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D.N. Nurgabyl[1,2], T.M. Seitova[2]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;
[2]Zhetysu State University named after I.Zhansugurov, Taldykorgan, Kazakhstan.
E-mail: kebek.kz@mail.ru, terasyltolganai@mail.ru

INTRODUCING METHOD OF GENERALIZED DERIVATIVE CONCEPT IN MATHEMATICS

Abstract. In this paper, we consider a technique for introducing the concept of a generalized derivative function of one variable. Some assertions of combinatorics are given and proved; the concept of the derivative of the natural order is introduced using the limit of the sequence. Using the above statements, the concept of a fractional derivative is introduced. The basic properties of the fractional derivative are formulated and proved. Examples are given

Key words: mathematical analysis, derivative, combination, limit transition, fractional derivative.

Introduction

Mathematical analysis is a field of mathematics related to the concepts of function, derivative and integral.

The great English physicist, astronomer and mathematician Isaac Newton and the German mathematician and philosopher Gottfried Leibniz completed the construction of differential and integral calculi by the end of the 17th century. The discovery of differential and integral calculus was the beginning of a period of rapid development of mathematics.

Mathematics continues to develop rapidly. Various generalizations of the concepts of function, derivatives, and integral have a particular interest. For example, mathematicians, including Leibniz, Euler, Liouville, and Riemann, dealt with generalization of the concept of a derivative. Generalized functions and their derivatives find various applications in real processes of the economy and production [1-4].

Our goal is to develop a methodology for introducing various definitions of a generalized derivative function of one variable.

To achieve this goal, we first give some statements of combinatorics, introduce the concept of a derivative of the natural order using the limit of the sequence. Using these statements, we introduce the concept of a fractional derivative

1. Auxiliary definitions and formulas

As it is known [5], the number of combinations of n elements by k is equal

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad n \geq k . \tag{1}$$

Here it is convenient to assume that $0! = 1$. The following well-known equalities are justly:

$$C_n^k + C_n^{k+1} = C_{n+1}^{k+1}; \tag{2}$$

$$C_n^0 \cdot C_m^k + C_n^1 \cdot C_m^{k-1} + ... + C_n^{k-1} \cdot C_m^1 + C_n^k \cdot C_m^0 = C_{n+m}^k, \quad m, n \geq k . \tag{3}$$

In particular, in $n = m$ formula (3) takes the form:

$$\sum_{i=0}^{k} C_n^i C_n^{k-i} = C_{2n}^k, \quad n > k \tag{4}$$

Obviously, that the formula (1) one can be wtitten in the form of

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!}. \quad (5)$$

Now, we will consider a generalization of formulas (2), (3), (5). We notice that the right-hand side of formula (5) is defined for any real values n. Thus, by definition, we assume that

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!}, \quad (6)$$

where τ is a real number. For example:

$$C_{-2}^k = \frac{(-2-0)(-2-1)(-2-2)...(-2-(k-1))}{k!} = (-1)^k(k+1), \; C_{-1}^k = (-1)^k$$

Lemma 1. For any real values of τ, the equality

$$C_\tau^k + C_{\tau+1}^k = C_{\tau+1}^{k+1} \quad (7)$$

Proof: Using (4), we have

$$C_\tau^k + C_\tau^{k+1} = \frac{\tau(\tau-1)...(\tau-(k-1))}{k!} + \frac{\tau(\tau-1)...(\tau-(k-1))(\tau+1)}{(k+1)!} =$$
$$= \frac{\tau(\tau-1)...(\tau-(k-1))}{k!} \cdot \left(1 + \frac{\tau-k}{k+1}\right) = \frac{\tau(\tau-1)...(\tau-(k-1))(\tau+1)}{(k+1)!} =$$
$$= \frac{(\tau+1-0)(\tau+1-1)(\tau+1-2)..(\tau+1-k)}{(k+1)!} = C_{\tau+1}^{k+1}.$$

Lemma 1 is proved.

Lemma 2. For any real values of τ, the equality

$$C_\tau^0 C_\tau^k + C_\tau^1 C_\tau^{k-1} + ... + C_\tau^{k-1} C_\tau^1 + C_\tau^k C_\tau^0 = C_{2\tau}^k \quad (8)$$

Proof. We use the following statement [6].

If the polynomials $P(x)$ and $Q(x)$, whose degrees do not exceed n, have equal values for more than n different of values unknowns, then $P(x) = Q(x)$.

We write the equalities $P(\tau) = \sum_{i=0}^{k} C_\tau^i C_\tau^{k-i}$, $Q(\tau) = C_{2\tau}^k$, where $P(\tau)$ and $P(\tau)$ are polynomials of degree k.

By virtue of formula (4), the polynomials $P(\tau)$ and $Q(\tau)$ have equal values at $\tau = n > k$. Then, by based on the above statement, it is easy to verify the validity of formula (8) for any real values τ. Lemma 2 is proved.

2. About a definition of a natural order derivative

For a function of one variable, we define a derivative of the natural order in a slightly different way. Let the function $f(x)$ be defined and continuous on the interval $[a, b], a < 0, b > 0$, at that $f(x) = 0$ for $x < 0$, where x is a fixed point of this interval.

We divide the segment $[0, x]$ into equal n parts by points $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$. We define the increment of the function $y = f(x)$ at the point x in the form $\Delta y = f(x) - f(x - \frac{x}{n})$. As it is known, if the ratio

$$\frac{\Delta y}{\frac{x}{n}} = \frac{f(x) - f(x - \frac{x}{n})}{\frac{x}{n}}$$

has a limit is at $n \to \infty$, then this limit is called the derivative of the function $f(x)$ at the point x and is denoted by:

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f\left(x - \frac{x}{n}\right)}{\frac{x}{n}} .$$

Now we define the increment $\Delta^2 y$:

$$\Delta^2 y = f(x) - f(x - \frac{x}{n}) - \left[f(x - \frac{x}{n}) - f(x - \frac{2x}{n}) \right] =$$

$$= f(x) - 2f(x - \frac{x}{n}) + f(x - \frac{2x}{n}) = C_2^0 f(x) + C_2^1 f(x - \frac{x}{n}) + C_2^2 f(x - \frac{2x}{n}) =$$

$$= \sum_{k=0}^{2} (-1)^k \cdot C_2^k f\left(x - \frac{kx}{n}\right),$$

etc. continuing this process, we obtain:

$$\Delta^m y = \sum_{k=0}^{m} (-1)^k C_m^k f(x - \frac{kx}{n}) . \qquad (9)$$

Let m be a fixed positive integer. We choose a positive integer n so that $n > m$. Then the formula (9) will take the form:

$$\Delta^m y = \sum_{k=1}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right), \qquad (10)$$

as $C_m^k = 0;\ k > m$. Using (10), we obtain

$$f^{(m)}(x) = \lim_{n\to\infty} \frac{\Delta^m y}{\left(\frac{x}{n}\right)^m} = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-m} \sum_{k=0}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right). \qquad (11)$$

As an example, we consider the function $f(x) = x^2$:

$$(x^2)'' = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-2} \left[x^2 - 2(x - \frac{x}{n})^2 + (x - \frac{2x}{n})^2 \right] = \lim_{n\to\infty} \frac{2\frac{x^2}{n^2}}{\frac{x^2}{n^2}} = 2 \text{, где } C_2^k = 0 \text{, } k = \overline{3, n} .$$

3. Definition of a fractional order derivative. Properties

Based on the fact that the expression C_τ^k according to (6) is defined for any real values of τ, then the right-hand side of formula (10) is determined for any real values of τ.

Now, using [7], for any real number τ we define the derivative of the τ th order:

$$f^{(\tau)}(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{x}{n}\right)^{\tau}}\sum_{k=0}^{n}(-1)^k c_\tau^k f\left(x-\frac{kx}{n}\right). \qquad (12)$$

Since any real number is an infinite decimal fraction, we call the derivative (14) fractional.

For example, given that $C_{-1}^{k}=(-1)^k$, from (12) the formula we find:

$$f^{(-1)}(x)=\lim_{n\to\infty}\frac{1}{\left(\frac{x}{n}\right)^{-1}}\sum_{k=0}^{n} f\left(x-\frac{kx}{n}\right)=\lim_{n\to\infty}\sum_{k=0}^{n} f\left(x-\frac{kx}{n}\right)\cdot\left(\frac{x}{n}\right) \qquad (13)$$

Example: We define the (-1)st order derivative for function $f(x)=x$. From (13) we get:

$$f^{(-1)}(x)=\lim_{n\to\infty}\frac{x}{n}\sum_{k=0}^{n}\left(x-\frac{kx}{n}\right)=\lim_{n\to\infty}\frac{x}{n}\left[(x-0)+\left(x-\frac{x}{n}\right)+\left(x-\frac{2x}{n}\right)+\cdots+\left(x-\frac{kx}{n}\right)\right]=$$

$$=\lim_{n\to\infty}\frac{x}{n}\left[(x-1)x-\frac{x}{n}\cdot\frac{(n+1)n}{2}\right]=\lim_{nn\to\infty}\left[\frac{x^2(n+1)}{n}-\frac{x^2(n+1)n}{2n^2}\right]=x^2-\frac{x^2}{2}=\frac{x^2}{2}.$$

In this way, $(x)^{-1}=\frac{x^2}{2}$.

Now we give the main properties of the τ th derivative for any real of values of τ.

Theorem 1. Right

$$(\alpha\cdot f(x)+\beta\cdot g(x))^{(\tau)}=\alpha\cdot f^{(\tau)}(x)+\beta\cdot g^{(\tau)}(x), \qquad (14)$$
$$\left[f^{(\tau_1)}(x)\right]^{(\tau_2)}=f^{(\tau_1+\tau_2)}(x).$$

To establish relations (14), it suffices to use representation (12) of the derivative of an arbitrary function.

From formula (15), we note that the sum $\sum_{л=0}^{m} f\left(x-\frac{kx}{n}\right)\cdot\frac{x}{n}$ represents the integral sum of the function $f(x)$ for a given partition $\left\{\frac{kx}{n}\right\}$, $k=\overline{1,n}$ of the segment $[0;x]$, $x\in[0,x]$.

Since the function $f(x)$, being continuous on the segment $[0,x]$, is integrable on $[0,x]$, therefore, the limit (13) gives us a definite integral $\lim_{n\to\infty}\sum_{k=0}^{n} f\left(x-\frac{kx}{n}\right)\cdot\frac{x}{n}=\int_0^x f(t)dt$ and thus, we obtain

$$f^{(-1)}(x)=\int_0^x f(t)dt. \qquad (15)$$

Theorem 2. For the natural value m, the derivative of the (–m)th order function is determined by the formula:

$$f^{(-m)}(x)=\frac{1}{(m-1)!}\int_0^x f(t)(x-t)^{m+1}dt. \qquad (16)$$

Proof: By hypothesis, the function $f(x)$ is continuous at any point $x \in [a,b]$. Therefore, the function $f^{(-1)}(x) = g(x) = \int_0^x f(t)dt$ is also continuous in the same $x \in [a,b]$. Therefore, for the point $x \in [0,x]$ we can find

$$f^{(-2)}(x) = \int_0^x f(t)(x-t)dt\ .$$

Next, by induction, we establish that

$$f^{(-k)}(x) = \frac{1}{(k-1)!}\int_0^x f(t)(x-t)^{k-1}dt\ . \tag{17}$$

Theorem 2 is proved.

Conclusions

Thus, in this paper, we propose a technique for introducing the concept of a fractional derivative. Using the limit of the sequence, the notion of a derivative of the natural order is introduced, the definition of a fractional derivative is given for any real values of x. The basic properties of a fractional derivative are proved. Examples are given.

Practice has shown that this approach of introducing the concept of a generalized derivative contributes to the effective assimilation by students of various definitions of generalized functions.

Д.Н. Нұрғабыл[1,2], Т.М. Сеитова[2]

[1] Математика және математикалық модельдеу институты, Алматы, Қазақстан;
[2] І. Жансүгіров атындағы Жетісу мемлекеттік университеті, Талдықорған, Қазақстан

МАТЕМАТИКА КУРСЫНДА ЖАЛПЫЛАНҒАН ТУЫНДЫ ҰҒЫМЫН БЕРУ ӘДІСТЕМЕСІ

Қазіргі математикада функция, туынды және интеграл ұғымдарының әртүрлі жалпылама тұжырымдары ерекше қызығушылық тудыруда. Мысалы, туынды ұғымның жалпылауымен математиктер, оның ішінде Лейбниц, Эйлер, Лиувиль және Риман айналысқан. Жалпыланған функциялар мен олардың туындылары экономика мен өндірістің нақты процестерінде әртүрлі қолданыстарын табуда. Бұл жұмыстың мақсаты бір айнымалы функцияның жалпыланған туындысының әр түрлі анықтамаларын енгізу әдістемесін жасау болып табылады. Аталған мақсатқа жету үшін бұл мақалада алдымен комбинаториканың тұжырымдарын, тізбек шегі ұғымын пайдалана отырып, натурал ретті туынды ұғымы енгізілген. Осы тұжырымдарды қолдана отырып, бөлшек туынды деген ұғым енгізіледі.

Комбинаторикада n элементтен k бойынша алынған терулер саны

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!} \tag{1}$$

формуласымен анықталатыны белгілі. (1) формуланың оң жағы кезкелген $n=\tau$ нақты саны саны үшін де анықталатынын байқаймыз. Онда анықтама бойынша

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!}\ . \tag{2}$$

формуласын қабылдауымызға болады. Осы формуланы пайдалана отырып бір айнымалы функцияның натурал ретті туындысын сәл басқаша енгізелік. $f(x)$ функциясы $[a,\ b],\ \ a<0,\ \ b>0$ аралығында анықталсын және осы интервалдың кезкелген нақты x нүктесі үшін $f(x)=0$, мұнда $x<0$. $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$ нүктелерімен $[0,\ x]$ кесіндісін n бірдей бөліктерге бөлелік. Сонда

$$\frac{\Delta y}{\frac{x}{n}} = \frac{f(x) - f(x-\frac{x}{n})}{\frac{x}{n}}$$

қатысының $n \to \infty$ -да шегі бар болса, онда бұл шекті $f(x)$ функциясының x нүктесіндегі туындысы деп атаймыз. Сонымен:

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f\left(x - \frac{x}{n}\right)}{\frac{x}{n}} \; ; \quad f^{(m)}(x) = \lim_{n\to\infty} \frac{\Delta^m y}{\left(\frac{x}{n}\right)^m} = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-m} \sum_{k=0}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right). \qquad (3)$$

Сонда (2) формулаға сәйкес C_τ^k өрнегі кезкелген τ нақты саны үшін анықталатын болғандықтан (3) формуланың оң жақтары кезкелген τ нақты саны үшін де анықталатын болады. Олай болса, кезкелген τ нақты саны үшін $f(x)$ функциясының x нүктесіндегі τ –ші ретті туындысы

$$f^{(\tau)}(x) = \lim_{n\to\infty} \frac{1}{\left(\frac{x}{n}\right)^\tau} \sum_{k=0}^{n} (-1)^k c_\tau^k f\left(x - \frac{kx}{n}\right). \qquad (4)$$

Формуласымен анықталады. Ал кезкелген нақты сан шексіз ондық бөлшек болғандықтан, (4) –ші туындыны бөлшек ретті туынды деп атаймыз .

Теорема . m натурал саны үшін $f(x)$ функциясының ($-m$)-ші ретті туындысы:

$$f^{(-m)}(x) = \frac{1}{(m-1)!} \int_0^x f(t)(x-t)^{m+1} dt \cdot$$

формуласымен анықталады.

Түйін сөздер: математикалық анализ, туынды, теру, шекке көшу, бөлшек ретті туынды

Д.Н. Нургабыл[1,2], Т.М.Сеитова[2]

[1]Институт математики и математического моделирования, Алматы, Казахстан;
[2]Жетысуский государственный университет им. И.Жансугурова,Талдыкорган, Казахстан

МЕТОДИКА ВВЕДЕНИЯ ПОНЯТИЯ ОБОБЩЕННОЙ ПРОИЗВОДНОЙ В КУРСЕ МАТЕМАТИКИ

Аннотация. В современной математике особый интерес представляют различные обобщения понятий функции, производных и интеграла. Например, вопросами об обобщении понятия производной занимались математики, в том числе Лейбниц, Эйлер, Лиувилл и Риман. Обобщенные функции и их производные находят различные применения в реальных процессах экономики и производства.

Цель настоящей работы – разработка методики введения различных определений обобщенной производной функции одной переменной.

Для достижения этой цели вначале приведены некоторые утверждения комбинаторики, введены понятие производной натурального порядка с помощью предела последовательности. Используя эти утверждения, введены понятие производной дробного порядка.

В комбинаторике число сочетаний из n элементов по k определяется по формуле

$$C_n^k = \frac{n(n-1)...(n-k+1)}{k!}. \qquad (1)$$

Заметим, что правая часть формулы (1) определена при любых вещественных значениях $n = \tau$. Тогда по определению можем положить, что

$$C_\tau^k = \frac{(\tau-0)(\tau-1)...(\tau-k+1)}{k!}. \qquad (2)$$

Используя эту формулу определим для функции одной переменной производную натурального порядка несколько иным образом.

Пусть функция $f(x)$ определена и непрерывна на промежутке $[a, b]$, $a < 0,\ b > 0$, причем $f(x) = 0$ при $x < 0$, где x -фиксированнная точка этого интервала.

Отрезок $[0,\ x]$ разобьем на n равных частей точками $0, \frac{x}{n}, \frac{2x}{n}, ..., \frac{nx}{n}$. Тогда, если отношение

$$\frac{\Delta y}{\frac{x}{n}} = \frac{f(x) - f(x - \frac{x}{n})}{\frac{x}{n}}$$

имеет предел при $n \to \infty$, то этот предел называется производной функции $f(x)$ в точке x и обозначается:

$$f'(x) = \lim_{n\to\infty} \frac{f(x) - f\left(x - \frac{x}{n}\right)}{\frac{x}{n}}, \quad f^{(m)}(x) = \lim_{n\to\infty} \frac{\Delta^m y}{\left(\frac{x}{n}\right)^m} = \lim_{n\to\infty} \left(\frac{x}{n}\right)^{-m} \sum_{k=0}^{n} (-1)^k C_m^k \cdot f\left(x - \frac{kx}{n}\right) \tag{3}$$

На основании того, что выражение C_τ^k согласно (2) определено при любых вещественных значениях τ, то правая часть формулы (3) определяется при любых вещественных значениях τ.

Следовательно, для любого вещественного числа τ производная τ–го порядка функции $f(x)$ в точке x определяется в виде:

$$f^{(\tau)}(x) = \lim_{n\to\infty} \frac{1}{\left(\frac{x}{n}\right)^\tau} \sum_{k=0}^{n} (-1)^k c_\tau^k f\left(x - \frac{kx}{n}\right). \tag{4}$$

Так как любое вещественное число это бесконечная десятичная дробь, то производную (4) назовем дробным .

Теорема . Для натурального значения m производная функции $f(x)$ $(-m)$-го порядка определяется по формуле:

$$f^{(-m)}(x) = \frac{1}{(m-1)!} \int_0^x f(t)(x-t)^{m+1} dt \cdot$$

Ключевые слова: математический анализ, производная, сочетание, предельный переход, дробная производная.

Information about the authors:

Nurgabyl D. N. - doctor of physic-mathematical Sciences, Professor: Zhetysu State University named after I.Zhansugurov, Taldykorgan and Kazakh National Women's Teacher Training University, Almaty; Leading Researcher: Institute of Mathematics and Mathematical Modeling, Almaty, https://orcid.org/0000-0002-8111-862X

Seitova T.M. - Master student of Zhetysu State University named after I. Zhansugurov, Taldykorgan, https://orcid.org/0000-0003-0175-7067

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