace{0\,...\,0}_{d-6}294, \qquad \beta_d = 45\underbrace{0\,...\,0}_{d-3}9\,.$$

Proposition 1. For all integer values $n \in N$ of $d(n)$ where $d(n) > 10$, numbers α_d and β_d satisfy Equation 1. Hence, there is an infinite number of solutions to the equation K (n) = M (n).

Proof. 1). We can express a solution from the type α_d in general form as follows:

$$\alpha_d = 246\underbrace{0\,...\,0}_{d-6}294.$$

We can multiply it by 2, and get the following:

$$2\alpha_d = 492\underbrace{0\,...\,0}_{d-6}588.$$

Then, we know that $2 * S(\alpha_d) = 2(2 + 4 + 6 + 2 + 9 + 4) = 54$,

and $\overline{\alpha_d} = 492\underbrace{0\,...\,0}_{d-6}642.$

Recall the simplified equation above, and substitute n with α_d :

$$2\alpha_d = -2 * S(\alpha_d) + \overline{\alpha_d}.$$

Algebraically, we show that the equality holds true and hence, we prove Proposition 1.

Consequently, $K(\alpha_d) = M(\alpha_d)$ when $d(n) > 10$, which means we are convinced that there are infinitely many solutions to the equation K(n) = M(n).

2) We can express a solution from the type β_d in general form as follows:

$$\beta_d = 45\underbrace{0 \dots 0}_{d-3}9.$$

We can multiply it by 2, and get the following:

$$2\beta_d = 9\underbrace{0 \dots 0}_{d-3}18.$$

Then, we know that

$$2 * S(\beta_d) = 2(4 + 5 + 9) = 36,$$

and

$$\overline{\beta_d} = 9\underbrace{0 \dots 0}_{d-3}54.$$

Again, recall the simplified equation, and substitute n with β_d :

$$2\,\beta_d = -2 * S(\beta_d) + \overline{\beta_d}.$$

Algebraically, we show that the equality holds true and hence, we prove Proposition 1.

Consequently, $K(\beta_d) = M(\beta_d)$ for $d(n) > 10$, which is another evidence that there are infinitely many solutions to the equation K(n) = M(n).

Conjecture 2. Equation 1 does not have any other solutions except α_d and β_d when $d(n) > 10$.

4. Arithmetic progression with $\boldsymbol{n}$, $\boldsymbol{K(n)}$, $\boldsymbol{M(n)}$ terms. We are intrigued to know for what $n \in N$ do numbers $n,\ K(n),\ M(n)$ form an arithmetic progression in a certain order. Since $n < K(n)$ for all $n \in N$, there are only 3 distinct orders possible to form an increasing arithmetic progression.

4.1. Let $n,\ K(n),\ M(n)$ be the order of an arithmetic progression. Then, by definition

$$n + M(n) = 2K(n),$$

which we can simplify to

$$n + \frac{1}{3}(n + S(n) + \bar{n}) = 2\big(n + S(n)\big).$$

Algebraically rearranging the equation, we can express it as follows:

$$2n = -5 * S(n) + \bar{n}\ . \qquad \text{(Equation 2)}$$

Using a C++ program, we found the following solutions to Equation 2 for numbers up to 10^{10}:

18	15003	186273	15000003	186000273
153	18873	1500003	18600273	1500000003
1503	150003	1860273	150000003	1860000273

Proposition 2. All numbers in the form $a_d = 15\underbrace{0 \dots 0}_{d-3}3$ and $b_d = 186\underbrace{0 \dots 0}_{d-6}273$, where $d(n) > 10$, satisfy Equation 2.

Hence, the numbers $a_d, K(a_d),\ M(a_d)$ and $b_d, K(b_d),\ M(b_d)$ form arithmetic progressions.

The proof is similar to the proof of Proposition 1: we recall the simplest form of the equation, plug in the values for a_d and b_d, and show the proof algebraically.

4.2. Let $n,\ M(n),\ K(n)$ be the order of an arithmetic progression. Then, by definition

$$n + K(n) = 2 * M(n),$$

which we can simplify to

$$n + \big(n + S(n)\big) = \frac{2}{3}(n + S(n) + \bar{n}).$$

Algebraically rearranging the equation, we can express it as follows:

$$4n = -S(n) + 2\bar{n}\ . \qquad \text{(Equation 3)}$$

Using a C++ program, we found numbers that satisfy Equation 3 for numbers up to 10^{10}:

387	36027	3600027	360000027
3627	360027	36000027	3600000027

Proposition 3. All numbers in the form $c_d = 36\underbrace{0 \ldots 0}_{d-4}27$, where $d(n) > 10$, satisfy Equation 3. Hence, the numbers $c_d, M(c_d),\ K\ (c_d)$ form an arithmetic progression.

Again**, the proof** is similar to the proof of Proposition 1.

4.3. Let *M(n), n, K(n)* be the order of an arithmetic progression. Then, by definition

$$M(n) + K(n) = 2n.$$

which we can simplify to

$$\frac{1}{3}(n + S(n) + \bar{n}) + \big(n + S(n)\big) = 2n.$$

Algebraically rearranging the equation, we can express it as follows:

$$2n = 4S(n) + \bar{n}\ . \qquad \text{(Equation 4)}$$

Using a C++ program, we found solutions for Equation 4 for numbers up to 10^{10}:

45	2124	42048	420048	2670435	21000024	47700459	420000048	2670000435
234	4248	210024	477459	4200048	26700435	210000024	477000459	4200000048
468	21024	267435	2100024	4770459	42000048	267000435	2100000024	4770000459

Proposition 4. When $d(n) > 10$, all numbers of the following types

$$\mathrm{e}_d = 21\underbrace{0 \ldots 0}_{d-4}24\ , \qquad \mathrm{f}_d = 267\underbrace{0 \ldots 0}_{d-6}435\ ,$$

$$\mathrm{g}_d = 42\underbrace{0 \ldots 0}_{d-4}48\ , \qquad \mathrm{h}_d = 477\underbrace{0 \ldots 0}_{d-6}459$$

are solutions to Equation 4.

Consequently, the following numbers:

$\mathrm{M}(\mathrm{e}_d),\ e_{d,}\ \mathrm{K}\,(\mathrm{e}_d)$; $\quad \mathrm{M}(\mathrm{f}_d),\ f_{d,}\ \mathrm{K}\,(\mathrm{f}_d)$; $\quad \mathrm{M}(\mathrm{g}_d),\ g_{d,}\ \mathrm{K}\,(\mathrm{g}_d)$;

$\mathrm{M}(\mathrm{h}_d),\ h_d,\ \mathrm{K}\,(\mathrm{h}_d)$ - all form an arithmetic progression when $d(n) > 10$.

The proof is similar to the proof of Proposition 1.

According to our results for equations 2, 3, and 4, we can devise the following conjecture:

Conjecture 3. When $d(n) > 10$, equations 2, 3, and 4 don't have any other solutions except solutions specified in propositions 2, 3, 4.

С. Мақышов

М-ФУНКЦИЯ САНДАРЫ: ЦИКЛДАР ЖӘНЕ БАСҚА ЗЕРТТЕУЛЕР

Аннотация. Индиялық математик Д.Р. Капрекар ашқан "Капрекар Константасы" - 6174 санымен аса танымал.

Капрекардың тағы бір ашқан жаңалығы өзіндік туындаған сандар класы белгілі американдық ғылым насихаттаушысы Мартин Гарднердің «Уақыт бойынша саяхат» [1] атты кітабында баяндалған. Кез келген натурал n санын аламыз және ол санға цифрларының қосындысы $S(n)$−ді қосамыз. Шыққан сан $K(n)$=n+$S(n)$ туындаған сан, ал алғашқы сап n – оның генераторы деп аталады. Мысалы, егер 53 санын алсақ, онда туындаған сан 53+3+5=61 саны болады.

Туындаған санның генераторларының саны бірден артық болуы мүмкін. Екі генераторы бар ең кіші сан 101, ал оның генераторлары 91 және 100 сандары. Өзіндік туындаған сандар – генераторлары жоқ сандар. «The American Mathematical Monthly» [2] журналында жарияланған мақалада өзіндік туындаған сандардың шексіз көп екендігі және өзіндік туындаған сандар туындаған сандарға қарағанда өте сирек кездесетіндігі дәлелденген.

Капрекар ашқан осы жаңалықтар көптеген математиктерді қызықтырды. Әртүрлі елдерде «Капрекар Константасының», өзіндік туындаған және туындаған сандары жиындарының жаңа қасиеттерін жан-жақты зерттеген көптеген мақалалар, математикалық ғылыми жобалар мен программалық өнімдер жарық көрді.

Мен Капрекарға сүйене отырып, натурал сандарды алудың жаңа әдісін таптым: n→$M(n)$=13(n+$S(n)$+$\overline{n}$), мұндағы $\overline{n}$ – сол цифрлармен, бірақ кері бағытта жазылған сан. $M(n)$ саны әрқашан бүтін сан болады, себебі n,$S(n)$,$\overline{n}$ сандарының 3-ке бөлгендегі қалдықтары әрқашан тең болады. Егер Капрекар жағдайында n<$K(n)$ теңсіздігі кез келген натурал n сандарында орындалса, менің құрған функциямда әртүрлі қатынастар болады, яғни барлық 3 жағдай да орын алады : n<$M(n)$, n=$M(n)$ и n>$M(n)$.

Мен тапқан жаңа натурал сан алу функциясы n→$M(n)$ әрі қарапайым, табиғи және Капрекар n→$K(n)$ функциясының аналогы.

Мақаланың 1-бөлімінде m-циклдарының таралуы және олардың қасиеттері зерттеледі (егер $Ml(n)$=n теңдігі орындалатындай ең кіші натурал сан l болса, онда n→$M(n)$→$M2(n)$→ ...→Ml−1(n)→ $Ml(n)$→n сандары m-циклді құрайды. Ал $K(n)$ жағдайында циклдар туындамайды, себебі . n<$K(n)$<$K2(n)$….). Сонымен қатар $K(n)$=$M(n)$ функцияларының теңдігі сұрағы және n, $K(n)$ және $M(n)$ сандарының кандай да бір ретпен арифметикалық прогрессия құрайтын сұрақтары зерттелген.

Мақаланың 2-бөлімінде n,$K(n)$ және $M(n)$ сандарының арасындағы еселік қатынастар қарастырылған. Яғни, қандай натурал t сандарында, t≥2, $K(n)$=$tM(n)$, $tK(n)$=$M(n)$, n=t $M(n)$, n·t=$M(n)$ теңдіктері орындалатындығы зерттелген (айта кетейік, n және $K(n)$ сандары арасында еселік қатынастар болуы мүмкін емес). Сонымен қатар m−туындаған сандар жиынының таралуы және қасиеттері зерттелген (m−туындаған сандар m−өзіндік туындаған сандарға қарағанда өте сирек кездеседі. Сондықтан m−туындаған сандар жиынын зерттеу m−өзіндік туындаған сандар жиынына қарағанда маңыздырақ). Осы бөлімде "көрші", яғни қатарлас тұрған m−туындаған сандар жиыны зерттелген.

Зерттеу барысында 1 мәселе және 9 гипотеза тұжырымдалған.

Түйін сөздер: М-функция, Д.Р. Капрекар, өзіндік туындаған сандар.

С. Макышов

ЧИСЛА М-ФУНКЦИИ: ЦИКЛЫ И ДРУГИЕ ИССЛЕДОВАНИЯ

Аннотация. Индийский математик Д.Р. Капрекар особенно известен своим открытием "Константу Капрекара"- числом 6174. Другое выдающееся открытие Капрекара, описанное известным американским популяризатором науки Мартином Гарднером в своей книге "Путешествие во времени"[1] – это класс самопорожденных чисел. Выберем любое натуральное число n и прибавим к нему сумму его цифр $S(n)$. Полученное число $K(n) = n + S(n)$ называется *порожденным*, а исходное число n – его *генератором*. Например, если выберем число 53, порожденное им число равно $53 + 3 + 5 = 61$.

Порожденное число может иметь более одного генератора. Наименьшее число с двумя генераторами равно 101, и его генераторами являются числа 91 и 100. Самопорожденное число – это число, у которого нет генератора. В статье журнала «The American Mathematical Monthly»[2] доказывалось, что существует бесконечно много самопорожденных чисел, но встречаются они гораздо реже, чем порожденные числа.

Эти открытия Капрекара заинтересовали многих математиков, и в разных странах мира появились много научных статьей, научных проектов по математике, программных продуктов, в которых исследовались различные новые свойства “Константы Капрекара” и множеств самопорожденных чисел и порожденных чисел.

Следуя Капрекару, я нашел новый способ получения натуральных чисел: $n \to M(n) = \frac{1}{3}(n + S(n) + \bar{n})$, где $\bar{n}$ – число, записанное теми же цифрами, но в обратном порядке. Число $M(n)$ будет всегда целым, так как числа $n, S(n), \bar{n}$ дают одинаковые остатки при делении на 3. Если в случае Капрекара неравенство $n < K(n)$ выполняется при всех натуральных n, то в моем случае положение разнообразнее, т.е. возможны все 3 случая: $n < M(n)$, $n = M(n)$ и $n > M(n)$.

Моя функция получения новых натуральных чисел $n \to M(n)$ простая, естественная, и она является аналогом функции Капрекара $K(n)$. В 1-й части данной статьи исследованы распределение m-циклов и их свойства. (Если l – наименьшее натуральное такое, что $M^l(n) = n$, то числа $n \to M(n) \to M^2(n) \to \ldots \to M^{l-1}(n) \to M^l(n) \to n$ образуют m-цикл. В случае функции $K(n)$ циклы невозможны, т.к. $n < K(n) < K^2(n)$ ....). Также изучен вопрос равенства чисел $K(n) = M(n)$ и вопрос образования арифметической прогрессии в некотором порядке числами n, $K(n)$ и $M(n)$.

Во 2-й части статьи изучены кратные отношения между числами n, $K(n)$ и $M(n)$. Т.е. исследованы вопросы: при каких t натуральном, $t \geq 2$, возможны равенства $K(n) = tM(n)$, $tK(n) = M(n)$, $n = t\,M(n)$, $n * t = M(n)$. (Отметим, что кратные отношения между числами n и $K(n)$ невозможны). Также исследованы распределение и свойства множества $m-$порожденных чисел ($m-$порожденных чисел встречаются гораздо реже, чем $m-$ самопорожденные, поэтому изучение множества $m-$порожденных намного важнее, чем изучение класса $m-$ самопорожденных чисел). В этой части исследовано множество “соседних”, т.е. последовательных $m-$порожденных чисел.

В процессе исследования сформулированы 1 проблема и 9 гипотез.

Ключевые слова: М-функция, Д.Р.Капрекар, самопорожденные числа.

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INVESTIGATION OF TWO FIXED CENTERS PROBLEM AND HENON-HEILES POTENTIAL BASED ON THE POINCARE SECTION

Abstract. In this paper, we study the Henon-Heiles potential and the problem of two fixed centers. In studies of nonlinear systems for which exact solutions are unknown, the Poincare section method is used. For the Henon-Heiles potential, Poincare sections were obtained. Next, the potential of two fixed centers was investigated. It was shown on the basis of the Poincare section that, in the case $\mu_1 = \mu_2 = 1$ the internal cross-sectional structure decomposes from the values $H = -1.7$, but the internal cross-sectional structure is preserved in the interval $H \in [-0.5, -1.6]$, in the case $\mu_1 = 0.9$ and $\mu_2 = 0.1$ the internal cross-sectional structure decomposes from the values $H = -0.9$ but the internal cross-sectional structure is preserved in the interval $H \in [-0.3, -0.8]$, in the case of $\mu_1 = 0.7$ and $\mu_2 = 0.3$ the internal cross-sectional structure decomposes from the values $H = -0.8$, but the internal cross-sectional structure is preserved in the interval $H \in [-0.2, -0.7]$. With increasing energy, many of these surfaces decay. It is assumed that the numerical results obtained will serve as the basis for comparison with analytical solutions.

Keywords: Henon-Heiles model, the problem of two fixed centers, Poincaré section, numerical solutions.

Introduction. Interest in the existence of the third integral of motion for stars moving in the potential of the galaxy revived in the late 50's and early 60's of the last century. Initially it was assumed that the potential has a symmetry and does not depend on time, therefore in cylindrical coordinates (r, θ, z) this will be only a function of r and z. There must be five integrals of motion that are constant for the six-dimensional phase space. However, the integrals can be either isolating or non-isolating. Non-isolating integrals usually fill all available phase spaces and do not restrict the orbit.

Henon and Heiles tried to find out if they could find any real proof that there must be a third isolating integral of the motion. Making numerical calculations, they did not complicate the astronomical meaning of the problem; they only demanded that the potential investigated by them be axially symmetric. The authors also suggested that the motion was tied to a plane and passed into the Cartesian phase space ($x, y, \dot{x}, \dot{y}$). After some tests they managed to find a real potential. This potential is analytically simple, so that the orbits can be calculated quite easily, but it is still quite complex, so that the types of orbits are nontrivial. This potential is now known as the potential of Henon and Heiles [1-3].

Some particular solutions to the three-body problem are known, but a general solution has not yet been found. One of the special cases of the three-body problem is the problem of two fixed centers. It was first considered by Euler in 1760 [4]. Jacobi showed that the equations of motion can be integrated in terms of elliptic functions [5]. This problem can be used as some first approximation in astronomical problems about the motion of minor planets and comets under the influence of gravity of the Sun and Jupiter. The period of revolution of Jupiter is about twelve years, and for a short period of time the motion of these celestial bodies can be considered in the framework of the problem of two fixed centers. Also, the problem of the motion of a spacecraft to the Moon can be considered within the framework of this task. The flight time of the spacecraft to the Moon is about four days. During this time, the Moon will move slightly in a circular orbit of the Earth. The study of the problem of two fixed centers was carried out in different directions [6-22]. For example, V.V. Kozlov and A.O. Harin considered a modification of the problem of two fixed centers on a sphere [23].

Methods and calculations. The Henon-Heiles potential is undoubtedly one of the simplest, classical and characteristic examples of open Hamiltonian systems with two degrees of freedom. The above topic was devoted to a large number of research scientists [24-26].

The potential of the Henon-Heiles system is determined by the formula:

$$U(x,y)=\frac{1}{2}(x^2+y^2+2x^2y-\frac{2}{3}y^3) \tag{1}$$

Equation (1) shows that the potential actually consists of two harmonic oscillators, which were connected by the perturbing terms $x^2y-\frac{1}{3}y^3$.

The basic equations of motion for a test particle with a unit mass $(m=1)$ are:

$$\begin{cases} \ddot{x}=-\frac{\partial U}{\partial x}=-x-2xy \\ \ddot{y}=-\frac{\partial U}{\partial y}=-y-x^2+y^2 \end{cases} \tag{2}$$

Consequently, the Hamiltonian of system (1) has the form:

$$H=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}\left(x^2+y^2\right)+x^2y-\frac{1}{3}y^3=h, \tag{3}$$

where $\dot{x}$ and $\dot{y}$ are the momenta per unit mass, x and y are the coordinates of the system; $h>0$ the numerical value of the Hamiltonian, which is conserved. It is seen that $h>0$ the Hamiltonian is symmetric with respect to $x \to -x$, and H also exhibits a symmetry of rotation at $2\pi\ /\ 3$.

Below are the dependencies of the coordinates of the functions in time for the systems of equations (2).

To study the Henon-Heiles system, the Poincaré section method is used. Advantages of this method are especially evident when we consider nonlinear systems for which exact solutions are unknown. In this case, the phase trajectories are calculated by numerical methods.

To solve the systems of equations (2), boundary conditions are chosen so that they satisfy equation (3). Further, the systems of equation (2) are solved on the basis of the Runge-Kutta method. To construct the Poincaré section, those values that intersect the plane $x=0$ are chosen. Below are the Poincaré sections for Henon-Heiles systems for different energy values: $E=1/12$, $E=1/8$. With increasing energy, the structure of the cross sections is destroyed. The results obtained are in agreement with other authors [1, 2].

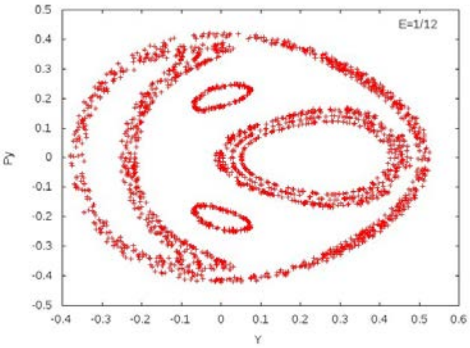

Figure 1 - Poincare section at $E=1/12$

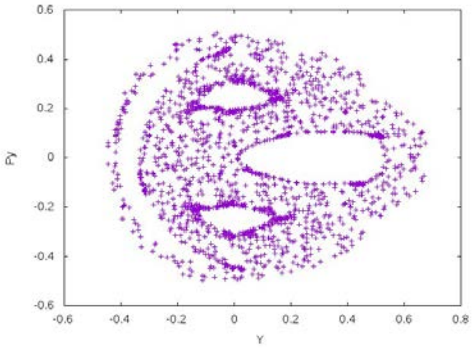

Figure 2 - Poincare section at $E=1/8$

Next, we study the problem of two fixed centers. Imagine that on the OXY plane there are two fixed points S1 and S2 with masses m1 and m2 under the influence of Newtonian attraction of which the material point S of mass m moves in the same plane. Thus, the equations of motion of a material point can be written in the following form [27]:

$$\begin{cases} \ddot{x} = \dfrac{\partial U}{\partial x} = -fm_1 \dfrac{x}{r_1^3} - fm_2 \dfrac{x}{r_2^3}, \\ \ddot{y} = \dfrac{\partial U}{\partial y} = -fm_1 \dfrac{y-c}{r_1^3} - fm_2 \dfrac{y+c}{r_2^3}, \end{cases} \tag{4}$$

Where $U = f(\frac{m_1}{r_1} + \frac{m_2}{r_2})$, f is gravitational constant.

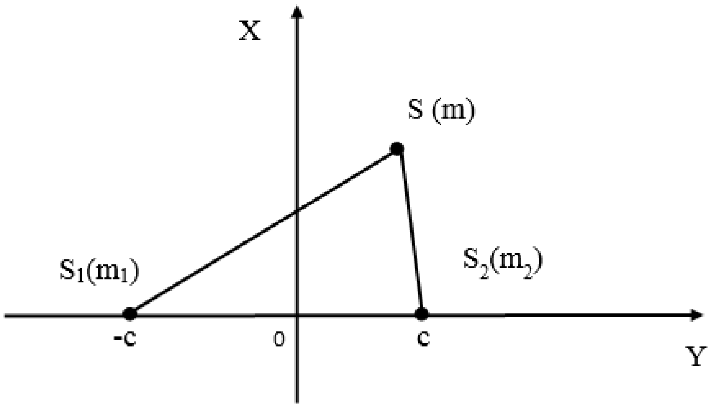

Figure 3 - Scheme of the task

Radius vectors are defined as follows:

$$r_1 = \sqrt{x^2 + (y-c)^2}, \ r_2 = \sqrt{x^2 + (y+c)^2} \tag{5}$$

The canonical equations of the problem of two fixed centers will have the form [28]:

$$\begin{cases} \dfrac{dx}{dt} = +\dfrac{\partial H}{\partial \dot{x}}, & \dfrac{dx}{dt} = +\dfrac{\partial H}{\partial \dot{y}} \\ \dfrac{d\dot{x}}{dt} = -\dfrac{\partial H}{\partial x}, & \dfrac{d\dot{y}}{dt} = -\dfrac{\partial H}{\partial y} \end{cases} \tag{6}$$

where the Hamiltonian is defined by the formula

$$H = T - U = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - f(\frac{m_1}{r_1} + \frac{m_2}{r_2}), \ H = const \tag{7}$$

We introduce the following notation: $\mu_1 = fm_1$, $\mu_2 = fm_2$. Consider the case $\mu_1 = \mu_2 = 1$, the second case $\mu_1 = 0.9$ and $\mu_2 = 0.1$, the third case $\mu_1 = 0.7$ and $\mu_2 = 0.3$. These parameters show different mass ratios of fixed centers. Now we study the Poincare section for the indicated model of the problem and parameters. Based on the results obtained, we can say that, in the case $\mu_1 = \mu_2 = 1$ the internal cross-sectional structure decomposes from the values $H = -1.7$, but the internal cross-sectional structure is preserved in the interval $H \in [-0.5, -1.6]$, in the case $\mu_1 = 0.9$ and $\mu_2 = 0.1$ the internal cross-sectional structure decomposes from the values $H = -0.9$, but the internal cross-sectional structure is preserved in the interval $H \in [-0.3, -0.8]$, in the case of $\mu_1 = 0.7$ and $\mu_2 = 0.3$ the internal cross-sectional structure decomposes from the values $H = -0.8$, but the internal cross-sectional structure is preserved in the interval $H \in [-0.2, -0.7]$.

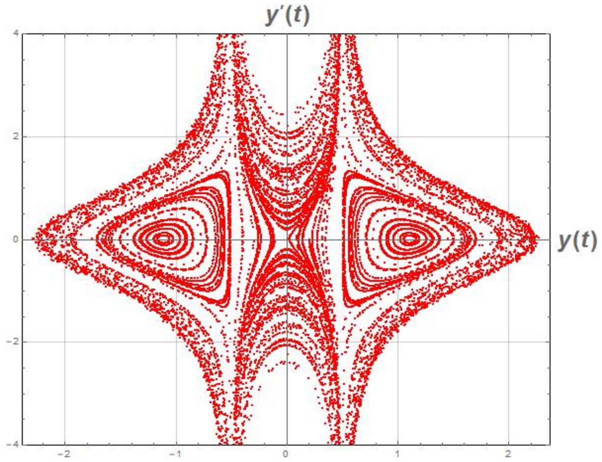

Figure 4 – Poincare section at $H = -0.9$, $c = 0.5$, $\mu_1 = 1.0$, $\mu_2 = 1.0$

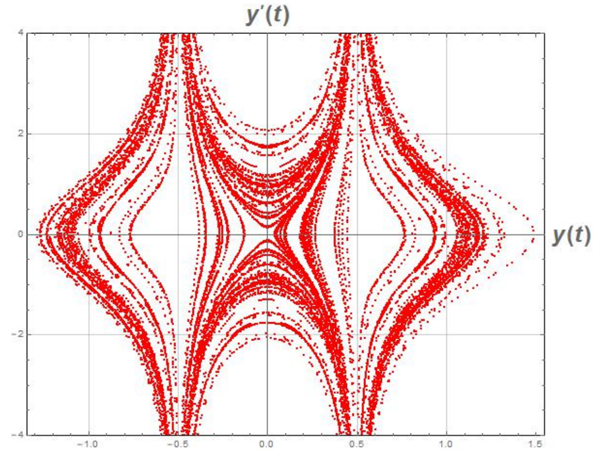

Figure 5 – Poincare section at $H = -1.7$, $c = 0.5$, $\mu_1 = 1.0$, $\mu_2 = 1.0$

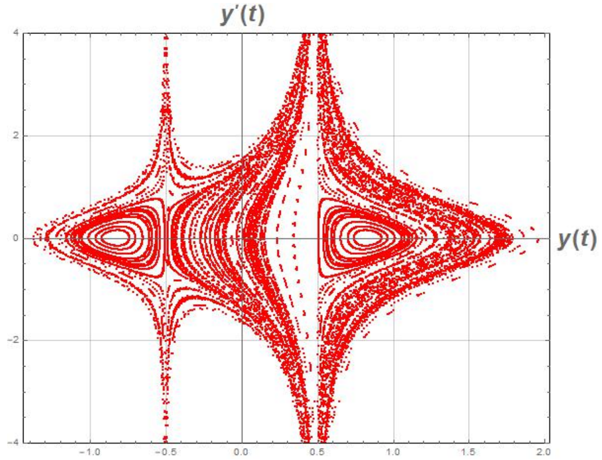

Figure 6 – Poincare section at $H = -0.6$, $c = 0.5$, $\mu_1 = 0.9$, $\mu_2 = 0.1$

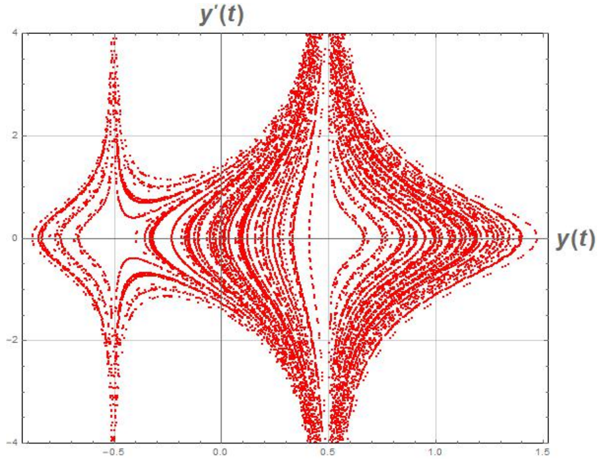

Figure 7 – Poincare section at $H = -0.9$, $c = 0.5$, $\mu_1 = 0.9$, $\mu_2 = 0.1$

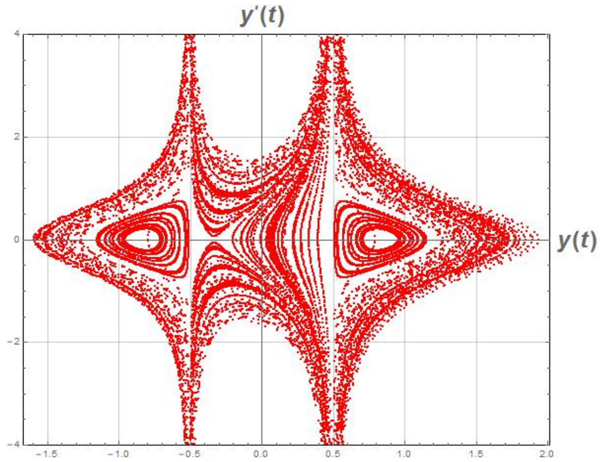

Figure 8 – Poincare section at $H = -0.6$, $c = 0.5$, $\mu_1 = 0.7$, $\mu_2 = 0.3$

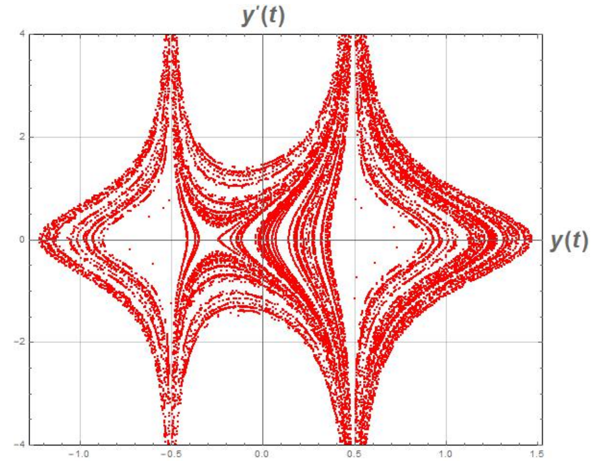

Figure 9 – Poincare section at $H = -0.8$, $c = 0.5$, $\mu_1 = 0.7$, $\mu_2 = 0.3$

Conclusion. Thus, the results obtained by the numerical method determine the structure of the Poincare sections for the model of the problem of two fixed centers and serve as the basis for comparative analysis in determining the analytical mapping.

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ПУАНКАРЕ ҚИМАСЫНЫҢ НЕГІЗІНДЕ ҚОЗҒАЛМАЙТЫН ЕКІ ЦЕНТР ЕСЕБІ МЕН ХЕНОН-ХЕЙЛЕС ПОТЕНЦИАЛЫН ЗЕРТТЕУ

Аннотация. Өткен ғасырдың 50-жылдарының аяғы мен 60-жылдарының басында галактика потенциалында қозғалатын жұлдыздар үшін үшінші интегралына қызығушылық туындай бастады. Бастапқыда потенциал симметриалы және уақытқа тәуелсіз деп қарастырылды, сондықтан цилиндрлік координатада (r, θ, z) функция тек r мен z-қа ғана тәуелді болады. Алты өлшемді фазалық кеңістікте тұрақты бес қозғалыс интергалы болуы керек. Бірақ, интегралдар шектелген немесе шектелмеген болуы қажет. Әдетте, шектелмеген интегралдар барлық фазалық кеңістікті толтырады және орбитаны шектемейді.

Хенон мен Хейлес үшінші шектелген қозғалыс интегралының бар екендігіне нақты дәлелдер табуға тырысты. Сандық есептеулер жүргізе отыра, олар бұл проблеманың астрономиялық мағынасын жеңілдетуге тырысты; олар зерттеліп отырған потенциалдың аксиальді-симметриялы болуын талап етті. Сонымен қатар авторлар, бұл қозғалыс жазықтыққа тәуелді және декарттық фазалық жазықтықта (x, y, $\dot{x}, \dot{y}$) жатады деп тапты. Бірнеше тәжірибелерден кейін, нақты потенциалды таба алды. Бұл потенциал аналитикалық тұрғыдан қарапайым, сондықтан орбиталарды анықтауға болады, сонымен қатар потенциал жеткілікті түрде қиын, сондықтан орбиталар тривиалды емес түрге жатады. Қазіргі таңда бұл потенциал Хенон-Хейлос потенциалы деп аталады.

Үш дене есебінің кейбір дербес шешімдері анықталған, бірақ толық шешімі жоқ. Үш дене есебінің дербес шешімдерінің бірі – қозғалмайтын екі центр есебі. Бұл есепті алғаш рет 1760 жылы Л. Эйлер қарастырды. Ал Якоби болса, қозғалыс теңдеулері эллиптикалық функциялар терминдерінде интегралданатынын көрсетті. Берілген есеп кейбір кіші планеталар мен кометалардың Күн және Юпитер гравитациясындағы қозғалысы жайлы астрономиялық есептерде бірінші жуықтауда қолданылады. Юпитердің Күнді айналу периоды 12 жылға жуық және осы уақыт аралығында кометалар мен кіші планеталардың қозғалысын қозғалмайтын екі центр есебі ретінде алуға болады. Берілген есепте, ғарыш кемесінің Айға ұшу қозғалысын қарастыруға болады. Ғарыштық кеменің Айға ұшу уақыты – 4 тәулікке жуық. Олай болса, осы уақытта Ай Жердің орбитасында кішкене ғана қозғалады. Қозғалмайтын екі центр есебі бірнеше бағытта зерттелген болатын.

Берілген мақалада Хенон-Хейлес потенциалы мен қозғалмайтын екі центр есебі қарастырылады. Сызықты емес жүйелердің нақты шешімдері белгісіз болғанда, Пуанкаре қима әдісі қолданылады. Хенон-Хейлес потенциалы үшін Пуанкаре қимасы алынды. Сонымен қатар қозғалмайтын екі центр есебі зерттелді. Пуанкаре қимасының негізінде қозғалмайтын екі центр есебіне келесідей тұжырымдамалар алынды: $\mu_1 = \mu_2 = 1$ кезінде $H = -1.7$ мәнінен бастап ішкі қима ыдырайды, ал $H \in [-0.5, -1.6]$ аралығында ішкі қима сақталады; $\mu_1 = 0.9$ және $\mu_2 = 0.1$ кезінде $H = -0.9$ мәнінен бастап ішкі қима ыдырайды, ал $H \in [-0.3, -0.8]$ аралығында ішкі қима сақталады; $\mu_1 = 0.7$ және $\mu_2 = 0.3$ кезінде $H = -0.8$ мәнінен бастап ішкі қима ыдырайды, ал $H \in [-0.2, -0.7]$ аралығында ішкі қима сақталады. Сонымен қатар, энергияның өсуімен, осы қималардың көпшілігі ыдырайды. Алынған сандық нәтижелер аналитикалық шешімдермен салыстыру үшін негіз болады деп болжануда.

Түйін сөздер: Хенон-Хейлес моделі, қозғалмайтын екі центр есебі, Пуанкаре қимасы, сандық шешімдер.

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ИССЛЕДОВАНИЕ ЗАДАЧИ ДВУХ НЕПОДВИЖНЫХ ЦЕНТРОВ И ПОТЕНЦИАЛА ХЕНОНА-ХЕЙЛЕСА НА ОСНОВЕ СЕЧЕНИЯ ПУАНКАРЕ

Аннотация. Интерес к существованию третьего интеграла движения для звезд, движущихся в потенциале галактики, возродился еще в конце 50-х и начале 60-х годов прошлого столетия. Первоначально предполагалось, что потенциал имеет симметрию и не зависит от времени, поэтому в цилиндрических координатах (r, θ, z) это будет только функция от r и z. Должны существовать пять интегралов движения, постоянных для шестимерного фазового пространства. Однако интегралы могут быть либо изолирующими, либо неизолирующими. Неизолирующие интегралы обычно заполняют все доступные фазовые пространства и не ограничивают орбиту.

Хенон и Хейлес попытались выяснить, могут ли они найти какое-либо реальное доказательство того, что должен существовать третий изолирующий интеграл движения. Проводя численные вычисления, они не слишком усложняли астрономический смысл проблемы; они требовали только, чтобы исследованный ими потенциал был аксиально-симметричным. Авторы также предположили, что движение привязано к плоскости и перешли в декартово фазовое пространство (x, y, $\dot{x}, \dot{y}$). После некоторых испытаний им удалось найти действительный потенциал. Этот потенциал аналитически прост, так что орбиты можно вычислить довольно легко, но он все еще достаточно сложный, так что типы орбит нетривиальны. Этот потенциал теперь известен как потенциал Хенона и Хейлеса.

Известны некоторые частные решения задачи трех тел, но общее решение еще не найдено. Одним из частных случаев задачи трех тел является задача двух неподвижных центров. Она была впервые рассмотрена Эйлером 1760 г. Якоби показал, что уравнения движения могут быть интегрированы в терминах эллиптических функций. Данная задача может быть использована как некоторое первое приближение в астрономических задачах о движении малых планет и комет под действием гравитации Солнца и Юпитера. Период обращения Юпитера составляет около двенадцати лет, и в течение небольшого промежутка времени движение указанных небесных тел можно рассматривать в рамках задачи двух неподвижных центров. Также задачу о движении космического корабля к Луне можно рассматривать в рамках указанной задачи. Время полета космического корабля до Луны составляет около четырех суток. За это время Луна по круговой орбите Земли переместится незначительно. Исследование задачи двух неподвижных центров проводилось различных направлениях.

В данной работе исследуется потенциал Хенона-Хейлеса и задача двух неподвижных центров. При исследовании нелинейных систем, для которых неизвестны точные решения используется метод сечения Пуанкаре. Для потенциала Хенона-Хейлеса были получены сечения Пуанкаре. Далее был исследован потенциал задачи двух неподвижных центров. Было показано на основе сечения Пуанкаре, что в случае $\mu_1 = \mu_2 = 1$ внутренняя структура сечений распадается со значений $H = -1.7$, но внутренняя структура сечений сохраняется в отрезке $H \in [-0.5, -1.6]$, в случае $\mu_1 = 0.9$ и $\mu_2 = 0.1$ внутренняя структура сечений распадается со значений $H = -0.9$, но внутренняя структура сечений сохраняется в отрезке $H \in [-0.3, -0.8]$, в случае $\mu_1 = 0.7$ и $\mu_2 = 0.3$ внутренняя структура сечений распадается со значений $H = -0.8$, но внутренняя структура сечений сохраняется в отрезке $H \in [-0.2, -0.7]$. С увеличением энергии многие из этих поверхностей распадаются. Предполагается, что полученные численные результаты послужат основой для сравнения с аналитическими решениями.

Ключевые слова: модель Хенона-Хейлеса, задача двух неподвижных центров, сечение Пуанкаре, численные решения.

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GREEN TENSOR OF MOTION EQUATIONS OF TWO COMPONENTS BIOT'S MEDIUM BY STATIONARY VIBRATIONS

Abstract. Here processes of wave propagation in a two-component Biot's medium are considered which are generated by periodic forces actions. By use Fourier transformation of generalized functions, the Green tensor - a fundamental solutions of oscillation equations of this medium has been constructed. This tensor describes the process of propagation of harmonic waves of a fixed frequency in spaces of dimension N = 1,2,3 under the action of power sources concentrated at the coordinates origin, described by a singular delta -function. Based on it, generalized solutions of these equations are constructed under the action of various sources of periodic perturbations, which are described by both regular and singular generalized functions. For regular acting forces, integral representations of solutions are given that can be used to calculate the stress-strain state of a porous water-saturated medium.

Key words: Biot's medium, solid and liquid components, fundamental solution, generalized direct and inverse Fourier transform, regularization.

Various mathematical models of deformable solids mechanics are used in the study of seismic processes in the earth's crust. The processes of waves propagation are most studied in elastic media. But these models do not take into account many real properties of the ambient array. These are, for example, the presence of groundwater, which complicates the construction and operation of surface and underground structures, affect the magnitude and distribution of stresses. Models, which take into account the water saturation form the earth's crust structures, the presence of gas bubbles, etc., are multi-component medium. A variety of multicomponent media, the complexity of the processes associated with their deformation, lead to a large difference in the methods of analysis and modelling used in the solution of wave problems.

Porous medium saturated with liquid or gas, from the point of view of continuum mechanics, is essentially a two-phase continuous medium, one phase of which is particles of liquid (gas), other solid particles is its elastic skeleton. There are various mathematical models of such media, developed by various authors. The most famous of them are the models of M. Biot, V.N. Nikolaevsky, L.P. Horoshun [1-7]. However, the class of solved tasks to them is very limited and mainly associated with the construction and study of particular solutions of these equations based on the methods of full and partial separation of variables and theory of special functions in the works of Rakhmatullin, H. A., Saatov Ya. U., Filippov I. G., Artykov T. U. [6,7], Erzhanov Zh. S, Ataliev Sh.M., Alexeyeva L.A., Shershnev V.V. [8,9] etc. In this regard, it is important to develop effective methods of solution of boundary value problems for such media with use of modern mathematical methods.

Periodic on time processes are very widespread in practice. By this cause here we consider the process of wave propagation in the Biot's medium, posed by the periodic forces of different types. Based on Fourier transformation of generalized functions we constructed fundamental solutions of oscillation equations of Biot's medium. It is Green's tensor, which describes the process of propagation of harmonic waves at a fixed frequency in the space-time of dimension N=1,2,3, under the action of concentrated at the coordinates origin. By use this tensor we construct generalized solutions of these equations for arbitrary sources of periodic disturbances, which can be described both regular and singular distributions. They can be used to calculate the stress-strain state of a porous water-saturated medium by seismic waves propagation.

1 The parameters and motion equations of a two-components M. Biot medium

The equations of motion of a homogeneous isotropic two-component M. Biot medium are described by the following system of second-order hyperbolic equations [1-3]:

$$\begin{gathered}(\lambda+\mu)\operatorname{grad}\operatorname{div}u_s+\mu\Delta u_s+Q\operatorname{grad}\operatorname{div}u_f+F^s(x,t)=\rho_{11}\ddot{u}_s+\rho_{12}\ddot{u}_f\\ Q\operatorname{grad}\operatorname{div}u_s+R\operatorname{grad}\operatorname{div}u_f+F^f(x,t)=\rho_{12}\ddot{u}_s+\rho_{22}\ddot{u}_f\end{gathered}\quad(1)$$

$(x,t)\in R^N\times[0,\infty)$. Here N is the dimension of the space. At a plane deformation *N*=2, the total spatial deformation corresponds to *N*=3, at N=1 the equations describe the dynamics of a porous liquid-saturated rod. We denote $u_s=u_{sj}(x,t)e_j$ a displacements vector of the elastic skeleton, $u_f=u_{fj}(x,t)e_j$ is the displacements vector of a liquid, e_j $(j=1,...,N)$ are the basic orts of the Lagrangian Cartesian coordinate system (everywhere by repeating indices there is summation from 1 to *N*).

Constants $\rho_{11},\rho_{12},\rho_{22}$ have the dimension of mass density and are associated with the density of the masses of the particles, composing a skeleton ρ_s and a fluid ρ_f , by relationships:

$$\rho_{11}=(1-m)\rho_s-\rho_{12},\quad \rho_{22}=m\rho_f-\rho_{12},$$

where *m* is a porosity of medium. The constant of the attached density ρ_{12} is related to the dispersion of the deviation of the micro-velocities of the fluid particles in the pores from the average velocity of the fluid flow and depends on the geometry of the pores. Elastic constants λ,μ are the Lame parameters of an isotropic elastic skeleton, and *Q*, *R* characterize the interaction of the skeleton with a liquid on the basis of *Biot law for stresses*:

$$\begin{gathered}\sigma_{ij}=(\lambda\partial_k u_k+Q\partial_k U_k)\delta_{ij}+\mu(\partial_i u_i+\partial_j u_i)\\ \sigma=-mp=R\partial_k U_k+Q\partial_k u_k\end{gathered}\quad(2)$$

Here $\sigma_{ij}(x,t)$ are the stress tensor in the skeleton, $p(x,t)$ is a pressure in the fluid. Further we use the notations for partial derivatives: $\partial_k=\frac{\partial}{\partial x_k}, u_{j,k}=\partial_k u_j$, $\Delta=\partial_k\partial_k$ is Laplace operator. The external mass forces acting on the skeleton $F^s=F_j^s(x,t)e_j$ and on the liquid component $F^f=F_j^f(x,t)e_j$.

There are three sound speeds in this medium:

$$c_1^2=\frac{\alpha_1+\sqrt{\alpha_1^2-4\alpha_2\alpha_3}}{2\alpha_2},\quad c_2^2=\frac{\alpha_1-\sqrt{\alpha_1^2-4\alpha_2\alpha_3}}{2\alpha_2},\quad c_3=\sqrt{\frac{\rho_{22}\mu}{\alpha_2}}\quad(3)$$

where the next constants were introduced:

$$\alpha_1=(\lambda+2\mu)\rho_{22}+R\rho_{11}-2Q\rho_{12},\quad \alpha_2=\rho_{11}\rho_{22}-(\rho_{12})^2,\quad \alpha_3=(\lambda+2\mu)R-Q^2$$

The first two speeds $c_1, c_2 \left(c_1 > c_2\right)$ describe the velocity of propagation of two types of *dilatational waves*. The second slower dilatation wave is called *the repackaging wave*. A third velocity c_3 corresponds to *shear waves* and at $\rho_{12} = 0$ coincides with velocity of transverse waves propagation in an elastic skeleton ($c_3 < c_1$).

We introduce also two velocities of propagation of dilatational waves in corresponding elastic body and in an ideal compressible fluid:

$$c_s = \sqrt{\frac{\lambda + 2\mu}{\rho_{11}}}, \qquad c_f = \sqrt{\frac{R}{\rho_{22}}}$$

2 Problems of periodic oscillations of the Biot's medium

Construction of motion equation solutions by periodic oscillations is very important for practice since existing power sources of disturbances are often periodic in time and therefore can be decomposed into a finite or infinite Fourier series in the form:

$$F^s\left(x,t\right) = \sum_n F_n^s\left(x\right) e^{-i\omega_n t}, \quad F^f\left(x,t\right) = \sum_n F_n^f\left(x\right) e^{-i\omega_n t} \tag{4}$$

where periods of oscillation of each harmonic $T_n = 2\pi / \omega_n$ are multiple to the general period *T* of oscillation. Therefore, it is enough to consider the case of stationary oscillations, when the acting forces are periodic on time with an oscillation frequency ω:

$$F^s\left(x,t\right) = F^s\left(x\right) e^{-i\omega t}, \quad F^f\left(x,t\right) = F^f\left(x\right) e^{-i\omega t} \tag{5}$$

The solution of the equations (1) can be represented in the similar form:

$$u_s\left(x,t\right) = u_s\left(x\right) e^{-i\omega t}, \quad u_f\left(x\right) = u_f\left(x\right) e^{-i\omega t} \tag{6}$$

where the complex amplitudes of the displacements $u_s\left(x\right)$, $u_f\left(x\right)$ must be determined. If the solution has been known for any frequency щ, then we get similar decomposition for the displacements of the medium:

$$u^s\left(x,t\right) = \sum_n u_n^s\left(x\right) e^{-i\omega_n t}, \quad u^f\left(x,t\right) = \sum_n u_n^f\left(x\right) e^{-i\omega_n t} \tag{7}$$

which give us the solution of problem for forces (4).

We get equations for complex amplitudes by stationary oscillations, substituting (6) into the system (1):

$$\left(\lambda + \mu\right) \operatorname{grad}\operatorname{div} u^s + \mu \Delta u^s + Q \operatorname{grad}\operatorname{div} u^f + F^s\left(x\right) = -\rho_{11}\omega^2 u^s - \rho_{12}\omega^2 u^f$$

$$Q \operatorname{grad}\operatorname{div} u^s + R \operatorname{grad}\operatorname{div} u^f + F^f\left(x\right) = -\rho_{12}\omega^2 u^s - \rho_{22}\omega^2 u^f \tag{8}$$

To construct the solutions of this system we define Green tensor of it.

3 Green tensor of Biot equations by stationary oscillations

Let's construct fundamental solutions of the system (1) in the form:

$$\begin{pmatrix} F^s \\ F^f \end{pmatrix} = \begin{pmatrix} \delta_k^{[j]} e_k \\ \delta_{k+N}^{[j]} e_k \end{pmatrix} \delta\left(x\right) e^{-i\omega t}, \quad k = 1,...,N,\ j = 1,...,2N \tag{9}$$

where δ_k^j is Kronecker symbol. They describe the motion of Biot medium at the action of sources of stationary oscillations, concentrated in the point *x*=0. The upper index of this tensor fixes the current concentrated force and its direction. The lower index corresponds to component of the movement of the skeleton and fluid, respectively $k=1,...,N$ and $k=N+1,...,2N$.

Their complex amplitudes $U_m^j(x,\omega)$ $(j,m=1,...,2N)$ satisfy to the next system of equation:

$$\left(\lambda+\mu\right)U_{j,ji}^k+\mu U_{i,jj}^k+\omega^2\rho_{11}U_i^k+QU_{j,ji}^{k+N}-\omega^2\rho_{12}U_i^{k+N}+\delta\left(x\right)\delta_j^k=0$$
$$QU_{j,ji}^k+\rho_{12}\omega^2U_i^k,_{tt}+RU_{j,ji}^{k+N}+\rho_{22}\omega^2U_i^{k+N}+\delta\left(x\right)\delta_{j+N}^k=0 \qquad (10)$$
$$j=1,...,2N, \quad k=1,...,2N$$

Since fundamental solutions are not unique, we'll construct such, which tend to zero at infinity:

$$U_i^j(x,\omega)\to 0 \quad \text{at} \quad \|x\|\to\infty \qquad (11)$$

and satisfy to radiation condition of type of Somerfield radiation conditions [10].
Matrix of such fundamental equations is names *Green tensor* of Eq. (8).

4 Fourie transform of fundamental solutions

To construct $U_m^j(x,\omega)$ we use the Fourier transformation, which for regular functions has the form:

$$F[\varphi\left(x\right)]=\overline{\varphi}\left(\xi\right)=\int\limits_{R^N}\varphi\left(x\right)e^{i\left(\xi,x\right)}dx_1...dx_N$$

$$F^{-1}[\overline{\varphi}\left(\xi\right)]=\varphi\left(x\right)=\frac{1}{\left(2\pi\right)^N}\int\limits_{R^N}\overline{\varphi}\left(\xi\right)e^{-i\left(\xi,x\right)}d\xi_1....d\xi_N$$

where $\xi=(\xi_1,...,\xi_N)$ are Fourier variables. Let's apply Fourier transformation to Eqs (10), and use property of Fourier transform of derivatives [10]:

$$\frac{\partial}{\partial x_j} \quad \leftrightarrow \quad -i\xi_j \qquad (12)$$

Then we get the system of 2N linear algebraic equations for the Fourier components of this tensor:

$$-\left(\lambda+\mu\right)\xi_j\xi_j\overline{U}_j^k-\mu\|\xi\|^2\overline{U}_j^k-Q\xi_j\xi_j\overline{U}_{j+N}^k+\rho_{11}\omega^2\overline{U}_j^k+\rho_{12}\omega^2\overline{U}_{j+N}^k+\delta_j^k=0,$$
$$-Q\xi_j\xi_j\overline{U}_j^k-R\xi_j\xi_l\overline{U}_{j+N}^k+\rho_{12}\omega^2\overline{U}_j^k+\rho_{22}\omega^2\overline{U}_{j+N}^k+\delta_{j+N}^k=0,$$
$$j=1,...,N, \quad k=N+1,...,2N \qquad (13)$$

By use gradient-divergence method this system has been solved by us. For this the next basic function were introduced

$$f_{0k}\left(\xi,\omega\right)=\frac{1}{c_k{}^2\|\xi\|^2-\omega^2}, \quad f_{jk}\left(\xi,\omega\right)=\frac{f_{(j-1)k}\left(\xi,\omega\right)}{-i\omega}, \quad j=1,2; \qquad (14)$$

and the next theorem was proved [11,12].

Theorem 1. *Components of Fourier transform of fundamental solutions have the form*

$$j=\overline{1,N},\quad k=\overline{1,N},$$

$$\overline{U}_j^k=\left(-i\xi_j\right)\left(-i\xi_k\right)\left[\beta_1 f_{21}+\beta_2 f_{22}+\beta_3 f_{23}\right]+\frac{1}{\alpha_2}\left(\rho_{12}\delta_{j+N}^k-\rho_{22}\delta_j^k\right)f_{03}$$

$$\overline{U}_{j+N}^k=\left(-i\xi_j\right)\left(-i\xi_k\right)\left[\gamma_1 f_{21}+\gamma_2 f_{22}+\gamma_3 f_{23}\right]-\frac{\mu}{\alpha_2}\delta_{j+N}^k\left\|\xi\right\|^2 f_{23}-\frac{1}{\alpha_2}\left(\rho_{11}\delta_{j+N}^k+\rho_{12}\delta_j^k\right)f_{03}$$

$$j=1,\dots,N\quad k=N+1,\dots,2N$$

$$\overline{U}_j^k=\left(-i\xi_j\right)\left(-i\xi_{k-N}\right)\left[\eta_1 f_{21}+\eta_2 f_{22}+\eta_3 f_{23}\right]+\frac{1}{\alpha_2}\left(\rho_{12}\delta_{j+N}^k-\rho_{22}\delta_j^k\right)f_{03}$$

$$\overline{U}_{j+N}^k=\left(-i\xi_j\right)\left(-i\xi_{k-N}\right)\left[\varsigma_1 f_{21}+\varsigma_2 f_{22}+\varsigma_3 f_{23}\right]-\frac{\mu}{\alpha_2}\delta_{j+N}^k\left\|\xi\right\|^2 f_{23}-\frac{1}{\alpha_2}\left(\rho_{11}\delta_{j+N}^k+\rho_{12}\delta_j^k\right)f_{03}$$

where the next constants have been introduced:

$$D_1=\frac{1}{\alpha_2\upsilon_{12}},\quad \upsilon_{lm}=c_l^2-c_m^2,\ q_1=Q\rho_{12}-(\lambda+\mu)\rho_{12},\quad q_2=\rho_{11}R-Q\rho_{12},$$

$$d_1=(\lambda+\mu)\rho_{22}-Q\rho_{12},\ d_2=Q\rho_{22}-R\rho_{12},\quad d_{3j}=\rho_{12}c_j^2-Q\ \ (j=1,2)$$

$$\beta_j=(-1)^{(j+1)}\frac{D_1c_j^2}{\alpha_2\upsilon_{3j}}\left(d_1b_{sj}+d_2d_{3j}\right),\quad \beta_3=-\frac{c_3^2}{\alpha_2\upsilon_{31}\upsilon_{32}}\left(d_1b_{s3}+d_2d_{33}\right);$$

$$\gamma_j=(-1)^{j+1}\frac{D_1c_j^2}{\alpha_2\upsilon_{3j}}\left(q_1b_{fj}+q_2d_{3j}\right),\quad \gamma_3=-\frac{D_1c_3^2\upsilon_{12}}{\alpha_2\upsilon_{31}\upsilon_{32}}\left(q_1b_{f3}+q_2d_{33}\right);$$

$$\eta_j=(-1)^{j+1}\frac{D_1c_j^2}{\alpha_2\upsilon_{3j}}\left(d_jd_{3j}+d_2b_{js}\right),\quad \eta_3=-\frac{c_3^2\upsilon_{12}}{\alpha_2\upsilon_{31}\upsilon_{32}}\left(d_1d_{33}+d_2b_{3s}\right);$$

$$\varsigma_j=(-1)^{j+1}\frac{D_1c_j^2}{\alpha_2\upsilon_{3j}}\left(q_1d_{3j}+q_2b_{(4-j)s}\right),\quad \varsigma_3=-\frac{c_3^2\upsilon_{12}}{\alpha_2\upsilon_{31}\upsilon_{32}}\left(q_1d_{33}+q_2b_{3s}\right)b_{fj}=\rho_{22}\upsilon_{fj},b_{sj}=\rho_{11}\upsilon_{js}.$$

This form is very convenient for constructing originals of Green tensor.

5 Green tensor construction. Radiation conditions

At first let's construct the originals of basic function:

$$\Phi_{0m}(x,\omega)=F^{-1}\left[f_{0m}(\xi,\omega)\right],$$

which, in accordance to its definition (14), satisfy to the equation

$$\left(c_m^2\left\|\xi\right\|^2-\omega^2\right)f_{0m}=1. \qquad (15)$$

Using the property (12) for derivatives from here we get Helmholtz equation for fundamental solution (accurate within a factor c_m^{-2}) :

$$\left(\Delta + k_m^2\right)\Phi_{0m} + c_m^{-2}\delta\left(x\right) = 0, \quad k_m = \frac{\omega}{c_m} \tag{16}$$

Fundamental solutions of Helmholtz equation which satisfy to Sommerfeld conditions of radiation: at $r \to \infty$

$$\varPhi'_{0m}(r) - ik_m\varPhi_{0m}(r) = O(r^{-1}), \qquad N = 3,$$
$$\varPhi'_{0m}(r) - ik_m\varPhi_{0m}(r) = O(r^{-1/2}), \quad N = 2.$$

are well known [10]. They are unique. Using them we obtain:

for $N = 3$ $$\varPhi_{0m} = \frac{1}{4\pi rc^2}e^{ik_m r}, \quad k_m = \frac{\omega}{c_m};$$

for $N = 2$ $$\varPhi_{0m} = \frac{i}{4c^2}H_0^{(1)}\left(k_m r\right),$$

where $H_j^{(1)}\left(k_m r\right)$ is cylindrical Hankel function of the first kind;

for $N = 1$ $$\varPhi_{0m} = \frac{\sin k_m\left|x\right|}{2k_m c_m^2}.$$

These functions (subject to factor $e^{-i\omega t}$)) describe harmonic waves which move from x=**0** to infinity and decay at infinity. Last property is true only for N=2,3. In the case N=1 all fundamental solutions of Eq. (16) :

$$\left(\frac{d^2}{dx^2} + k_m^2\right)\Phi_{0m} + c_m^{-2}\delta\left(x\right) = 0,$$

don’t decay at infinity.

From theorem 1 the next theorem follows.

Theorem 2. The components of Green tensor of Biot’s equations at stationary oscillations with frequency щ, satisfying the conditions of radiation, have the form:

$$for \quad j = \overline{1,N}, \quad k = \overline{1,N},$$
$$U_j^k = -\omega^{-2}\sum_{m=1}^{3}\beta_m\frac{\partial^2\Phi_{0m}}{\partial x_j\partial x_k} + \frac{1}{\alpha_2}\left(\rho_{12}\delta_{j+N}^k - \rho_{22}\delta_j^k\right)\Phi_{03},$$

$$U_{j+N}^k = -\omega^{-2}\sum_{m=1}^{3}\gamma_m\frac{\partial^2\Phi_{0m}}{\partial x_j\partial x_k} + \frac{\mu}{\alpha_2\omega^2}(c_3^{-2}\delta\left(x\right) + k_3^2\Phi_{0m})\delta_{j+N}^k - \frac{1}{\alpha_2}\left(\rho_{11}\delta_{j+N}^k + \rho_{12}\delta_j^k\right)\Phi_{03};$$
$$for \quad j = 1,\ldots,N \quad k = N+1,\ldots,2N$$

$$U_j^k = -\omega^{-2}\sum_{m=1}^{3}\eta_m\frac{\partial^2\Phi_{0m}}{\partial x_j\partial x_k} + \frac{1}{\alpha_2}\left(\rho_{12}\delta_{j+N}^k - \rho_{22}\delta_j^k\right)\Phi_{03}$$

$$U_{j+N}^k = -\omega^{-2}\sum_{m=1}^{3}\varsigma_m\frac{\partial^2\Phi_{0m}}{\partial x_j\partial x_k} + \frac{\mu}{\alpha_2\omega^2}(c_3^{-2}\delta\left(x\right) + k_3^2\Phi_{0m})\delta_{j+N}^k - \frac{1}{\alpha_2}\left(\rho_{11}\delta_{j+N}^k + \rho_{12}\delta_j^k\right)\Phi_{03}$$

where

$$\frac{d^2\Phi_{0m}}{dx^2} = \frac{1}{2c_m^2 k_m}\left(k_m{}^2\left(\sin k_m |x|\right) - 2k_m\delta(x)\right) \quad for \;\; N=1,$$

$$\frac{\partial^2\Phi_{0m}}{\partial x_j \partial x_k} = -\frac{i}{4c_m^2}\left(0.5k_m^2\left(H_0(k_m r) - H_2(k_m r)\right)r,_j r,_k + kH_1^1(k_m r)r,_{jk}\right) \quad for \;\; N=2,$$

$$\frac{\partial\Phi_{0m}}{\partial x_j \partial x_k} = \frac{1}{4\pi r c_m^2}e^{ikr}\left\{r,_j r,_k\left(\left(ik_m - \frac{1}{r}\right)^2 + \frac{1}{r^2}\right) + r,_{jk}\left(ik_m - \frac{1}{r}\right)\right\} \quad for \;\; N=3;$$

$$k_m = \frac{\omega}{c_m},\; r = \|x\|, \quad r,_j = \frac{x_j}{r}, \quad r,_{ij} = \frac{1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right).$$

Proof. By use originals of basic functions , property (12) of derivatives, we can obtain from formulas for $\overline{U}_j^k$ in theorem 1 the originals of all addends, beside that which contain factor $\|\xi\|^2$. But using (16) we have

$$-\Delta\Phi = c_m^{-2}\delta(x) + k_m^2\Phi_{0m} \quad \leftrightarrow \quad \|\xi\|^2 f_{0m} = c_m^{-2} + k_m^2 f_{0m}$$

Then formulas of Theorem 2 follow from formulas of Theorem 1.

Conclusion. Under the action of arbitrary mass forces with frequency ω in the Biot's medium, the solution for complex amplitudes has the form of a tensor-functional convolution:

$$u_j(x,t) = U_j^k(x,\omega) * F_k(x)e^{-i\omega t},\; j,k = \overline{1,2N} \qquad (17)$$

Note that mass forces may be different from the space of generalized vector-function, singular and regular. Since Green tensor is singular, contains delta-functions, this convolution are calculated on the rule of convolution in generalized function space. If a support of acting forces are bounded (contained in a ball of finite radius), then all convolutions exist. If supports are not bounded, then the existence condition (17) require some limitations on behavior of forces at infinity which depend on the type of mass forces.

The obtained solutions allow us to study the dynamics of porous water and gas-saturated media at the action of periodic sources of disturbances of a sufficiently arbitrary form. In particular, under the action of certain forces on surfaces, for example cracks, in porous media that can be simulated by simple and double layers on the crack surface.

There is another interesting feature of the Green tensor of the Biot equations, which contains, as one of the terms, the delta function what complicates the application of this tensor for solving boundary value problems based on the boundary element method or boundary untegral equations method [13,14]. Here, when constructing the model, the viscosity of the liquid is not taken into account, which, apparently, leads to the presence of such terms, and requires improvement of this model taking into account the viscosity.

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СТАЦИОНАР ТЕРБЕЛІСТЕР КЕЗІНДЕГІ ЕКІ КОМПОНЕНТТІ М. БИО ОРТАСЫНЫҢ ҚОЗҒАЛЫС ТЕҢДЕУІНІҢ ГРИН ТЕНЗОРЫ

Аннотация. Қатты серпімді және идеалды сұйықтан тұратын екі компонентті Био ортаны қарастырамыз. Мұндай ортаның қозғалысы қатты және сұйық компоненттердің орын ауыстыруы үшін гиперболалық типтегі екінші ретті дифференциалдық теңдеулердің біріккен жүйесімен сипатталады. Әртүрлі типтегі периодты күштер тудыратын био ортадағы толқындардың таралу процестері зерттелген.

Бұл жүйенің стационарлы шешімдері N = 1,2,3 өлшемдер кеңістігіндегі гармоникалық тербелістерді сипаттайды. Бұл жағдайда күрделі тербеліс амплитудасы үшін теңдеулер жүйесі эллиптикалық түрде болады. Жалпыланған функциялардың және оның қасиеттерінің Фурье түрлендіруі негізінде ортаға тербеліс теңдеулерінің іргелі шешімі – сингулярлы дельта функциясымен сипатталған шығу көзіне шоғырланған күш көздерінің әсерінен тұрақты жиіліктегі уақыттық-гармоникалық толқындардың таралуын сипаттайтын Био орта – Грин тензоры тұрғызылды.

Тұрақты және сингулярлы жалпыланған функциялармен сипатталатын периодтық бұзулардың әртүрлі көздерінің әсері кезінде, осы теңдеулердің жалпыланған шешімдері тұрғызылды. Алынған нәтижелерді газ және сұйық қаныққан кеуекті ортадағы толқындық процестерді зерттеу үшін қолдануға болады.

Түйін сөздер: Био ортасы, қатты және сұйық компоненттер, іргелі шешім, жалпыланған тура және кері Фурье түрлендірулері, регуляризация.

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ТЕНЗОР ГРИНА УРАВНЕНИЙ ДВИЖЕНИЯ ДВУХКОМПОНЕНТНОЙ СРЕДЫ М. БИО ПРИ СТАЦИОНАРНЫХ КОЛЕБАНИЯХ

Аннотация. Рассматривается двухкомпонентная среда Био, содержащия твердую упругую компоненту и идеальную жидкую. Движение такой среды описывается связанной системой дифференциальных уравнений в частных производных второго порядка гиперболического типа для перемещений твердой и жидкой компоненты. Исследуются процессы распространения волн в среде Био, порождаемые действующими периодическими силами различного типа.

Строятся стационарные решения этой системы, описывающие гармонические колебания в пространствах размерности N = 1,2,3. В этом случае система уравнений для комплексных амплитуд колебаний является эллиптической. На основе преобразования Фурье обобщенных функций и его свойств построено фундаментальное решение уравнений колебаний среды Био - тензор Грина, который описывает процесс распространения гармонических по времени волн фиксированной частоты при действии сосредоточенных в начале координат силовых источников, описываемых сингулярной дельта-функцией.

На его основе построены обобщенные решения этих уравнений при действии разнообразных источников периодических возмущений, которые описываются как регулярными, так и сингулярными обобщенными функциями. Полученные решения можно использовать для исследования волновых процессов в газо- и жидконасыщенных пористых средах.

Ключевые слова: среда Био, твердая и жидкая компоненты, фундаментальное решение, обобщенное прямое и обратное преобразование Фурье, регуляризация.

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