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AN INITIAL-BOUNDARY VALUE PROBLEM FOR A HIGHER-ORDER PARTIAL DIFFERENTIAL EQUATION

Abstract. The initial-boundary value problem for higher-order partial differential equations is considered. We study the existence of its classical solutions, and also propose a method for finding approximate solutions. Paper establishes sufficient conditions for the existence and uniqueness of the classical solution of the problem under consideration. Introducing a new unknown function, we reduce the considered problem to an equivalent problem consisting of a nonlocal problem for second-order hyperbolic equations with functional parameters and integral relations. An algorithm for finding an approximate solution to the problem under study is proposed and its convergence is proved. Sufficient conditions for the existence of a unique solution to an equivalent problem with parameters are established. The conditions for the unique solvability of the initial-boundary value problem for higher-order partial differential equations are obtained in terms of the initial data. Solvability of the initial-boundary value problem for higher-order partial differential equations is connected with solvability of the nonlocal problem for second-order hyperbolic equations.

Keywords: the higher order partial differential equations, initial-boundary value problem, nonlocal problem, hyperbolic equations of second order, solvability, algorithm.

Introduction. It is well known that initial-boundary value problems for higher-order partial differential equations belong to one of the most important classes of problems in mathematical physics. To study various boundary value problems for higher-order partial differential equations, along with classical methods of mathematical physics, such as the Fourier method, the Green's function method, the Poincare metric concept, the method of differential inequalities, and other methods of the qualitative theory of ordinary differential equations are also often applied. Based on these methods, the solvability conditions of the considered boundary value problems were obtained and ways to solve them were proposed in [1-14]. However, the search for effective signs of the unique solvability of initial-boundary value problems, an analogue of multipoint boundary value problems for higher-order partial differential equations, still remains relevant. The article by T. I. Kiguradze and T. Kusano is one of the first works to fill this gap [4]. This paper establishes an equivalence between the well-posedness of the initial-boundary value problem for a higher-order hyperbolic equation and the existence of only trivial solutions of the corresponding family of homogeneous boundary value problems for ordinary differential equations. Based on it, a criterion is established for the well-posedness of initial-boundary value problems for one class of partial differential equations of higher-order hyperbolic type. It is known that by means of substitution, an ordinary differential equation of higher order can be reduced to a system of ordinary differential equations of the first order. Using the methods of the qualitative theory of differential equations, the solvability conditions for the resulting system can be formulated in terms of the fundamental matrix of the differential part or the right side of the system. A similar approach can be applied to higher-order hyperbolic equations with two independent variables, and by replacement, the equations can be reduced to a system of second order hyperbolic equations with mixed derivatives. Then, using well-known methods for solving boundary value problems for systems of hyperbolic equations with mixed derivatives, the solvability conditions can be established in different terms.

Mathematical modeling of many problems of physics, mechanics, chemistry, biology, and other sciences has resulted into the necessity of studying multipoint and nonlocal boundary value problems for higher-order partial differential equations of hyperbolic type. Applying the methods of the qualitative theory of differential equations directly to these problems, we can establish the conditions for their solvability [4, 8, 14]. Multipoint and nonlocal boundary value problems for high-order partial differential equations of hyperbolic type by replacement are reduced to nonlocal boundary value problems for systems of second-order hyperbolic equations. The theory of nonlocal boundary value problems for systems of second-order hyperbolic equations has been developed in many papers. To date, various solvability conditions for nonlocal boundary value problems for hyperbolic equations have been obtained.

The criteria for the unique solvability of some classes of linear boundary value problems for hyperbolic equations with variable coefficients were obtained relatively recently [15-21]. In [15], a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations by introducing new unknown functions is reduced to a problem consisting of a family of boundary value problems with an integral condition for systems of ordinary differential equations and functional relations. It is established that the well-posedness of a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations is equivalent to the well-posedness of a family of two-point boundary value problems for a system of ordinary differential equations. In terms of the initial data, a criterion is obtained for the well-posedness of a nonlocal boundary value problem with an integral condition for systems of hyperbolic equations.

In present paper, we consider a higher-order partial differential equation defined in a rectangular domain. The boundary conditions for the time variable are specified as a combination of values from the partial derivatives of the desired solution in rows $t = t_j$, $j=\overline{1,l}$. We also study the existence and uniqueness of the classical solution to the initial-boundary value problem for a higher-order partial differential equation and its applications.

1. Methods. To solve the problem under consideration, we use the method of introducing additional functional parameters [15-33] and reduce the original problem to an equivalent problem consisting of a nonlocal problem for a second-order hyperbolic equation with functional parameters and integral relations. We establish sufficient conditions for the unique solvability of the problem under study in terms of the initial data. Algorithms for finding a solution to an equivalent problem are constructed. The conditions for the unique solvability of the initial-boundary-value problem for a system of fourth-order partial differential equations are established in terms of the coefficients of the system and the boundary matrices. Separately, the result is given for an initial periodic-time boundary value problem. Note that in [34–36] a similar approach was applied to the initial-boundary value problem and the nonlocal problem for a system of partial differential equations of the third and fourth orders.

2. Statement of problem. At the domain $\Omega=[0,T]\times[0,\omega]$, we consider the initial-boundary value problem for the higher-order partial differential equation of the following form:

$$\frac{\partial^{m+1}u}{\partial t\partial x^m}=\sum_{i=1}^{m}\left\{A_i(t,x)\frac{\partial^i u}{\partial x^i}+B_i(t,x)\frac{\partial^i u}{\partial t\partial x^{i-1}}\right\}+C(t,x)u+f(t,x),\ (t,x)\in\Omega, \tag{1}$$

$$\sum_{j=1}^{l}\sum_{i=1}^{m}\left\{P_{ij}(x)\frac{\partial^i u(t,x)}{\partial x^i}+S_{ij}(x)\frac{\partial^i u(t,x)}{\partial t\partial x^{i-1}}\right\}\Big|_{t=t_j}=\varphi(x),\ x\in[0,\omega], \tag{2}$$

$$u(t,0)=\psi_0(t),\ \frac{\partial u(t,x)}{\partial x}\Big|_{x=0}=\psi_1(t),\ \ldots,\ \frac{\partial^{m-1}u(t,x)}{\partial x^{m-1}}\Big|_{x=0}=\psi_{m-1}(t),\ t\in[0,T], \tag{3}$$

where $u(t,x)$ is an unknown function, the functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ are continuous on Ω, the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\varphi(x)$ are continuous on $[0,\omega]$, $0\le t_1<t_2<\ldots<t_{l-1}<t_l\le T$, the functions $\psi_s(t)$, $s=\overline{0,m-1}$, are continuously differentiable on $[0,T]$. The initial data satisfy the matching condition.

A function $u(t,x) \in C(\Omega,R)$ having partial derivatives $\frac{\partial^{p+r}u(t,x)}{\partial t^r \partial x^p} \in C(\Omega,R)$, $p=\overline{1,m}$, $r=0,1$, is called a classical solution to problem (1) - (3) if it satisfies equation (1) for all $(t,x)\in\Omega$, and the initial-boundary conditions (2), (3).

We will investigate the questions of the existence and uniqueness of classical solutions to the initial-boundary value problem for a higher-order partial differential equation (1) - (3) and the construction of its approximate solutions. For these purposes, we apply the method of introducing additional functional parameters proposed in [15–33] for solving various nonlocal problems for systems of hyperbolic equations with mixed derivatives. The considered problem is reduced to a nonlocal problem for second-order hyperbolic equations, including additional functions, and integral relations. An algorithm for finding an approximate solution to the problem under study is proposed and its convergence is proved. Sufficient conditions for the existence of a unique classical solution to problem (1) - (3) are obtained in terms of the initial data.

3. Scheme of the method and reduction to equivalent problem. We introduce new unknown functions

$$v(t,x)=\frac{\partial^{m-1}u(t,x)}{\partial x^{m-1}},\ v_1(t,x)=u(t,x),\ v_2(t,x)=\frac{\partial u(t,x)}{\partial x},\ \ldots,\ v_{m-1}(t,x)=\frac{\partial^{m-2}u(t,x)}{\partial x^{m-2}} \quad (4)$$

and re-write problem (1)--(3) in the following form:

$$\frac{\partial^2 v}{\partial t\partial x}=A_m(t,x)\frac{\partial v}{\partial x}+B_m(t,x)\frac{\partial v}{\partial t}+A_{m-1}(t,x)v+f(t,x)+$$
$$+\sum_{r=1}^{m-2}A_r(t,x)v_{r+1}(t,x)+\sum_{s=1}^{m-1}B_r(t,x)\frac{\partial v_r(t,x)}{\partial t}+C(t,x)v_1(t,x),\ (t,x)\in\Omega, \quad (5)$$

$$\sum_{j=1}^{l}\left\{P_{m,j}(x)\frac{\partial v(t,x)}{\partial x}+S_{m,j}(x)\frac{\partial v(t,x)}{\partial t}+P_{m-1,j}(x)v(t,x)\right\}\Big|_{t=t_j}=\varphi(x)-$$
$$+\sum_{j=1}^{l}\left\{\sum_{r=1}^{m-2}P_{r,j}(x)v_{r+1}(t,x)+\sum_{s=1}^{m-1}S_{s,j}(x)\frac{\partial v_s(t,x)}{\partial t}\right\}\Big|_{t=t_j},\ x\in[0,\omega], \quad (6)$$

$$v(t,0)=\psi_{m-1}(t),\ t\in[0,T], \quad (7)$$

$$v_s(t,x)=\sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}v(t,\xi)d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega. \quad (8)$$

Here the conditions (3) are taken into account in (9).

Differentiating (8) by t, we obtain

$$\frac{\partial v_s(t,x)}{\partial t}=\sum_{p=0}^{s-1}\dot{\psi}_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}\frac{\partial v(t,\xi)}{\partial t}d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega. \quad (9)$$

A system of functions $(v(t,x),v_1(t,x),v_2(t,x),\ldots,v_{m-1}(t,x))$, where the function $v(t,x)\in C(\Omega,R)$ has partial derivatives $\frac{\partial v(t,x)}{\partial x}\in C(\Omega,R)$, $\frac{\partial v(t,x)}{\partial t}\in C(\Omega,R)$, and $\frac{\partial^2 v(t,x)}{\partial t\partial x}\in C(\Omega,R^n)$, functions $v_s(t,x)\in C(\Omega,R)$ have partial derivatives $\frac{\partial v_s(t,x)}{\partial t}\in C(\Omega,R)$, $s=\overline{1,m-1}$, is called a solution to problem (5)--(8) if it satisfies the second-order hyperbolic equation of (5) for all $(t,x)\in\Omega$, boundary conditions (6) and (7) and integral relations (8).

For fixed $v_s(t,x)$, $s=\overline{1,m-1}$, problem (5)--(7) is a nonlocal problem for the hyperbolic equation with respect to $v(t,x)$ on Ω. Integral relations (8) allow us to determine unknown functions $v_s(t,x)$, $s=\overline{1,m-1}$ for all $(t,x)\in\Omega$.

4. Algorithm. We determine the unknown function $v(t,x)$ from the nonlocal problem for hyperbolic equations (5)--(7). Unknown functions $v_s(t,x)$, $s=\overline{1,m-1}$, will be found from integral relations (8).

If we know the functions $v_s(t,x)$, $s=\overline{1,m-1}$, then from the nonlocal problem (5)--(7) we find the function $v(t,x)$. And, conversely, if we know the function $v(t,x)$, then from the integral conditions (8) we find the functions $v_s(t,x)$, $s=\overline{1,m-1}$. Since both functions $v(t,x)$, $v_s(t,x)$, $s=\overline{1,m-1}$, are unknown, then to find a solution to problem (5)--(8) we use an iterative method. The solution to problem (5)--(8) is the system of functions $(v^*(t,x), v_1^*(t,x), v_2^*(t,x), \ldots, v_{m-1}^*(t,x))$, which we defined as the limit of the sequence of systems $(v^{(k)}(t,x), v_1^{(k)}(t,x), v_2^{(k)}(t,x), \ldots, v_{m-1}^{(k)}(t,x))$, $k=0,1,2,\ldots$, according to the following algorithm:

Step 0. 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=\sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!}$ and $\frac{\partial v_s(t,x)}{\partial t}=\sum_{p=0}^{s-1}\dot{\psi}_k(t)\frac{x^p}{p!}$, $s=\overline{1,m-1}$. From nonlocal problem (5)--(7) we find the initial approximation $v^{(0)}(t,x)$ and its partial derivatives $\frac{\partial v^{(0)}(t,x)}{\partial x}$ and $\frac{\partial v^{(0)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$;

2) From integral relations (8) and (9) under $v(t,x)=v^{(0)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(0)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(0)}(t,x)$ and $\frac{\partial v_s^{(0)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$.

Step 1. 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=v_s^{(0)}(t,x)$ and $\frac{\partial v_s(t,x)}{\partial t}=\frac{\partial v_s^{(0)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$. From nonlocal problem (5)--(7) we find the first approximation $v^{(1)}(t,x)$ and its partial derivatives $\frac{\partial v^{(1)}(t,x)}{\partial x}$ and $\frac{\partial v^{(1)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$.

2) From integral relations (8) and (9) under $v(t,x)=v^{(1)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(1)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(1)}(t,x)$ and $\frac{\partial v_s^{(1)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$.

And so on.

Step k. 1) Suppose in the right-hand side of equation (5) we have $v_s(t,x)=v_s^{(k-1)}(t,x)$ and $\frac{\partial v_s(t,x)}{\partial t}=\frac{\partial v_s^{(k-1)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$. From nonlocal problem (6)--(7) we find the k-th approximation $v^{(k)}(t,x)$ and its partial derivatives $\frac{\partial v^{(k)}(t,x)}{\partial x}$ and $\frac{\partial v^{(k)}(t,x)}{\partial t}$ for all $(t,x)\in\Omega$:

$$\frac{\partial^2 v^{(k)}}{\partial t\partial x}=A_m(t,x)\frac{\partial v^{(k)}}{\partial x}+B_m(t,x)\frac{\partial v^{(k)}}{\partial t}+A_{m-1}(t,x)v^{(k)}+f(t,x)+$$

$$+\sum_{r=1}^{m-2}A_r(t,x)v_{r+1}^{(k-1)}(t,x)+\sum_{s=1}^{m-1}B_r(t,x)\frac{\partial v_r^{(k-1)}(t,x)}{\partial t}+C(t,x)v_1^{(k-1)}(t,x),\ (t,x)\in\Omega, \tag{10}$$

$$\sum_{j=1}^{l}\left\{P_{m,j}(x)\frac{\partial v^{(k)}(t,x)}{\partial x}+S_{m,j}(x)\frac{\partial v^{(k)}(t,x)}{\partial t}+P_{m-1,j}(x)v^{(k)}(t,x)\right\}\Big|_{t=t_j}=\varphi(x)-$$

$$+\sum_{j=1}^{l}\left\{\sum_{r=1}^{m-2}P_{r,j}(x)v_{r+1}^{(k-1)}(t,x)+\sum_{s=1}^{m-1}S_{s,j}(x)\frac{\partial v_s^{(k-1)}(t,x)}{\partial t}\right\}\Big|_{t=t_j},\ x\in[0,\omega],\tag{11}$$

$$v^{(k)}(t,0)=\psi_{m-1}(t),\ t\in[0,T].\tag{12}$$

2) From integral relations (8) and (9) under $v(t,x)=v^{(k)}(t,x)$ and $\frac{\partial v(t,x)}{\partial t}=\frac{\partial v^{(k)}(t,x)}{\partial t}$, respectively, we find the functions $v_s^{(k)}(t,x)$ and $\frac{\partial v_s^{(k)}(t,x)}{\partial t}$, $s=\overline{1,m-1}$, for all $(t,x)\in\Omega$:

$$v_s^{(k)}(t,x)=\sum_{p=0}^{s-1}\psi_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}v^{(k)}(t,\xi)d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega.\tag{13}$$

$$\frac{\partial v_s^{(k)}(t,x)}{\partial t}=\sum_{p=0}^{s-1}\dot{\psi}_k(t)\frac{x^p}{p!}+\int_0^x\frac{(x-\xi)^{m-1-s}}{(m-1-s)!}\frac{\partial v^{(k)}(t,\xi)}{\partial t}d\xi,\ s=\overline{1,m-1},\ (t,x)\in\Omega.\tag{14}$$

Here $k=1,2,3,\ldots$.

5. The main results. The following theorem provides conditions for the feasibility and convergence of the constructed algorithm, as well as conditions for the existence of a unique solution to problem (5)--(8). The functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ are continuous on Ω, the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\varphi(x)$ are continuous on $[0,\omega]$, the functions $\psi_s(t)$, $s=\overline{0,m-1}$, are continuously differentiable on $[0,T]$.

Theorem 1. *Let*

i) the functions $A_i(t,x)$, $B_i(t,x)$, $i=\overline{1,m}$, $C(t,x)$, and $f(t,x)$ be continuous on Ω;

ii) the functions $P_{ij}(x)$, $S_{ij}(x)$, $i=\overline{1,m}$, $j=\overline{1,l}$, and $\phi(x)$ be continuous on $[0,\omega]$;

iii) the functions $\psi_s(t)$, $s=\overline{0,m-1}$, be continuously differentiable on $[0,T]$;

iv) the function $Q(x)=\sum_{j=1}^{l}P_{m,j}(x)\exp\left[\int_0^{t_j}A_m(\tau,x)d\tau\right]\neq 0$ *for all $x\in[0,\omega]$.*

Then the nonlocal problem for the hyperbolic equation with parameters and integral conditions (5)--(8) has a unique solution $(v^*(t,x),v_1^*(t,x),v_2^*(t,x),\ldots,v_{m-1}^*(t,x))$ as a limit of sequences $(v^{(k)}(t,x),v_1^{(k)}(t,x),v_2^{(k)}(t,x),\ldots,v_{m-1}^{(k)}(t,x))$ determined by the algorithm proposed above for $k=0,1,2,\ldots$.

The proof of Theorem 1 is similar to the proof of Theorem 1 in [36].

The equivalence of problems (5)—(8) and (1)—(3) implies

Theorem 2. *Let conditions i) - iv) of Theorem 1 be fulfilled.*

Then the initial-periodic boundary value problem for the higher-order partial differential equation (1)--(3) has a unique classical solution $u^*(t,x)$.

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ЖОҒАРҒЫ РЕТТІ ДЕРБЕС ТУЫНДЫЛЫ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН БАСТАПҚЫ- ШЕТТІК ЕСЕП ТУРАЛЫ

Аннотация. Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есептер математикалыө физика мәселелерінің барынша маңызды кластарына жататыны жақсы белгілі. Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін әралуан есептерді зерттеу үшін, математикалық изиканың классикалық әдістерімен, мысалға Фурье әдісі, Грин функциялары әдісі сияқты, қатар, көп жағдайда Пуанкаренің метрикалық концепциясы, дифференциалдық теңсіздіктер әдісі және басқа да жәй дифференциалдық теңдеулердің сапалық теориясының әдістері қолданылды. Осы әдістер негізінде қарастырылып отырған шеттік есептердің шешілімділік шарттары алынды және оларды шешу тәсілдері ұсынылды. Осыған қарамастан, жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есептердің бірмәнді шешілімділігінің тиімді белгілерін іздеу мәселесі әлі де ашық әрі өзекті болып отыр. Бұрынырақ жоғарғы ретті гиперболалық теңдеу үшін бастапқы-шеттік есептің қисындылығы мен жоғарғы ретті жәй дифференциалдық теңдеу үшін сәйкес біртекті шеттік есептер әулетінің тек тривиалды шешімдерінің бар болуының арасындағы пара-парлық орнатылған болатын. Соған негізделе отырып, жоғарғы ретті гиперболалық тектес дербес туындылы дифференциалдық теңдеулердің бір класы үшін бастапқы-шеттік есептердің қисындылық критерийі тағайындалды. Жоғарғы ретті жәй дифференциалдық теңдеуді алмастырулар көмегімен бірінші ретті жәй дифференциалдық теңдеулер жүйесіне келтіруге болатыны баршаға белгілі. Дифференциалдық теңдеулердің сапалық теориясының әдістерін пайдалана отырып, алынған жүйенің шешілімділік шарттары осы жүйенің дифференциалдық бөлігінің фундаменталдық матрицасы терминінде тұжырымдалуы мүмкін. Осыған ұқсас тәсілді екі айнымалылы жоғарғы ретті гиперболалық теңдеулерге қолдануға болады және олар аралас туындылы екінші ретті гиперболалық теңдеулер жүйесіне келтірілуі мүмкін. Сонда, аралас туындылы гиперболалық теңдеулер жүйелері үшін шеттік есептерді шешудің белгілі әдістерін пайдалана отырып, шешілімділік шарттары әртүрлі терминдерде тағайындалуы мүмкін.

Жаратылыстанудың көптеген есептерін математикалық моделдеу жоғарғы ретті гиперболалық тектес дербес туындылы теңдеулер үшін көпнүктелі және бейлокал шеттік есептерді зерттеу қажеттілігіне алып келді. Дифференциалдық теңдеулердің сапалық теориясының әдістерін осы есептерге тікелей қолдану арқылы олардың шешілімділігін орнатуға болады. Жоғарғы ретті гиперболалық тектес дербес туындылы теңдеулер үшін көпнүктелі және бейлокал шеттік есептер алмастыру жолымен екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептерге келтіріледі. Екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептер теориясы көптеген жұмыстарда дамытылған. Бүгінгі кезеңде екінші ретті гиперболалық теңдеулер жүйелері үшін бейлокал шеттік есептердің шешілімділігінің әралуан шарттары алынған. Айнымалы коэффициенттері бар гиперболалық теңдеулер үшін сызықты шеттік есептердің бірмәнді шешілімділігінің критерийлері салыстырмалы түрде жақында алынды. Авторлардың біреуінің жұмысында гиперболалық теңдеулер жүйелері үшін интегралдық шарты бар бейлокал шеттік есеп жаңа белгісіз функциялар енгізу арқылы жәй дифференциалдық теңдеулер жүйелері үшін интегралдық шарты бар шеттік есептер әулеті мен функционалдық қатынастардан тұратын есепке келтіріледі. Гиперболалық теңдеулер жүйесі үшін бейлокал есептің қисынды шешілімділігі жәй дифференциалдық теңдеулер жүйесі үшін шеттік есептер әулетінің қисынды шешілімділігіне пара-пар екені орнатылды. Гиперболалық теңдеулер жүйелері үшін интегралдық шарты бар бейлокал шеттік есептің қисынды шешілімділігі критерийі бастапқы берілімдер терминінде алынды.

Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер үшін бастапқы-шеттік есеп қарастырылады. Оның классикалық шешімдерінің бар болуы мәселелері мен жуық шешімдерін табу әдістері зерттелген. Жаңа белгісіз функциялар енгізу жолымен зерттеліп отырған есеп гиперболалық теңдеулер үшін параметрлері бар бейлокал есептен және интегралдық қатынастардан тұратын пара-пар есепке келтірілген. Зерттеліп отырған есептің жуық шешімін табу алгоритмдері ұсынылған және олардың жинақтылығы дәлелденген. Параметрлері бар пара-пар есептің жалғыз шешімінің бар болуының жеткілікті шарттары тағайындалған. Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер жүйесі үшін бастапқы-шеттік есептің бірмәнді шешілімділігінің шарттары бастапқы берілімдер терминінде алынған.

Түйін сөздер: Жоғарғы ретті дербес туындылы дифференциалдық теңдеулер, бастапқы-шеттік есеп, бейлокал есеп, екінші ретті гиперболалық теңдеулер, шешілімділік, алгоритм.

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О НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧЕ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ВЫСОКОГО ПОРЯДКА

Аннотация. Хорошо известно, что начально-краевые задачи для дифференциальных уравнений в частных производных высокого порядка относятся к наиболее важным классам задач математической физики. Для исследования различных задач для дифференциальных уравнений в частных производных высокого порядка, наряду с классическими методами математической физики, таких как метод Фурье, метод функций Грина, также часто применялись метрическая концепция Пуанкаре, метод дифференциальных неравенств и другие методы качественной теории обыкновенных дифференциальных уравнений. На основе этих методов были получены условия разрешимости рассматриваемых краевых задач и предложены способы их решения. Несмотря на это, поиск эффективных признаков однозначной разрешимости начально-краевых задач для дифференциальных уравнений в частных производных высокого порядка все еще остается открытым. Ранее была установлена эквивалентность между корректностью начально-краевой задачи для гиперболического уравнения высокого порядка и существованием только тривиальных решений соответствующего семейства однородных краевых задач для обыкновенных дифференциальных уравнений высокого порядка. Основываясь на этом, установлен критерий корректности начально-краевых задач для одного класса дифференциальных уравнений в частных производных гиперболического типа высокого порядка. Известно, что с помощью замены обыкновенное дифференциальное уравнение высокого порядка может быть сведено к системе обыкновенных дифференциальных уравнений первого порядка. Используя методы качественной теории дифференциальных уравнений, условия разрешимости полученной системы могут быть сформулированы в терминах фундаментальной матрицы дифференциальной части системы. Аналогичный подход может быть применен к гиперболическим уравнениям высокого порядка с двумя независимыми переменными и они могут быть сведены к системе гиперболических уравнений второго порядка со смешанными производными. Тогда, используя известные методы решения краевых задач для систем гиперболических уравнений со смешанными производными, условия разрешимости могут быть установлены в различных терминах.

Математическое моделирование многих задач естествознания привело к необходимости изучения многоточечных и нелокальных краевых задач для уравнений в частных производных высокого порядка гиперболического типа. Применяя методы качественной теории дифференциальных уравнений непосредственно к этим задачам, можно установить условия их разрешимости. Многоточечные и нелокальные краевые задачи для уравнений с частными производными высокого порядка гиперболического типа путем замены сводятся к нелокальным краевым задачам для систем гиперболических уравнений второго порядка. Теория нелокальных краевых задач для систем гиперболических уравнений второго порядка развита во многих работах. К настоящему времени получены различные условия разрешимости нелокальных краевых задач для гиперболических уравнений. Критерии однозначной разрешимости некоторых классов линейных краевых задач для гиперболических уравнений с переменными коэффициентами были получены сравнительно недавно. В работе одного из авторов нелокальная краевая задача с интегральным условием для систем гиперболических уравнений путем введения новых неизвестных функций сводится к задаче, состоящей из семейства краевых задач с интегральным условием для систем обыкновенных дифференциальных уравнений и функциональных отношений. Установлено, что корректная разрешимость нелокальной задачи для системы гиперболических уравнений эквивалентна корректной разрешимости семейства краевых задач для системы обыкновенных дифференциальных уравнений. Получен критерий корректной разрешимости нелокальной краевой задачи с интегральным условием для систем гиперболических уравнений в терминах исходных данных.

Рассматривается начально-краевая задача для дифференциальных уравнений в частных производных высокого порядка. Исследуются вопросы существования ее классических решений и предлагаются методы нахождения приближенных решений. Установлены достаточные условия существования и единственности классического решения рассматриваемой задачи. Введя новые неизвестные функции мы сводим исследуемую задачу к эквивалентной задаче, состоящей из нелокальной задачи для гиперболических уравнений второго порядка с функциональными параметрами и интегральных соотношений. Предложены алгоритмы нахождения приближенного решения исследуемой задачи и доказана их сходимость. Установлены достаточные условия существования единственного решения эквивалентной задачи с параметрами. Условия однозначной разрешимости начально-краевой задачи для дифференциальных уравнений в частных производных высокого порядка получены в терминах исходных данных.

Ключевые слова: дифференциальные уравнения в частных производных высокого порядка, начально-краевая задача, нелокальная задача, гиперболические уравнения второго порядка, разрешимость, алгоритм.

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