

ISSN 2518-1726 (Online),  
ISSN 1991-346X (Print)

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ  
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ

ӘЛЪ-ФАРАБИ АТЫНДАҒЫ  
ҚАЗАҚ ҰЛТТЫҚ УНИВЕРСИТЕТІНІҢ

# Х А Б А Р Л А Р Ы

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## ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК  
РЕСПУБЛИКИ КАЗАХСТАН

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ  
УНИВЕРСИТЕТ ИМЕНИ АЛЬ-ФАРАБИ

## NEWS

OF THE NATIONAL ACADEMY OF SCIENCES  
OF THE REPUBLIC OF KAZAKHSTAN

AL-FARABI KAZAKH  
NATIONAL UNIVERSITY

ФИЗИКА-МАТЕМАТИКА СЕРИЯСЫ



СЕРИЯ ФИЗИКО-МАТЕМАТИЧЕСКАЯ



PHYSICO-MATHEMATICAL SERIES

## 5 (321)

ҚЫРКҮЙЕК – ҚАЗАН 2018 ж.  
СЕНТЯБРЬ – ОКТЯБРЬ 2018 г.  
SEPTEMBER – OCTOBER 2018

1963 ЖЫЛДЫҢ ҚАҢТАР АЙЫНАН ШЫҒА БАСТАҒАН  
ИЗДАЕТСЯ С ЯНВАРЯ 1963 ГОДА  
PUBLISHED SINCE JANUARY 1963

ЖЫЛЫНА 6 РЕТ ШЫҒАДЫ  
ВЫХОДИТ 6 РАЗ В ГОД  
PUBLISHED 6 TIMES A YEAR

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ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Меншіктенуші: «Қазақстан Республикасының Ұлттық ғылым академиясы» РҚБ (Алматы қ.)  
Қазақстан республикасының Мәдениет пен ақпарат министрлігінің Ақпарат және мұрағат комитетінде  
01.06.2006 ж. берілген №5543-Ж мерзімдік басылым тіркеуіне қойылу туралы куәлік

Мерзімділігі: жылына 6 рет.  
Тиражы: 300 дана.

Редакцияның мекенжайы: 050010, Алматы қ., Шевченко көш., 28, 219 бөл., 220, тел.: 272-13-19, 272-13-18,  
[www.nauka-nanrk.kz](http://www.nauka-nanrk.kz) / [physics-mathematics.kz](http://physics-mathematics.kz)

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Типографияның мекенжайы: «Аруна» ЖК, Алматы қ., Муратбаева көш., 75.

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**«Известия НАН РК. Серия физико-математическая».**

**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Собственник: РОО «Национальная академия наук Республики Казахстан» (г. Алматы)

Свидетельство о постановке на учет периодического печатного издания в Комитете информации и архивов  
Министерства культуры и информации Республики Казахстан №5543-Ж, выданное 01.06.2006 г.

Периодичность: 6 раз в год.

Тираж: 300 экземпляров.

Адрес редакции: 050010, г. Алматы, ул. Шевченко, 28, ком. 219, 220, тел.: 272-13-19, 272-13-18,  
[www.nauka-nanrk.kz](http://www.nauka-nanrk.kz) / [physics-mathematics.kz](http://physics-mathematics.kz)

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**News of the National Academy of Sciences of the Republic of Kazakhstan. Physical-mathematical series.**

**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Owner: RPA "National Academy of Sciences of the Republic of Kazakhstan" (Almaty)

The certificate of registration of a periodic printed publication in the Committee of information and archives of the Ministry of culture and information of the Republic of Kazakhstan N 5543-Ж, issued 01.06.2006

Periodicity: 6 times a year

Circulation: 300 copies

Editorial address: 28, Shevchenko str., of. 219, 220, Almaty, 050010, tel. 272-13-19, 272-13-18,  
[www.nauka-nanrk.kz/physics-mathematics.kz](http://www.nauka-nanrk.kz/physics-mathematics.kz)

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Address of printing house: ST "Aruna", 75, Muratbayev str, Almaty

NEWS

OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN  
PHYSICO-MATHEMATICAL SERIES

ISSN 1991-346X

<https://doi.org/10.32014/2018.2518-1726.9>

Volume 5, Number 321 (2018), 68 – 74

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## PROBLEM FROM THE THEORY OF BRIDGE EROSION

**Abstract.** In this paper, we represent the exact solution of a two phase Stefan problem. Radial heat polynomials and integral error function are used for solving bridge problem. The recurrent expressions for the coefficients of these series are presented. The mathematical models describe the dynamics of contact opening and bridging.

**Keywords:** radial heat polynomials, Stefan problem.

### Introduction

In consideration of the heat transfer, the shape of the liquid bridge plays an important role. The overwhelming majority of researchers proceed from the fact that the visible part of the bridge has the shape of a cylinder whose axis is directed perpendicular to the plane of the electrodes [1]. In the general case, in the result of the action of surface tension and the pinch effect, the bridge takes the form of a certain surface of revolution about the z- axis. In this problem, we consider a symmetric model of the bridge, where the shape of the bridge is described by a surface  $y(z, t) = z^{\nu/2}$ . For a liquid bridge in we consider the generalized heat equation for and for solid contact we use the spherical Holm model [2].

### Preliminaries

The fundamental solution for the equation

$$\frac{\partial \theta}{\partial t} = a^2 \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\nu}{x} \frac{\partial \theta}{\partial x} \right) \quad (1)$$

can be obtained by the solution of this equation with the initial condition containing delta-function using the Laplace transform in the form [3]

$$G(x, y, t) = \frac{C_\nu}{2t} (xy)^{-\beta} e^{-\frac{x^2+y^2}{4t}} I_\beta \left( \frac{xy}{2t} \right), \quad \beta = \frac{\nu-1}{2}, \quad C_\nu = 2^{-\beta} \Gamma(\beta+1) \quad (2)$$

If we consider the corresponding heat potentials for this solution

$$Q_{n,\nu}(x, t) = 2^{-\beta} \Gamma(\beta+1)^{-1} \int_0^\infty G(x, y, t) y^{2n+\nu} dy \quad (3)$$

and integrating by parts we obtain the explicit formula for the heat polynomials

$$Q_{n,\nu}(x, t) = \sum_{k=0}^n 2^{2k} \frac{n! \Gamma(\beta+1)}{k!(n-k)! \Gamma(\beta+1+n-k)} x^{2n-2k} t^k \quad (4)$$

It is more convenient for applications to multiply both sides of this formula by  $\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)}$ .

$$\text{Then } R_{n,\nu}(r,t) = \frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1)} Q_{n,\nu}(x,t) = \sum_{k=0}^n 2^{2k} \frac{n! \Gamma(\beta+1)}{k!(n-k)! \Gamma(\beta+1+n-k)} x^{2n-2k} t^k \quad (5)$$

### Mathematical model

The heat equations for each zone are

$$\frac{\partial \theta_1}{\partial t} = a_1^2 \left( \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\nu}{x} \frac{\partial \theta_1}{\partial x} \right) \quad \alpha(t) < x < 0 \quad (6)$$

$$\frac{\partial \theta_2}{\partial t} = a_1^2 \left( \frac{\partial^2 \theta_2}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_2}{\partial r} \right) \quad r_0 < r < \beta(t) \quad (7)$$

$$\frac{\partial \theta_3}{\partial t} = a_2^2 \left( \frac{\partial^2 \theta_3}{\partial r^2} + \frac{2}{r} \frac{\partial \theta_3}{\partial r} \right) \quad \beta(t) < r < \infty \quad (8)$$

with boundary and initial conditions:

$$\alpha(0) = 0, \beta(0) = r_0, \theta_1(0,0) = \theta_2(r_0,0) = \theta_m, \theta_3(r,0) = f(r), f(r_0) = \theta_m \quad (9)$$

$$-\lambda \pi \alpha^\nu(t) \frac{\partial \theta_1}{\partial x} \Big|_{x=\alpha(t)} = Q(t) \quad (10)$$

$$\theta_1(0,t) = \theta_2(r_0,t) \quad (11)$$

$$\lambda_1 \frac{\partial \theta_1(0,t)}{\partial x} = 2\lambda_2 \frac{\partial \theta_2(r_0,t)}{\partial r} \quad (12)$$

$$\theta_2(\beta(t),t) = \theta_m \quad (13a)$$

$$\theta_3(\beta(t),t) = \theta_m \quad (13b)$$

The Stefan's condition

$$-\lambda_1 \frac{\partial \theta_2}{\partial r} \Big|_{r=\beta(t)} = -\lambda_2 \frac{\partial \theta_3}{\partial r} \Big|_{r=\beta(t)} + L\gamma \frac{d\beta}{dt} \quad (14)$$

$$\theta_3(\infty,t) = 0 \quad (15)$$

### Method of solution

We represent solution of the problem (6)-(15) in the form

$$\theta_1(x,t) = \sum_{n=0}^{\infty} A_n \sum_{k=0}^n \zeta_{n,k} x^{2n-2k} t^k \quad (16)$$

$$\theta_2(r,t) = \sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} x^{2n-2k} t^k + \sum_{n=0}^{\infty} D_n \frac{(2a_1 \sqrt{t})^{2n+1}}{r} \left( i^{2n+1} \operatorname{erfc} \frac{-(r-r_0)}{2a_1 \sqrt{t}} - i^{2n+1} \operatorname{erfc} \frac{(r-r_0)}{2a_1 \sqrt{t}} \right) \quad (17)$$

$$\theta_3(r,t) = \sum_{n=0}^{\infty} E_n \sum_{k=0}^n \zeta_{n,k,2} x^{2n-2k} t^k + \sum_{n=0}^{\infty} G_n \frac{(2a_2 \sqrt{t})^{2n+1}}{r} \left( i^{2n+1} \operatorname{erfc} \frac{-(r-r_0)}{2a_2 \sqrt{t}} - i^{2n+1} \operatorname{erfc} \frac{(r-r_0)}{2a_2 \sqrt{t}} \right) \quad (18)$$

$$\text{Where } \zeta_{n,k} = \frac{2^{2k} n! \Gamma\left(\frac{\nu-1}{2} + n + 1\right)}{k!(n-k)! \Gamma\left(\frac{\nu-1}{2} + n + 1 - k\right)} \text{ and } \zeta_{n,k,2} = \frac{2^{2k} n! \Gamma\left(\frac{3}{2} + n\right)}{k!(n-k)! \Gamma\left(\frac{3}{2} + n - k\right)},$$

and coefficients  $A_n, C_n, D_n, E_n, G_n$  have to be determined. Free boundary  $\beta(t)$  represent in the form

$$[4] \beta(t) = \sum_{n=0}^{\infty} \beta_n t^{n/2}.$$

Using the boundary condition (10) we get

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^n \zeta_{n,k} (n-k) \alpha(t)^{2n-2k-1} t^k = F(t) \tag{19}$$

where

$$F(t) = -\frac{Q(t)}{2\lambda\pi\alpha^\nu(t)}$$

Taking into account that  $\alpha(t)$  is given and using properties of raising the power series to a power [5]

$$\alpha(t)^{2n-2k-1} = \sum_{m=0}^{\infty} \beta(\alpha)_m t^m \tag{20}$$

Where coefficients  $\beta_m(\alpha)$  determined by

$$\beta_0(\alpha) = \alpha_0^{2n-2k-1}$$

$$\beta(\alpha)_i = \frac{1}{i\alpha_0} \sum_{m=1}^i [2m(n-k) - i] \alpha_m \beta(\alpha)_{i-m} \quad i \geq 1$$

Substituting the formula (20) into (19) we obtain

$$\sum_{n=0}^{\infty} A_n \sum_{k=0}^n \zeta_{n,k} (n-k) \sum_{m=0}^{\infty} \beta(\alpha)_m t^{m+k} = F(t) \tag{21}$$

$F(t)$  is given, can be expanded in Maclaourin series thus to derive  $A_m$ , we take both sides of (21),  $l$ -times derivative at  $t = 0$  we have

$$\sum_{n=0}^l A_{n+1} \hbar_{1,n} + \sum_{n=l}^{\infty} A_{n+1} \hbar_{2,n} = l! F_n \tag{22}$$

where

$$\hbar_{2,n} = \sum_{i=0}^l \zeta_{n+1,i} \beta(\alpha)_{l-i,i} l!(n-i+1)$$

$$\hbar_{1,n} = \sum_{i=0}^n \zeta_{n+1,i} \beta(\alpha)_{l-i,i} l!(n-i+1)$$

from (22) we can find  $A_n$ .

Satisfying the boundary conditions of conjugations of temperature (11) and heat flux (12) we get

$$\sum_{n=0}^{\infty} A_n \zeta_{n,n} t^m = \sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} r_0^{2n-2k} t^k \tag{23}$$

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} r_0^{2n-2k-1} t^k + \frac{2}{r_0} \sum_{n=0}^{\infty} D_n (2a_1 \sqrt{t})^{2n} i^{2n} \operatorname{erfc} 0 = 0 \tag{24}$$

from taking the  $m$ -times derivative of (23) and (24) we get

$$A_m \zeta_{m,m} = \sum_{i=m}^{\infty} C_i \zeta_{i,m,2} r_0^{2n-2m} \tag{23*}$$

$$\sum_{i=0}^{\infty} C_n \zeta_{i+m+1,m,2} r_0^{2i+1} (i+1) = -\frac{1}{r_0} D_m (2a_1)^{2m} i^{2m} \operatorname{erfc} 0 \tag{25}$$

From expression (13a) when we put  $r = \beta(t)$  we have

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} + \sum_{n=0}^{\infty} \frac{1}{\beta(\tau)} (2a_1 \tau)^{2n+1} (i^{2n+1} \operatorname{erfc}(-\gamma(\tau)) - i^{2n+1} \operatorname{erfc}(\gamma(\tau))) = \theta_m \tag{26}$$

where  $\tau = \sqrt{t}$  and  $\frac{\beta(\tau) - \beta_0}{2a_1 \tau} = \frac{1}{2a_1} \sum_{n=0}^{\infty} \gamma_{n+1} \tau^n$

Multiplying both sides of (26) by  $\beta(\tau)$  we get

$$\sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} \sum_{m=0}^{\infty} \eta(\beta)_m \tau^{m+2k} + \sum_{n=0}^{\infty} D_n (2a_1 \tau)^{2n+1} (i^{2n+1} \operatorname{erfc}(-\gamma(\tau)) - i^{2n+1} \operatorname{erfc}(\gamma(\tau))) = \theta_m \beta(\tau) \tag{27}$$

To comparing coefficients in (27) we apply Leibniz, Faa Di Bruno's formula and Bell polynomials. Using Leibniz we have

$$\left. \frac{\partial^l \left[ (2a_1)^{2n+1} \tau^{2n+1} i^{2n+1} \operatorname{erfc}(\gamma) \right]}{\partial \tau^l} \right|_{\tau=0} = \begin{cases} 0 & , \text{ for } l < 2n+1 \\ \frac{(2a_1)^{2n+1} l!}{(l-2n-1)!} \left[ i^{2n+1} \operatorname{erfc}(\delta) \right]^{(l-2n-1)} & \end{cases}$$

Using Faa Di Bruno's formula and Bell polynomials for a derivative of a composite function we have

$$\left. \frac{\partial^{l-2n-1}}{\partial \tau^{l-2n-1}} \left\{ i^{2n+1} \operatorname{erfc}(\pm \delta) \right\} \right|_{\tau=0} = \sum_{m=1}^{l-2n-1} \left( i^{2n+1} \operatorname{erfc}(\pm \delta) \right)^{(m)} \Big|_{\delta=0} B_{l-2n-1,m}$$

where

$$B_{l-2n-1,m} = \sum \frac{(l-2n-1)!}{j_1! j_2! \dots j_{l-2n-m}!} \zeta_1^{j_1} \zeta_2^{j_2} \zeta_3^{j_3} \dots \zeta_{l-2n-m}^{j_{l-2n-m}}$$

and  $j_1, j_2, \dots$  satisfy the following equations

$$\begin{aligned} j_1 + j_2 + \dots + j_{l-2n-m} &= m \\ j_1 + 2j_2 + \dots + (l-2n-m)j_{l-2n-m} &= l-2n-1 \end{aligned}$$

by taking both sides of (27)  $l$ -times derivative at  $\tau = 0$  we get

$$\begin{aligned} & \sum_{n=0}^{\lfloor \frac{l}{2} \rfloor - 1} C_n \sum_{i=0}^n \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + \sum_{n=\lfloor \frac{l}{2} \rfloor}^{\infty} C_n \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + \\ & + D_n \frac{2^{2n+1} l!}{(l-2n-1)!} \sum_{m=1}^{l-2n-1} \left( i^{2n+1-m} \operatorname{erfc}(-\gamma_1) + (-1)^{2n+1-m} \operatorname{erfc}(\gamma_1) \right) \beta_{l-2n-1,m} = \theta_m l! \beta_l \end{aligned} \tag{28}$$

from (28), (23\*) and (25) we can find  $C_n$  and  $D_n$ .

By the properties of Integral error functions [6] and condition (9) we get

$$\sum_{n=0}^{\infty} \left\{ E_n r^{2n} + \frac{G_n}{r} \frac{2}{(2n+1)!} r^{2n+1} \right\} = f(r)$$

Suggesting that the initial function can be expanded in  $f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n$

we have



$$E_n + G_n \frac{2}{(2n+1)!} = \frac{f^{(2n)}(0)}{2n!} \quad (29)$$

From condition (13b) we have

$$\sum_{n=0}^{\infty} E_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} + \sum_{n=0}^{\infty} G_n \frac{(2a_1 \tau)^{2n+1}}{\beta(\tau)} (i^{2n+1} \operatorname{erfc}(-\xi(\tau)) - i^{2n+1} \operatorname{erfc}(\xi(\tau))) = \theta_m \quad (30)$$

where

$$\xi(\tau) = \frac{\beta(\tau)}{2a_2}$$

As previously by taking by both sides of (30),  $l$  – times derivatives at  $\tau = 0$  by using Leibniz, Faa Di Bruno's formulas and Bell polynomials we have

$$\sum_{n=0}^{\lfloor \frac{l}{2} \rfloor - 1} E_n \sum_{i=0}^n \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + \sum_{n=\lfloor \frac{l}{2} \rfloor}^{\infty} E_n \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \zeta_{n,i,2} \eta(\beta)_{l-2i,i,2} + G_n \frac{2^{2n+1} l!}{(l-2n-1)!} \sum_{m=1}^{l-2n-1} (i^{2n+1-m} \operatorname{erfc}(-\xi_1) + (-1)^{2n+1-m} \operatorname{erfc}(\xi_1)) \beta_{l-2n-1,m} = \theta_m l! \beta_l \quad (31)$$

From expression (29) and (31) we can find  $E_n$  and  $G_n$

Satisfying Stefan's condition (14) and substituting  $\sqrt{t} = \tau$  we have

$$\begin{aligned} & -\lambda_1 \left\{ \sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k-1} \tau^{2k} - \frac{1}{\beta^2(\tau)} \sum_{n=0}^{\infty} D_n (2a_1 \tau)^{2n+1} (i^{2n+1} \operatorname{erfc}(-\gamma(\tau)) - i^{2n+1} \operatorname{erfc}(\gamma(\tau))) - \frac{\lambda_1}{\beta(\tau)} \sum_{n=0}^{\infty} D_n (2a_1 \tau)^{2n} (i^{2n} \operatorname{erfc}(-\gamma(\tau)) + i^{2n} \operatorname{erfc}(\gamma(\tau))) \right\} = \\ & = -\lambda_2 \left\{ \sum_{n=0}^{\infty} E_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k-1} \tau^{2k} - \frac{1}{\beta^2(\tau)} \sum_{n=0}^{\infty} G_n (2a_2 \tau)^{2n+1} (i^{2n+1} \operatorname{erfc}(-\xi(\tau)) - i^{2n+1} \operatorname{erfc}(\xi(\tau))) + \frac{1}{\beta(\tau)} \sum_{n=0}^{\infty} D_n (2a_2 \tau)^{2n} (i^{2n} \operatorname{erfc}(-\xi(\tau)) + i^{2n} \operatorname{erfc}(\xi(\tau))) \right\} + L\gamma\beta'(\tau) \end{aligned} \quad (32)$$

multiply both sides of (32) by  $\beta(\tau)$  and using conditions (13a), (13b) we have the following expression

$$\begin{aligned} & -2\lambda_1 \sum_{n=0}^{\infty} C_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} - \lambda_1 \sum_{n=0}^{\infty} D_n (2a_1 \tau)^{2n} (i^{2n} \operatorname{erfc}(-\gamma(\tau)) + i^{2n} \operatorname{erfc}(\gamma(\tau))) \\ & = -2\lambda_2 \sum_{n=0}^{\infty} E_n \sum_{k=0}^n \zeta_{n,k,2} \beta(\tau)^{2n-2k} \tau^{2k} - \lambda_2 \sum_{n=0}^{\infty} G_n (2a_2 \tau)^{2n} (i^{2n} \operatorname{erfc}(-\xi(\tau)) + i^{2n} \operatorname{erfc}(\xi(\tau))) \\ & \quad + (\lambda_2 - \lambda_1) \theta_m + L\gamma'\psi(\tau) \end{aligned} \quad (33)$$

where

$$\psi(\tau) = \beta'(\tau)\beta(\tau) = \frac{1}{2} \frac{d}{d\tau} \beta^2(\tau)$$

$$(\beta(\tau))^2 = \sum_{n=0}^{\infty} \mu(\beta)_n \tau^n$$

$$\mu(\beta) = \beta_0^2 \quad \mu(\beta)_m = \frac{1}{m\beta_0} \sum_{k=1}^m (3k - m)\beta_k \cdot \mu(\beta)_{m-k} \quad m \geq 1$$

Previously by taking both sides of (33)  $l$  – times derivatives  $\tau = 0$  and for  $l \geq 1$  we have

$$-2\lambda_1 \left( \sum_{n=0}^{\lfloor \frac{l}{2} \rfloor - 1} C_n \chi_{1,n} + \sum_{n=\lfloor \frac{l}{2} \rfloor}^{\infty} C_n \chi_{2,n} \right) - \lambda_1 D_n \frac{2^{2n} l!}{(l - 2n)!} \sum_{m=1}^{l-2n} (i^{2n-m} \operatorname{erfc}(-\gamma_1) + (-1)^m i^{2n-m} \operatorname{erfc}(\gamma_1)) \beta_{l-2n,m}$$

$$= -2\lambda_2 \left( \sum_{n=0}^{\lfloor \frac{l}{2} \rfloor - 1} E_n \chi_{21n} + \sum_{n=\lfloor \frac{l}{2} \rfloor}^{\infty} E_n \chi_{2,n} \right) - \lambda_2 G_n \frac{2^{2n} l!}{(l - 2n)!} \sum_{m=1}^{l-2n} (i^{2n-m} \operatorname{erfc}(-\xi_1) + (-1)^m i^{2n-m} \operatorname{erfc}(\xi_1)) \beta_{l-2n,m}$$

$$+ \frac{L\gamma'}{2} l! \mu(\beta)_{l+1} \quad (34)$$

For  $l = 0$  we have

$$\theta_m - D_0 (\operatorname{erfc}(-\gamma_1) + \operatorname{erfc}(\gamma_1)) - 2 \sum_{n=0}^{\infty} C_n \zeta_{n,0,2} r_0^{2n} = \frac{\lambda_2}{\lambda_1} \left( \theta_m - G_0 (\operatorname{erfc}(-\xi_1) + \operatorname{erfc}(\xi_1)) - 2 \sum_{n=0}^{\infty} E_n \zeta_{n,0,2} r_0^{2n} \right)$$

where

$$\chi_{2,n} = \sum_{i=0}^{\lfloor \frac{l}{2} \rfloor} \zeta_{n,i,2} \nu(\beta)_{l-2i,i,2}$$

$$\chi_{1,n} = \sum_{i=0}^n \zeta_{n,i,2} \nu(\beta)_{l-2i,i,2}$$

From this recurrent formula we can express  $\beta_n$ .

**Conclusion**

To summarize, the coefficients  $A_n, C_n, D_n, E_n, G_n$  are determined from equations (22),(23\*),(28),(29),(31) and (25), the moving boundary  $\beta(t)$  obtained from equation (34). For the convergence of temperature functions  $\theta_1, \theta_2, \theta_3$ , it is possible to follow the idea proposed in [6].

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### КӨПІР ЭРОЗИЯСЫНЫҢ ТЕОРИЯ ЕСЕБІ

**Аннотация.** Осы мақалада біз екі фазалық Стефан мәселесінің дәл шешімін ұсынамыз. Көпір мәселесін шешу үшін радиалды жылу полиномы және интегралдық қателіс функциясы қолданылады. Осы қатарлардың коэффициенттері үшін қайталанатын өрнектер ұсынылған. Математикалық модельдер байланыстың ашылу және көпіршікті динамикасын сипаттайды

**Түйін сөздер:** радиалды жылу полиномы, Стефан проблемасы, интеграл

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### ЗАДАЧА ИЗ ТЕОРИИ МОСТИКОВОЙ ЭРОЗИИ

**Аннотация.** В настоящей работе мы представляем точное решение двухфазной задачи Стефана. Для решения данной задачи использовали решение в виде радиальных тепловых полиномов и интегральной функции ошибок. Приводятся рекуррентные выражения для коэффициентов ряда. Математические модели описывают динамику размыкания металлических контактов.

**Ключевые слова:** радиальные тепловые полиномы, проблема Стефана.

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**МАЗМҰНЫ**

<i>Малков Е.А., Момынов С.Б.</i> Хенон-Хейлес потенциалының фазалық бейнесі.....	5
<i>Каракеев Т.Т., Мустафаева Н.Т. Ж.</i> Бірінші түрдегі сызықты емес интегралды Вольтерра теңдеулерін сандық шешу әдісі.....	10
<i>Харин С. Н., Қасабек С. А., Салыбек Д., Ашимов Т.</i> Эллипсоидтік координаттардағы стефан есебі.....	19
<i>Джумабаев Д.С., Бакирова Э.А., Кадирбаева Ж.М.</i> Параметрі бар дифференциалдық теңдеулер үшін басқару есебін шешудің бір алгоритмі туралы.....	25
<i>Асанова А.Т., Алиханова Б.Ж., Назарова К.Ж.</i> Үшінші ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін интегралдық шарттары бар бейлокал есептің корректілі шешілімділігі.....	33
<i>Сейтмұратов А.Ж., Тілеубай С.Ш., Тоқсанова С.К., Ибрагимова Н.Ж., Досжанов Б.А., Айтимов М.Ж.</i> Қатаң шекаралармен шектелген серпімді қабат тербелісі жайлы есеп.....	42
<i>Ахметов Дж. Ш., Сейтова С.М., Тойбазаров Д.Б., Кадырбаева Г.Т., Даулеткулова А.У., Исаева Г. Б.</i> Техникалық құрылғылардың жауапкершілікті және аталған мүмкіндіктерді тиімділікті арқылы.....	49
<i>Калимолдаев М.Н., Абдилдаева А.А., Ахметжанов М.А., Галиева Ф.М.</i> Электр энергетикалық жүйелерді тиімді басқару мәселесін математикалық модельдеу.....	62
<i>Харин С. Н., Қасабек С. А., Слямхан М.</i> Көпір эрозиясының теория есебі.....	69

СОДЕРЖАНИЕ

<i>Малков Е.А., Момынов С.Б.</i> Фазовые портреты потенциала Хенона-Хейлеса.....	5
<i>Каракеев Т.Т., Мустафаева Н.Т.</i> Метод численного решения нелинейных интегральных уравнений Вольтерра первого рода.....	10
<i>Харин С. Н., Касабек С. А., Салыбек Д., Ашимов Т.</i> Задача Стефана в эллипсоидальных координатах.....	19
<i>Джумабаев Д.С., Бакирова Э.А., Кадирбаева Ж.М.</i> Об одном алгоритме решения задачи управления для дифференциальных уравнений с параметром.....	25
<i>Асанова А.Т., Алиханова Б.Ж., Назарова К.Ж.</i> Корректная разрешимость нелокальной задачи с интегральными для системы дифференциальных уравнений в частных производных третьего порядка .....	33
<i>Сейтмуратов А.Ж., Тилеубай С.Ш., Токсанова С.К., Ибрагимова Н.Ж., Досжанов Б.А., Айтимов М.Ж.</i> Задача о колебании упругого слоя ограниченные жесткими границами .....	42
<i>Ахметов Дж.Ш., Сейтова С.М., Тойбазаров Д.Б., Кадырбаева Г.Т., Даулеткулова А.У., Исаева Г. Б.</i> Проверка технических устройств надежности через решение вероятности неисправности и исправности.....	49
<i>Калимолдаев М.Н., Абдилдаева А.А., Ахметжанов М.А., Галиева Ф.М.</i> Математическое моделирование задачи оптимального управления электроэнергетическими системами.....	62
<i>Харин С. Н., Касабек С. А., Слямхан М.</i> Задача из теории мостиковой эрозии.....	69

## CONTENTS

<i>Malkov E.A., Momynov S.B.</i> Phase portraits of the Henon-Heiles potential.....	5
<i>Karakeev T.T., Mustafayeva N.T.</i> The method of numerical solution of nonlinear Volterra integral equations of the first kind.....	10
<i>Kharin S.N., Kassabek S.A., Salybek D., Ashymov T.</i> Stefan problem in ellipsoidal coordinates.....	19
<i>Dzhumabaev D.S., Bakirova E.A., Kadirbayeva Zh.M.</i> An algorithm for solving a control problem for a differential equation with a parameter.....	25
<i>Assanova A.T., Alikhanova B.Zh., Nazarova K.Zh.</i> Well-posedness of a nonlocal problem with integral conditions for third order system of the partial differential equations.....	33
<i>Seitmuratov A., Tileubay S., Toxanova S., Ibragimova N., Doszhanov B., Aitimov M.Zh.</i> The problem of the oscillation of the elastic layer bounded by rigid bouhdaries.....	42
<i>Akmetov J.W., Seitova S.M., Toibazarov D.B., Kadyrbayeva G.T., Dauletkulova A.U., Issayeva G. B.</i> Verification of reliability technical devices through resolving probability of failure and failure.....	49
<i>Kalimoldayev M.N., Abdildayeva A.A., Akhmetzhanov M.A., Galiyeva F.M.</i> Mathematical modeling of the problem of optimal control of electric power systems.....	62
<i>Kharin S.N., Kassabek S.A., Slyamkhan M.</i> Problem from the theory of bridge erosion.....	69

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**ISSN 2518-1726 (Online), ISSN 1991-346X (Print)**

Редакторы *М. С. Ахметова, Т.А. Апендиев, Д.С. Аленов*  
Верстка на компьютере *А.М. Кульгинбаевой*

Подписано в печать 11.10.2018.

Формат 60x881/8. Бумага офсетная. Печать – ризограф.  
4,9 п.л. Тираж 300. Заказ 5.