NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2020.2518-1726.13**
Volume 2, Number 330 (2020), 40 – 49

UDC 517.598:536.25
MRNTI 27.35.45

S.N. Kharin[1,2], T.A. Nauryz[1,2,3,4]

[1]Kazak-British Technical University, Almaty, Kazakhstan;
[2]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;
[3]Al-Farabi Kazakh National University, Almaty, Kazakhstan;
[4]Satbayev University, Almaty, Kazakhstan.
E-mail: staskharin@yahoo.com, targyn.nauryz@gmail.com

TWO-PHASE STEFAN PROBLEM FOR GENERALIZED HEAT EQUATION

Abstract. The generalized heat equation is very important for modeling of the heat transfer in bodies with a variable cross section, when the radial component of the temperature gradient can be neglected in comparison with the axial component. Such models can be applied in the theory of the heat- and mass transfer in the electrical contacts. In particular, the temperature field in a liquid metal bridge appearing at the opening electrical contacts can be modelled by the above considered Stefan problem for the generalized heat equation. The exact solution in the case when the temperature field in a liquid bridge is modelled by the generalized heat equation, while for the temperature of the solid contact zone is modelled by the spherical model, can be represented in the form of series with radial heat polynomials and integral error functions. The recurrence formulas for the coefficients of these series are given in papers published earlier in "News of the National Academy of Sciences of the Republic of Kazakhstan, Physic-mathematical series".

The two-phase Stefan problem for the generalized heat equation is considered in this paper for the case when one of the phases collapses into a point at the initial time. That creates a serious difficulty for the solution by the standard method of reduction of the problem to the integral equations because these equations are singular. Another method is used here in the case, when the functions appearing in the initial and boundary conditions are analytical and can be expanded into Taylor series. Then the solution of the problem can be represented in the form of series for special functions (Laguerre polynomials and the confluent hypergeometric function) with undetermined coefficients. These special functions have a close link with the heat polynomials introduced by P.C. Rosenbloom and D.V. Widder.

Keywords: Stefan problem, special functions, Laguerre polynomial, Faa-di Bruno formula.

1. Introduction

These special functions have a close link with the heat polynomials introduced by P.C. Rosenbloom and D.V. Widder [1]. The similar approach was used for the solution of other free boundary problems [2], [3].

Let us consider the equation

$$x\frac{d^2\varphi}{dx^2}+(\frac{\nu+1}{2}-x)\frac{d\varphi}{dx}+\frac{\beta}{2}\varphi=0\ ,\ \nu=0\,,\ -\infty<\beta<\infty \tag{1}$$

It is well known that this equation has two linearly independent solutions

$$\varphi_1(x)=\Phi(-\frac{\beta}{2},\frac{\nu+1}{2};x)\,,\ \varphi_2(x)=x^{\frac{1-\nu}{2}}\Phi(\frac{1-\beta-\nu}{2},\frac{3-\nu}{2};x) \tag{2}$$

where $\Phi(a,b;x)$ is the confluent (degenerate) hypergeometric function . Setting $T(z)=\varphi(x)$, where $x=-z^2$, one can find that $T(z)$ satisfies the equation

$$\frac{d^2T}{dz^2}+(\frac{\nu}{z}+2z)\frac{dT}{dz}-2\beta T(z)=0$$

Using this equation one can check up that the function

$$\theta(z,t)=(2a\sqrt{t})^{\beta}T(\frac{z}{2a\sqrt{t}})$$

satisfies the equation

$$\frac{\partial\theta}{\partial t}=a^2(\frac{\partial^2\theta}{\partial z^2}+\frac{\nu}{z}\frac{\partial\theta}{\partial z}) \quad (3)$$

Hence the functions

$$S_{\beta,\nu}{}^{(1)}(z,t)=(2a\sqrt{t})^{\beta}\Phi(-\frac{\beta}{2},\frac{\nu+1}{2};-\frac{z^2}{4a^2t}), \tag{4}$$

$$S_{\beta,\nu}{}^{(2)}(z,t)=(2a\sqrt{t})^{\beta}(\frac{z^2}{4a^2t})^{\frac{1-\nu}{2}}\Phi(\frac{1-\nu-\beta}{2},\frac{3-\nu}{2};-\frac{z^2}{4a^2t})$$

satisfy the equation (3).

If β is an even integer, $\beta=2n$, the function $S_{\beta,\nu}(z,t)$ can be expressed in terms of the generalized Laguerre polynomials

$$S_{2n,\nu}{}^{(1)}(z,t)=(4a^2t)^n\Phi\left(-n,\mu,-\frac{z^2}{4a^2t}\right)=\frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}\left(4a^2t\right)^n L_n^{(\mu-1)}\left(-\frac{z^2}{4a^2t}\right) \tag{5}$$

$$S_{2n,\nu}{}^{(2)}(z,t)=4a^2t^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{z^2}{4a^2t}\right)=\frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}\left(4a^2t\right)^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}L_n^{(\mu-1)}\left(-\frac{z^2}{4a^2t}\right) \tag{6}$$

where $\mu=\frac{\nu+1}{2}$. It should be noted that this formula is valid for $\mu>0$ only.

Properties. Using the integral representation for the degenerate hypergeometric function

$$\Phi(-\frac{\beta}{2},\mu;-z^2)=\frac{2\Gamma(\mu)}{\Gamma(\mu+\frac{\beta}{2})}\exp(-z^2)z^{-\mu+1}\int_0^{\infty}\exp(-x^2)x^{\mu+\beta}I_{\mu-1}(2zx)dx \tag{7}$$

and the asymptotic formula

$$\lim_{z\to\infty}\frac{e^{-z}I_{\nu}(z)}{\sqrt{2\pi z}}=1$$

it is possible to show that

$$\lim_{z\to\infty}\frac{1}{z^{\beta}}\Phi(-\frac{\beta}{2},\mu;-z^2)=\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{\beta}{2})} \tag{8}$$

In particular,

$$\lim_{z\to\infty}\frac{1}{z^{\beta}}\Phi(-\frac{\beta}{2},1;-z^{2})=\frac{1}{\Gamma(1+\frac{\beta}{2})} \tag{9}$$

For $\nu=1$ both functions (1) coincide :

$$S_{\beta,1}{}^{(1)}(z,t)=S_{\beta,1}{}^{(2)}(z,t)=(2a\sqrt{t})^{\beta}\,\Phi(-\frac{\beta}{2},1;-\frac{z^{2}}{4a^{2}t})$$

In this case, the second linearly independent solution of the equation (3) is [4]

$$\varphi_2(x)=\Phi\left(-\frac{\beta}{2},1,x\right)\ln x+\sum_{k=1}^{\infty}M_k x^k \tag{10}$$

where

$$M_k=\binom{k}{-\beta/2}\frac{1}{k!}\sum_{m=0}^{k-1}\left(\frac{1}{m-\beta/2}+\frac{2}{m+1}\right).$$

Stefan problem for spherical case when $\nu=2$ is considered in works [5]-[7]. The case when we represented spherical model as introduced in R. Holm [8] of heat transferring zones. In Stefan problem with generalized heat equation we can represent solution in heat polynomials [9], but in this work we represent in linear combination of two special functions. About special functions and their applications in heat transfer problems we can see in [10].

The generalized heat equation can be used to describe the heat transfer in a bar with the variable cross-section in the case when the radial component of the temperature gradient can be neglected in comparison with the axial component. Such mathematical model is very useful for some applied problems, in particular, for the dynamics of the heating with phase transformation in electrical contacts. Such approach was used in the papers [11] and [12] for the calculation of the temperature fields in a liquid metal bridge appearing at the contact opening, which was modelled by the generalized heat equation, and the solid contact zone modelled by the spherical heat equation. The exact solution in this case was represented in the form of radial heat polynomials and integral error functions.

2. Problem definition

Let us consider the two-phase Stefan problem for the equations

$$\frac{\partial\theta_1}{\partial t}=a_1{}^2\left(\frac{\partial^2\theta_1}{\partial z^2}+\frac{\nu}{r}\frac{\partial\theta_1}{\partial r}\right),\qquad 0<r<\alpha(t),\quad \alpha(0)=0,\quad 0<\nu<1, \tag{11}$$

$$\frac{\partial\theta_2}{\partial t}=a_2{}^2\left(\frac{\partial^2\theta_2}{\partial z^2}+\frac{\nu}{r}\frac{\partial\theta_2}{\partial r}\right),\qquad \alpha(t)<r<\infty \tag{12}$$

with the initial conditions

$$\theta_1(0,0)=\theta_m \tag{13}$$

$$\theta_2(r,0)=\varphi(r),\qquad \varphi(0)=\theta_m \tag{14}$$

the boundary conditions

$$\theta_1(0,t)=f(t),\qquad f(0)=\theta_m \tag{15}$$

$$\theta_1\left(\alpha(t),t\right)=\theta_2\left(\alpha(t),t\right)=\theta_m \tag{16}$$

$$\theta_2\left(\infty,t\right)=0 \tag{17}$$

and the Stefan condition

$$-\lambda_1\frac{\partial\theta_1\left(\alpha(t),t\right)}{\partial r}=-\lambda_2\frac{\partial\theta_2\left(\alpha(t),t\right)}{\partial r}+L\gamma\frac{d\alpha}{dt} \tag{18}$$

2. The method of solution

Suggesting that the initial and boundary functions can be expanded in Maclaurin series

$$f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n \qquad \varphi(r)=\sum_{n=0}^{\infty}\frac{\varphi^{(2n)}(0)}{(2n)!}r^{2n} \tag{19}$$

we represent the solution in the form

$$\theta_1(r,t)=\sum_{n=0}^{\infty}A_n\left(4a_1^2t\right)^nL_n^{(\mu-1)}\left(-\frac{r^2}{4a_1^2t}\right)+\sum_{n=0}^{\infty}B_n\left(4a_1^2t\right)^n\left(\frac{r^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{r^2}{4a_1^2t}\right) \tag{20}$$

$$\theta_2(r,t)=\sum_{n=0}^{\infty}C_n\left(4a_2^2t\right)^nL_n^{(\mu-1)}\left(-\frac{r^2}{4a_2^2t}\right)+\sum_{n=0}^{\infty}D_n\left(4a_2^2t\right)^n\left(\frac{r^2}{4a_2^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{r^2}{4a_2^2t}\right) \tag{21}$$

where $\frac{1}{2}<\mu=\frac{\nu+1}{2}<1$.

Satisfying the boundary condition (15) and using the formula (8) for $z=\frac{r}{2a_1\sqrt{t}}$, $\beta=2n$ we obtain

$$\sum_{n=0}^{\infty}A_n\left(4a_1^2t\right)^nL_n^{(\mu-1)}\left(0\right)=f(t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$$

and

$$A_n=\frac{f^{(n)}(0)}{n!\left(4a_1^2\right)^n\binom{n+\mu-1}{n}} \tag{22}$$

Using the initial condition (14) and the formula (8) for $z=\frac{r}{2a_1\sqrt{t}}$, $\beta=2n$ for the first term with C_n and $\beta=2(n+\mu-1)$ for the second term with D_n we get

$$\varphi(r)=\sum_{n=0}^{\infty}\frac{\varphi^{(2n)}(0)}{(2n)!}r^{2n}=\lim_{t\to 0}\theta_2(r,t)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}C_nr^{2n}+\frac{1}{\Gamma(n+\mu)}D_nr^{2n}\right],$$

Thus

$$\frac{(-1)^n}{n!}C_n+\frac{1}{\Gamma(n+\mu)}D_n=\frac{\varphi^{(2n)}(0)}{(2n)!} \tag{23}$$

Now we should use the conditions (16) and (18) to get additional three equations for the definition of all coefficients and the free boundary. Thus $\alpha(t)$ can be written in the form

$$\alpha(\tau)=\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}$$

where $\tau=\sqrt{t}$.

Now we rewrite the conditions (16) and (18) in terms of τ and compare the powers in the left and the right sides of equations using k-th differentiation and putting then $\tau=0$. We obtain

$$\left.\frac{\partial^k\theta_1\left(\alpha(\tau),\tau\right)}{\partial\tau^k}\right|_{\tau=0}=\left.\frac{\partial^k\theta_2\left(\alpha(\tau),\tau\right)}{\partial\tau^k}\right|_{\tau=0}=0,\quad k=0,1,2,... \tag{24}$$

$$-\lambda_1\frac{\partial^k\theta_{1r}\left(\alpha(\tau),\tau\right)}{\partial\tau^k}=-\lambda_2\frac{\partial^k\theta_{2r}\left(\alpha(\tau),\tau\right)}{\partial\tau^k}+L\gamma k!\alpha_k,\quad k=0,1,2,... \tag{25}$$

At first, we use Leibniz formula for k-th derivative for (24) equation and we obtain for the first term of $\theta_i(r,t),\quad i=1,2$.

$$\left.\frac{\partial^k\left[2^{2n}a_i^{\,2n}\tau^{2n}L_n^{\,(\mu-1)}(-\delta(\tau))\right]}{\partial\tau^k}\right|_{\tau=0}=2^{2n}a_i^{\,2n}\frac{k!}{(k-2n)!}\left.\frac{\partial^{k-2n}[L_n^{\,(\mu-1)}(-\delta(\tau))]}{\partial\tau^{k-2n}}\right|_{\tau=0}$$

and for second term we have

$$\left.\frac{\partial^k\left[2^{2n}a_i^{\,2n}\tau^{2n}(\delta(\tau))^{1-\mu}\Phi[1-\mu-n,2-\mu,-\delta(\tau)]\right]}{\partial\tau^k}\right|_{\tau=0}=$$

$$=2^{2n}a_i^{\,2n}\frac{k!}{(k-2n)!}\left.\frac{\partial^{k-2n}\left[(\delta(\tau))^{1-\mu}\Phi[1-\mu-n,2-\mu,-\delta(\tau)]\right]}{\partial\tau^k}\right|_{\tau=0}$$

where $\delta(\tau)=\dfrac{1}{4a_i^{\,2}}\left(\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}\right)^2,\quad i=1,2..$

In particular, when $k=0$ and $\tau=0$ we have

$$A_0=\theta_m,\quad B_0=0,\quad C_0=\theta_m,\quad D_0=0.$$

For this purpose, we use the Faa di Bruno formula (Arbogast formula) for a derivative of a composite function. For the first term of temperature equation we have

$$\left.\frac{\partial^{k-2n}[L_n^{\,(\mu-1)}(-\delta(\tau))]}{\partial\tau^{k-2n}}\right|_{\tau=0}=\sum_{m=0}^{k-2n}(-1)^m[L_{n-m}^{(\mu-1-m)}(-\delta_1)]\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}$$

$b_2,b_3,...$ satisfy the following equations

$$b_2+b_3+...+b_{k-2n-m+2}=m$$
$$2b_2+3b_3+...+(k-2n-m+2)b_{k-2n-m+2}=k-2n$$

where $$\delta_1 = \frac{\alpha_1^{\,2}}{4a_i^{\,2}}, \delta_2 = \frac{\alpha_2^{\,2}}{4a_i^{\,2}}, \ldots, \delta_{k-2n-m+2} = \frac{\alpha_{k-2n-m+2}^{\,2}}{4a_i^{\,2}}, \quad i = 1, 2$$

for the second term we have analogously.

Then from condition (16) we get the following recurrent formulas for determine coefficients A_n, B_n, C_n and D_n

$$\sum_{n=0}^{k} A_n (2a_1)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} (-1)^m [L_{n-m}^{(\mu-1-m)}(-\delta_1)] \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!\ldots b_{k-2n-m+2}!} +$$
$$+\sum_{n=0}^{k} B_n (2a_1)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} \frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)} \delta_1^{1-\mu-m} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!\ldots b_{k-2n-m+2}!} \cdot \tag{26}$$
$$\cdot \sum_{l=0}^{k-2n-m} [\Phi(1-\mu-n, 2-\mu, -\delta_1)]^{(l)} \sum_{b_i} \frac{(k-2n-m)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!\ldots b_{k-2n-m-l+2}!} = 0$$

$$\sum_{n=0}^{k} C_n (2a_2)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} (-1)^m [L_{n-m}^{(\mu-1-m)}(-\delta_1)] \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!\ldots b_{k-2n-m+2}!} +$$
$$+\sum_{n=0}^{k} D_n (2a_2)^{2n} \frac{k!}{(k-2n)!} \sum_{m=0}^{k-2n} \frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)} \delta_1^{1-\mu-m} \sum_{b_i} \frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!\ldots b_{k-2n-m+2}!} \cdot \tag{27}$$
$$\cdot \sum_{l=0}^{k-2n-m} [\Phi(1-\mu-n, 2-\mu, -\delta_1)]^{(l)} \sum_{b_i} \frac{(k-2n-m)!\delta_2^{b_2}\delta_3^{b_3}\ldots\delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!\ldots b_{k-2n-m-l+2}!} = 0$$

As coefficient A_n is known from (22), then by making substitution to (26) we can find coefficient B_n. From system of equations (24) and (27) we can determine the coefficients C_n and D_n

$$B_n = -\frac{f^{(n)}(0)\xi_1}{n!(2a_1)^{2n}\binom{n+\mu-1}{n}\xi_2} \tag{28}$$

$$C_n = \frac{\varphi^{(2n)}(0)n!}{(2n)!(-1)^n} - D_n \frac{n!}{(-1)^n\Gamma(n+\mu)}, \tag{29}$$

$$D_n = \frac{\varphi^{(2n)}(0)n!\xi_3}{(2n)!(-1)^{n+1}\left(\xi_4 - \frac{n!}{(-1)^n\Gamma(n+\mu)}\right)} \tag{30}$$

where $n = 1, 2, \ldots$ and

$$\xi_{1(3)}=(2a_{1(2)})^{2n}\frac{k!}{(k-2n)!}\sum_{m=0}^{k-2n}[L_n^{(\mu-1)}(-\delta_1)]^{(m)}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!},$$

$$\xi_{2(4)}=(2a_{1(2)})^{2n}\frac{k!}{(k-2n)!}\sum_{m=0}^{k-2n}\frac{(1-\mu)\Gamma(2-\mu)}{(1-\mu-k+2n)\Gamma(2-\mu-k+2n)}\delta_1^{1-\mu-m}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}\cdot$$

$$\cdot\sum_{l=0}^{k-2n-m}[\Phi(1-\mu-n,2-\mu,-\delta_1)]^{(l)}\sum_{b_i}\frac{(k-2n-m)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}.$$

From Stefan's condition (18) and (24) we have the recurrent formula for free boundary

$$\alpha_k=\frac{\lambda_2}{L\gamma k!}\left[\sum_{n=0}^{k}C_n(2a_2)^{2n}\frac{k!}{(k-2n)!}\sum_{m=0}^{k-2n}[L_{nr}^{(\mu-1)}(-\delta_1)]^{(m)}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}+\right.$$

$$+\sum_{n=0}^{k}D_n2^{2n-1}a_2^{2n-2}\frac{k!(1-\mu)}{(k-2n+2)!}\sum_{m=0}^{k-2n+2}\binom{k-2n+2}{m}\frac{(-1)^m\Gamma(\mu+1)}{\Gamma(1+\mu-m)}\delta_1^{-\mu-m}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}\cdot$$

$$\cdot\sum_{l=0}^{k-2n-m+2}\binom{k-2n-m+2}{l}\beta_1^{(l)}\sum_{b_i}\frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}$$

$$\left.\cdot\sum_{i=0}^{k-2n-m-l+2}[\Phi_r(1-\mu-n,2-\mu,-\delta_1)]^{(i)}\sum_{b_i}\frac{(k-2n-m-l+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l-i+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l-i+2}!}\right]-$$

$$-\frac{\lambda_1}{L\gamma k!}\left[\sum_{n=0}^{k}A_n(2a_1)^{2n}\frac{k!}{(k-2n)!}\sum_{m=0}^{k-2n}[L_{nr}^{(\mu-1)}(-\delta_1)]^{(m)}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}+\right.$$

$$+\sum_{n=0}^{k}B_n2^{2n-1}a_2^{2n-2}\frac{k!(1-\mu)}{(k-2n+2)!}\sum_{m=1}^{k-2n+2}\binom{k-2n+2}{m}\frac{(-1)^m\Gamma(\mu+1)}{\Gamma(1+\mu-m)}\delta_1^{-\mu-m}\sum_{b_i}\frac{(k-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{k-2n-m+2}^{b_{k-2n-m+2}}}{b_2!b_3!...b_{k-2n-m+2}!}\cdot$$

$$\cdot\sum_{l=0}^{k-2n-m+2}\binom{k-2n-m+2}{l}\beta_1^{(l)}\sum_{b_i}\frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}\cdot$$

$$\cdot\sum_{l=0}^{k-2n-m+2}\binom{k-2n-m+2}{l}\beta_1^{(l)}\sum_{b_i}\frac{(k-2n-m+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l+2}!}$$

$$\left.\cdot\sum_{i=0}^{k-2n-m-l+2}[\Phi_r(1-\mu-n,2-\mu,-\delta_1)]^{(i)}\sum_{b_i}\frac{(k-2n-m-l+2)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{k-2n-m-l-i+2}^{b_{k-2n-m-l+2}}}{b_2!b_3!...b_{k-2n-m-l-i+2}!}\right] \quad (31)$$

We can find coefficient A_n, B_n, C_n and D_n from (22), (28)-(30) and free boundary we can determine from (31).

3. Convergence of series

Convergence of series (20)-(21) can be proved as following. Let $\alpha(t_0)=\eta_0$ for any $t=t_0$. Then series (20) can be written as

$$\theta_1(r,t_0)=\sum_{n=0}^{\infty}A_n\left(4a_1^2t_0\right)^nL_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2t_0}\right)+\sum_{n=0}^{\infty}B_n\left(4a_1^2t_0\right)^n\left(\frac{\eta_0^2}{4a_1^2t_0}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\eta_0^2}{4a_1^2t_0}\right) \quad (32)$$

The series (20) and (21) must be convergence because $\theta_1(r,t)=\theta_2(r,t)=\theta_m$. Then there exists some constant E_1 independent of n and for the first term of (32) we have

$$|A_n|< E_1/(4a_1^2t_0)^n L_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2t_0}\right) \tag{33}$$

Since A_n bounded, then multiply both sides of (33) by $(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)$ we obtain

$$\sum_{n=0}^{\infty} A_n(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)< E_1\sum_{n=0}^{\infty}\frac{(4a_1^2t)^n L_n^{(\mu-1)}\left(-\frac{(\alpha(t))^2}{4a_1^2t}\right)}{(4a_1^2t_0)^n L_n^{(\mu-1)}\left(-\frac{\eta_0^2}{4a_1^2t_0}\right)}< E_1\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n \tag{34}$$

Similarly, for the second term of (33) we have some constant E_2 which satisfy

$$|B_n|< E_2/\left(4a_1^2t_0\right)^n\left(\frac{\eta_0^2}{4a_1^2t_0}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\eta_0^2}{4a_1^2t_0}\right) \tag{35}$$

Analogously, if we multiple both sides of (35) by $(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)$ we get

$$\begin{aligned}&\sum_{n=0}^{\infty} B_n(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)\\ &< E_2\sum_{n=0}^{\infty}\frac{(4a_1^2t)^n\left(\frac{(\alpha(t))^2}{4a_1^2t}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\alpha(t)^2}{4a_1^2t}\right)}{\left(4a_1^2t_0\right)^n\left(\frac{\eta_0^2}{4a_1^2t_0}\right)^{1-\mu}\Phi\left(1-\mu-n,2-\mu,-\frac{\eta_0^2}{4a_1^2t_0}\right)}< E_2\sum_{n=0}^{\infty}\left(\frac{t}{t_0}\right)^n\end{aligned} \tag{36}$$

These geometric series and $\theta_1(r,t)$ convergence for all $r<\mu_0$ and the same $\theta_2(r,t)$ convergence for all $r>\mu_0$ and $t<t_0$. Convergence for $\alpha(t)$ can be determined analogously from (31).

4. Conclusion

A mathematical model of describing heat distribution for generalized heat in electrical contacts on liquid and solid zone is constructed by two-phase Stefan problem. Temperature for liquid zone $\theta_1(r,t)$ and for solid zone $\theta_2(r,t)$ which given in the form summation two special functions as Laguerre polynomial and confluent hyper-geometric function are determined and their coefficients A_n, B_n, C_n and D_n are founded from equations (22) and (28)-(30) and free boundary on melting isotherm is described in recurrent formula (31). The convergence of series is proved.

5. Acknowledgement

The research work is supported by the grant project AP05133919 (2018-2020) from the Ministry of Science and Education of the Republic of Kazakhstan.

С.Н. Харин[1,2], Т.А. Наурыз[1,2,3,4]

[1]Қазақ-Британдық Техникалық Университеті, Алматы, Қазақстан;
[2]Математика жане математикалық модельдеу институты, Алматы, Қазақстан;
[3]Әл-Фараби атындағы қазақ ұлттық университеті, Алматы, Қазақстан;
[4] Satbayev University, Алматы, Қазақстан

ЖАЛПЫЛАНҒАН ЖЫЛУ ТЕҢДЕУІ ҮШІН ЕКІ ФАЗАЛЫ СТЕФАН ЕСЕБІ

Аннотация. Жалпыланған жылу теңдеуі температуралық градиенттің радиалды компонентін оның осьтік компонентімен салыстырғанда елемеуге болатын ауыспалы қимасы бар денелердегі жылу беруді модельдеу үшін маңызды. Мұндай модельдерді электр байланыстарындағы жылу және масса алмасу теориясында қолдануға болады. Атап айтқанда, ашық электр түйіспелерінде пайда болатын сұйық металл көпіріндегі температуралық өрісті жалпыланған жылу теңдеуі үшін жоғарыда қарастырылған Стефан есебімен модельдеуге болады. Сұйық көпірдің температуралық өрісі жалпыланған жылу теңдеуімен, ал қатты жанасу аймағының температурасы сфералық түрде модельденген кездегі нақты шешім радиалды термиялық полиномдар және қателіктердің интегралдық функциялары түрінде ұсынылуы мүмкін. Осы қатарлардың коэффициенттерінің қайталану формулалары ҚР ҰҒА-ның «Известия» физика-математикалық сериясында жарияланған еңбектерде келтірілген.

Бұл мақалада фазалардың біреуі бастапқы уақытта бір нүктеге төмендеген жағдайда жалпыланған жылу теңдеуі үшін екі фазалы Стефан есебі қарастырылады. Бұл есепті интегралдық теңдеулерге келтірудің стандартты әдісі арқылы шешуге айтарлықтай қиындықтар туғызады, өйткені бұл жағдайда теңдеулер жеке сипатқа ие болады. Бұл жұмыста бастапқы және шекаралық шарттарда пайда болатын функциялар аналитикалық болып, оларды Тейлор қатарына кеңейту мүмкін болған жағдайда біз басқа әдісті қолданамыз. Бұл жағдайда мәселенің шешімі белгісіз коэффициенттері бар арнайы функциялардағы қатарлар түрінде ұсынылуы мүмкін (Лагуерр полиномиясы және дегенеративті гипергеометриялық функция). Бұл арнайы функциялар Розенблум және Д.В. Виддер енгізген жылу полиномдарымен тығыз байланысты.

Алынған қатар априорлы түрде жылу теңдеуін қанағаттандырады, сондықтан бастапқы және шекаралық шарттарды, сондай-ақ еркін шекара үшін Стефан шартын қанағаттандыратын коэффициенттерді табу керек. Бұл тәсіл өте пайдалы болып көрінеді, өйткені шекаралық жағдайлар тек кейбір қателіктермен ғана қанағаттандырылса да, жылу теңдеуінің максималды принципі бойынша шешім қателігі шекара жағдайындағы қателіктен аспауы керек. Бұл ерітіндіні қабылдағанға дейін оны жақындатудың бағасын алуға мүмкіндік береді. Фа-ди-Бруно формуласын қолданып, бастапқы және шекаралық шарттардағы функцияларды қатарға көбейтіп, коэффициенттерді бір уақытта ретке келтіре отырып, біз белгісіз коэффициенттерді іздеу үшін қайталанатын теңдеулер жүйесін аламыз. Еркін шекараның коэффициенттері үшін теңдеулердің ұқсас жүйесін Стефан шартын қолдана отырып табуға болады. Алынған қатарлардың конвергенциясы еркін шекарадағы тұрақты температура жағдайын қолдана отырып дәлелденді. Бұл жағдай серияның геометриялық прогрессиямен масштабталуына және қатардың жақындасуы үшін қажетті бағаларды алуға мүмкіндік береді.

Түйін сөздер: Стефан есебі, арнайы функциялар, Лагерра полиномы, Фаа-ди Бруно формуласы.

С.Н. Харин[1,2], Т.А. Наурыз[1,2,3,4]

[1]Казахстанско-Британский Технический Университет, Алматы, Казахстан,
[2]Институт математики и математического моделирования, Алматы, Казахстан,
[3]Казахский национальный университет имени аль-Фараби, Алматы, Казахстан,
[4] Satbayev University, Алматы, Казахстан

ДВУХФАЗНАЯ ЗАДАЧА СТЕФАНА ДЛЯ ОБОБЩЕННОГО УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

Аннотация. Обобщенное уравнение теплопроводности имеет важное значение для моделирования теплообмена в телах с переменным поперечным сечением, когда радиальной составляющей градиента температуры можно пренебречь по сравнению с ее осевой составляющей. Такие модели могут быть использованы в теории тепло- и массопереноса в электрических контактах. В частности, температурное поле в жидкометаллическом мостике, возникающем в размыкающихся электрических контактах, может быть моделировано рассмотренной выше задачей Стефана для обобщенного уравнения теплопроводности. Точное решение для случая, когда температурное поле жидкого мостика моделируется обобщенным уравнением теплопроводности, в то время как температура твердой контактной зоны моделируется в сферическом варианте, может быть представлено в виде рядов по радиальным тепловым полиномам и интегральным функциям ошибок. Рекуррентные формулы для коэффициентов этих рядов даны в работах, опубликованных ранее в «Известиях НАН РК, серия физико-математическая».

В этой статье рассматривается двухфазная задача Стефана для обобщенного уравнения теплопровод-ности для случая, когда одна из фаз в начальный момент времени вырождается в точку. Это создает серьезные трудности для решения задачи стандартным методом ее сведения к интегральным уравнениям, поскольку уравнения в этом случае становятся сингулярными. В данной работе используется другой метод для случая, когда функции, фигурирующие в начальных и граничных условиях, являются аналитическими и могут быть разложены в ряды Тейлора. В этом случае решение задачи можно представить в виде рядов по специальным функциям (многочлены Лагерра и вырожденная гипергеометрическая функция) с неопределен-ными коэффициентами. Эти специальные функции имеют тесную связь с тепловыми полиномами, введенными П.С. Розенблюмом и Д.В. Уиддером.

Построенные ряды априори удовлетворяют уравнению теплопроводности, и нужно найти их коэффициенты, удовлетворяющие начальному и граничному условию, а также условию Стефана для свободной границы. Такой подход представляется весьма полезным, потому что даже если граничные условия выполняются лишь приближенно с некоторой ошибкой, то ошибка решения согласно принципу максимума для уравнения теплопроводности должна быть не больше, чем ошибка в граничных условиях. Это дает возможность получить оценку приближения решения до его получения. Используя формулу Фаа-ди Бруно и разлагая функции в начальных и граничных условиях на ряды и приравнивая коэффициенты в одном и том же порядке времени, можно получить систему рекуррентных уравнений для поиска неизвестных коэффициентов. Аналогичная система уравнений для коэффициентов свободной границы может быть найдена с использованием условия Стефана. Сходимость полученных рядов доказывается, используя условие постоянства температуры на свободной границе. Это обстоятельство позволяет мажорировать ряды геометрической прогрессией и получить оценки, необходимые для сходимости ряда.

Ключевые слова: задача Стефана, специальные функции, полином Лагерра, формула Фаа-ди Бруно.

Information about authors

Kharin S. N. – Academician of NAS RK, professor, Institute of Mathematics and Mathematical Modeling, Kazakh-British Technical University, Almaty, Kazakhstan. Email: staskharin@yahoo.com, https://orcid.org/0000-0002-9666-6514;

Nauryz T. A. – junior researcher, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan, PhD student, Al-Farabi Kazakh National University, Almaty, Kazakhstan, lector, Satbayev University, Almaty, Kazakhstan, Teaching Assistant, Kazakh-British Technical University, Almaty, Kazakhstan. Email: targyn.nauryz@gmail.com, https://orcid.org/0000-0003-0523-2081;

REFERENCES

[1] Rosenbloom P.C., Widder D.V. **(1959)** Expansions in terms of heat polynomials and associated functions, Trans. Amer. Math. Soc., Vol. 92, pp. 220-266.

[2] Kharin S.N., Sarsengeldin M.M., Nouri H. **(2016)** Analytical solution of two phase spherical Stefan problem by heat polynomials and integral error functions. AIP Conference Proceedings 1759, 020031; doi: http://dx.doi.org/10.1063/1.4959645

[3] M. Sarsengeldin, S.N. Kharin, **(2017)** Method of the Integral Error Functions for the Solution of the One- and Two-Phase Stefan Problems and Its Application, Filomat 31:4, 1017–1029 DOI 10.2298/FIL1704017S, Available at: http://www.pmf.ni.ac.rs/filomat

[4] Szego C. **(1962)** Orthogonal polynomials, GIFML, Moscow.

[5] Alexey A. Kavokin, Targyn Nauryz, Nazerke T. Bizhigitova **(2016)** Exact solution of two phase spherical Stefan problem with two free boundaries, AIP Conference Proceedings 1759, 020117, https://doi.org/10.1063/1.4959731.

[6] Sarsengeldin M.M., Erdogan A.S., Nauryz T.A., Nouri H. **(2017)** An approach for solving an inverse spherical two-phase Stefan problem arising in modeling of electric contact phenomena, Mathematical Methods in the Applied Sciences Vol. 41, Issue 2, https://doi.org/10.1002/mma.4649.

[7] Merey M. Sarsengeldin, Targyn A. Nauryz, Nazerke T. Bizhigitova, Alibek M. Orinbasar. **(2017)** Solution of an Inverse Two Phase Spherical Stefan Test Problem. Symposium Functional Analysis in Interdisciplinary Applications Springer,Cham, https://doi.org/10.1007/978-3-319-67053-937.

[8] Holm R. **(2000)** Electrical Contacts. Theory and Application. Symposium Functional Analysis in Interdisciplinary Applications Edition, Springer- Verlag, Berlin, New York.

[9] Kharin S.N., Jabbarkhanov K. **(2018)** "Method of the heat polynomials for the solution of free boundary problems for the generalized heat equation", Известия Межд. казахско-турецкого университета им. Х.А. Ясави. Сер. Математика, физика, информатика. No1 Vol. 4. pp.166-170.

[10] Stanislav N. Kharin. **(2018)** Special functions and heat polynomials for the solution of free boundary problems, AIP Conference Proceedings 1997, 020047; https://doi.org/10.1063/1.5049041.

[11] Kharin S.N., Kassabek S.A., Salybek D., Ashymov T. **(2018)** Stefan problem in ellipsoidal coordinates. News of the National Academy of Sciences of the Republic of Kazakhstan, physico-mathematical series. Volume 5, Number 321, pp 19-24. ISSN 1991-346X. https://doi.org/10.32014/2018.2518-1726.3

[12] Kharin S.N., Kassabek S.A., Slyamkhan M. **(2018)** Problem from the theory of bridge erosion. News of the National Academy of Sciences of the Republic of Kazakhstan, physico-mathematical series. Volume 5, Number 321, pp 68-74. ISSN 1991-346X. https://doi.org/10.32014/2018.2518-1726.9