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A.Sh. Shaldanbayev[1] , G.I. Beissenova[2], A.Zh. Beisebayeva[3] , A.A. Shaldanbayeva[4]

[1]Silkway International University, Shymkent, Kazakhstan;
[2,4]Regional social-innovative University, Shymkent, Kazakhstan;
[3]South Kazakhstan State University M.O.Auezov, Shymkent, Kazakhstan;
[4]International humanitarian and technical University, Shymkent, Kazakhstan.
E-mail: shaldanbaev51@mail.ru, gulia-74-74@mail.ru, akbope_a@mail.ru, altima_a@mail.ru

INVERSE PROBLEM OF THE STORM-LIUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

Abstract. Under the inverse tasks of spectral analysis understand tasks reconstruction of a linear operator from one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Ambartsumian. He proved the following theorem.

We denote by $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ the eigenvalues of the Sturm - Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2 (n = 0,1,2,\dots) \text{ то } q(x) \equiv 0.$$

The first mathematician who drew attention to the importance of this result of Ambartsumian was the Swedish mathematician Borg. He performed the first systematic study of one of the important inverse problems, namely, the inverse problem for the classical Sturm - Liouville operator of the form (1.1) with respect to spectra. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the result of Ambartsumian is an exception to the general rule. In the same work, Borg shows that two spectra of the Sturm - Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Borg's theorem.

Let the equations

$$-y'' + q(x)y = \lambda y,$$
$$-z'' + p(x)z = \lambda z,$$

have the same spectrum under boundary conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases}$$

and under boundary conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0, \pi]$ if

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Soon after Borg's work, important studies on the theory of inverse problems were carried out by Levinson, in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm-Liouville operator

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \quad (1.3)$$

is restored by one spectrum.

The inverse problems for differential operators with decaying boundary conditions are fairly well understood. More difficult inverse problems for Sturm - Liouville operators with unseparated boundary conditions have also been studied. In particular, the periodic boundary-value problem was considered in a number of papers. I.V. Stankevich proposed the formulation of the inverse problem and proved the corresponding uniqueness theorem.

The present work is devoted to a generalization of the theorems of Ambartsumian and Levinson, in particular, our results contain the results of these authors. In the paper, a uniqueness theorem is proved, for one spectrum, for the Sturm-Liouville operator with unseparated boundary conditions, a real continuous and symmetric potential. The research method differs from all previously known methods, and is based on the internal symmetry of the operator generated by invariant subspaces.

Note that the operator we are considering is non-self-adjoint, although the potential is real and symmetric, this moment plays an essential role for our method, because we construct a pair of Borg operators through the operator and its adjoint one. Other authors use the Leibenzon mapping method.

Keywords: Sturm-Liouville operator, spectrum, inverse Sturm-Liouville problem, Borg theorem, Hambardzumyan theorem, Levinson theorem, non-separated boundary value conditions, symmetric potential, invariant subspaces, differential operators, inverse spectral problems.

1. Introduction

We study the inverse spectral problem for the Sturm – Liouville operator:

$$Ly := y'' + q(x)y, \;\; x\epsilon(0,1),$$

on the finite interval (0,1) with non-separated boundary value conditions. Inverse problems consist in restoring the coefficients of differential operators by their spectral characteristics. Such problems often arise in mathematics and its applications.

Inverse problems for differential operators with decaying boundary value conditions have been thoroughly studied (see monographs [1–5] and references). More difficult inverse problems for Sturm – Liouville operators with non-separated boundary value conditions were studied in [6–9] and other works. In particular, periodic boundary-value problem was considered in [6, 7]. I. V. Stankevich [6] proposed formulation of the inverse problem and proved the corresponding uniqueness theorem. V. A. Marchenko and I. V. Ostrovsky [7] gave a characteristic of the spectrum of a periodic boundary-value problem in terms of special conformal mapping. The conditions proposed in [7] are difficult to verify. Another method used in [8] made it possible to obtain necessary and sufficient conditions for solvability of the inverse problem in the periodic case that are more convenient for verification. Similar results were obtained in [8] for another type of boundary conditions, namely

$$y'(0) - ay(0) + by(\pi) = y'(\pi) + dy(\pi) - by(0) = 0.$$

Later similar results were obtained in [9]. In [10], the case when the potential is $q -$ symmetric with respect to the middle of the interval, that is, $q(x) = q(\pi - x)$ a.e. on $(0, \pi)$, was investigated, and for this case, solution of the inverse spectral problem was constructed and the spectrum was characterized. The symmetric case requires nontrivial changes in the method and allows us to specify less spectral information than in the general case. Some results for the symmetric case were obtained in [11] - [13].

By inverse problems of spectral analysis, we understand the problems of reconstructing a linear operator by one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Hambardzumyan [14]. He proved the following theorem.

By $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ we denote eigenvalues of the Sturm - Liouville problem

$$-y'' + q(x)y = \lambda y, \quad (1.1)$$

$$y'(0) = 0, \;\; y'(\pi) = 0; \quad (1.2)$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2 \; (n = 0,1,2, \dots) \;\text{ then } q(x) \equiv 0.$$

The first mathematician who drew attention to the importance of this Hambardzumyan result was the Swedish mathematician Borg. He performed the first systematic study of one of the important inverse problems, namely, the inverse problem for the classical Sturm – Liouville operator of the form (1.1) by the spectra [15]. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the Hambartsumyan result is an exception to the general rule. In the same paper [15], Borg showed that two spectra of the Sturm – Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Soon after the Borg work, important studies on the theory of inverse problems were carried out by Levinson [18], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm – Liouville operator

$$-y'' + q(x)y = \lambda y,$$
$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases}$$

is reconstructed by one spectrum.

A number of works by B.M. Levitan [19] are devoted to the reconstruction the Sturm – Liouville operator by one and two spectra.

This work is devoted to a generalization of the theorems of Hambartsumian [14] and Levinson [16], in particular, our results contain the results of these authors. The research method of this work appeared under influence of [18] - [20], and differs from all previously known methods.

2. Research Methods.

Idea of this work is very simple. Having studied in detail contents of [14, 16], we realized that both of these operators have invariant subspaces. Generalization of this property of the Sturm - Liouville operator led us to the results presented below.

3. Research Results.

In the Hilbert space $H = L^2(0,\pi)$ we consider the Sturm - Liouville operator:

$$Ly = -y'' + q(x)y, \; x \in (0,\pi); \tag{1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{2}$$

where $q(x)$ is a continuous complex function, a_{ij} $(i = 1,2; \; j = 1,2,3,4)$ are arbitrary complex coefficients, and by Δ_{ij} $(i = 1,2; \; j = 1,2,3,4)$ we denote minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Assume that $\Delta_{13} \neq 0$, then the Sturm - Liouville operator (1) – (6) takes the following form:

$$Ly = -y'' + q(x)y, \; x \in (0,\pi); \tag{1}$$

$$\begin{cases} \Delta_{13}y(0) - \Delta_{32}y'(0) - \Delta_{34}y'(\pi) = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0, \end{cases} \tag{2'}$$

and its conjugate operator L^+ takes the form

$$L^+z = -z'' + \overline{q(x)}z, \; x \in (0,\pi); \tag{1$^+$}$$

$$\begin{cases} \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0. \end{cases} \tag{2$'^+$}$$

Let P and Q be projections, defined by the formulas

$$Pu(x) = \frac{u(x)+u(\pi-x)}{2}, \; Qv(x) = \frac{v(x)-v(\pi-x)}{2}$$

The main result of this work is the following theorem.

Theorem 3.1. If $\Delta_{13} \neq 0$, then

1) $PL = L^+P$; (3)
2) $LQ = QL^+$; (4)
3) $\Delta_{12} = -\Delta_{34}$; (5)

and the Sturm - Liouville operator (1) – (2′) is reconstructed by one spectrum.

4. Discussion.

In this section we prove Theorem 3.1, and discuss the obtained results. The following Lemmas 3.1 and 3.2 have and independent meaning.

Lemma 4.1. If for a linear discrete operator L we have

1) $PL = L^{+}P$; (3)

2) $LQ = QL^{+}$; (4)

3) $P + Q = I$; (6)

where P, Q are orthogonal projections, and I is unit operator, then all its eigenvalues are real.

The following lemma shows that the spectrum $\sigma(L)$ of the operator L consists of two parts, so the operator L, apparently, splits into two parts. In the future, we will see that this is exactly what happens, and moreover, under certain conditions, these parts form a Borg pair.

Lemma 4.2. If L is a linear discrete operator satisfying the conditions:

1) $PL = L^{+}P$; (3)

2) $LQ = QL^{+}$; (4)

3) $P + Q = I$; (6)

where P, Q are orthogonal projections, and I is unit operator, then

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2),$$

where $L_1 = PL,\ L_2 = LQ,\ \sigma(L)$ is a spectrum of the operator L.

Lemma 4.3. If

a) $\Delta_{13} \neq 0$; (7)

b) $PL = L^{+}P$; (3)

then for the Sturm - Liouville operator (1) – ($6^{/}$) we get

1) $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$; (8)

2) $\overline{\left(\frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}\right)} = \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} == \frac{\Delta_{34}-\Delta_{14}}{\Delta_{13}}$;

3) $q(\pi - x) = q(x), \overline{q(x)} = q(x)$. (9)

Moreover, operators L and L^{+} take the following forms:

a) $Ly = -y'' + q(x)y,\ x \in (0,\pi)$; (1)

$$\begin{cases} y(0) + \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} y'(0) + y(\pi) - \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} y'(\pi) = 0, \\ \frac{\Delta_{12}}{\Delta_{13}} y'(0) + y(\pi) + \frac{\Delta_{14}}{\Delta_{13}} y'(\pi) = 0. \end{cases} \quad (10)$$

b) $L^{+}z = -z'' + \bar{q}(x)z,\ x \in (0,\pi)$; $(1)^{+}$

$$\begin{cases} z(0) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}} z'(0) - z(\pi) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}} z'(\pi) = 0, \\ z(0) - \frac{\overline{\Delta_{32}}}{\overline{\Delta_{13}}} z'(0) - \frac{\overline{\Delta_{12}}}{\overline{\Delta_{13}}} z'(\pi) = 0. \end{cases} \quad (10)^{+}$$

Similar lemma holds with the projector Q.

Lemma 4.4. If

a) $\Delta_{13} \neq 0$; (7)

б) $LQ = QL^{+}$,

then for the Sturm - Liouville operator (1) – ($6^{/}$) we get

1) $\Delta_{12} + \Delta_{14} = \Delta_{32} + \Delta_{34}$; (8)

2) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right)} = \left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right) = \frac{\Delta_{32}+\Delta_{34}}{\Delta_{13}}$;

3) $q(\pi - x) = q(x), \bar{q}(x) = q(x).$ (9)

In this case operators L and L^+ take the following forms:

a) $Ly = -y'' + q(x)y,\ x \in (0,\pi);$ (1)

$$\begin{cases} y(0) + y(\pi) + \frac{\Delta_{12} - \Delta_{32}}{\Delta_{13}}[y'(0) - y'(\pi)] = 0, \\ \frac{\Delta_{12}}{\Delta_{13}}y'(0) + y(\pi) + \frac{\Delta_{14}}{\Delta_{13}}y'(\pi) = 0; \end{cases}$$

b) $L^+z = -z'' + \bar{q}(x)z,\ x \in (0,\pi);$ (1)+

$$\begin{cases} z(0) - z(\pi) - \frac{\overline{\Delta_{12}}+\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0) + z'(\pi)] = 0, \\ z(0) - \frac{\overline{\Delta_{32}}}{\overline{\Delta_{13}}}z'(0) - \frac{\overline{\Delta_{12}}}{\overline{\Delta_{13}}}z'(\pi) = 0. \end{cases}$$

Lemma 4.5. If $\Delta_{13} \neq 0$, and

a) $PL = L^+P$,

b) $LQ = QL^+$;

then the operators L and L^+ take the following forms:

$$Ly = -y'' + q(x)y,\ x \in (0,\pi);$$

$$\begin{cases} y(0) + y(\pi) + \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}[y'(0) - y'(\pi)] = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0. \end{cases}$$

$$L^+z = -z'' + \bar{q}(x)z,\ x \in (0,\pi);$$

$$\begin{cases} z(0) - z(\pi) - \frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}[z'(0) + z'(\pi)] = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0; \end{cases}$$

where

1) $q(\pi - x) = q(x)$;

2) $\bar{q}(x) = q(x)$,

3) $\overline{\left(\frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}}\right)} = \frac{\Delta_{12}-\Delta_{32}}{\Delta_{13}} = \frac{\Delta_{34}-\Delta_{14}}{\Delta_{13}}$;

4) $\overline{\left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right)} = \left(\frac{\Delta_{12}+\Delta_{14}}{\Delta_{13}}\right) = \frac{\Delta_{32}+\Delta_{34}}{\Delta_{13}}$.

Further, from the formula $PL = L^+P$ we note that the operator $L_1 = PL$ maps from the subspace $H_1 = PH$, where $H = L^2(0,\pi)$. Assuming

$$u(x) = Py(x) = \frac{y(x) + y(\pi - x)}{2},$$

we have

$$u'(x) = \frac{y'(x) - y'(\pi - x)}{2}.$$

Then from Lemma 4.5 it follows that

$$\begin{cases} \Delta_{13}u(0) + (\Delta_{12} - \Delta_{32})u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13} u(0) + (\Delta_{12} - \Delta_{32}) u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

Similarly, assuming that

$$v(x) = \frac{z(x) - z(\pi - x)}{2}, \text{ we get } v'(x) = \frac{z'(x) + z'(\pi - x)}{2}.$$

Then Lemma 4.5 implies that

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \overline{\Delta}_{13} v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{14}}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

Due to the lemmas,

$$\overline{\left(\frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}}\right)} = \frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}},$$

therefore, the last boundary condition has the form

$$\begin{cases} \Delta_{13} v(0) - (\Delta_{12} + \Delta_{14}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

Thus, the operators L_1 and L_2 take the following forms

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13} u(0) + (\Delta_{12} - \Delta_{32}) u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \overline{\Delta}_{13} v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{14}}) v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

where $\frac{\Delta_{12} - \Delta_{32}}{\Delta_{13}}$ and $\frac{\Delta_{12} + \Delta_{14}}{\Delta_{13}}$ are real quantities.

From the condition (10) of the proved Theorem 3.1 it follows that

$$\Delta_{12} - \Delta_{32} = -(\Delta_{12} + \Delta_{14}).$$

Assuming $\alpha = \Delta_{13}$, $\beta = \Delta_{12} - \Delta_{32}$, we rewrite the operators L_1 and L_2 as follows:

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \alpha u(0) + \beta u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \alpha v(0) + \beta v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

If a spectrum of the operator L is known, then by Lemma 2 spectra of the operators L_1 and L_2 are known. It is obvious that they form Borg pair in the interval $\left[0, \frac{\pi}{2}\right]$. By the Borg theorem spectra of these two operators uniquely determine the Sturm - Liouville operator on the segment $\left[0, \frac{\pi}{2}\right]$, and due to the formula $q(x) = q(\pi - x)$ on the whole segment $[0, \pi]$. Theorem 3.1 is proved.

А.Ш.Шалданбаев[1], Г.И. Бейсенова[2], А.Ж.Бейсебаева[3], А.А.Шалданбаева[4]

[1]Халықаралық Silkway университеті, Шымкент, Казақстан;
[2,4]Аймақтық әлеуметтік-инновациялық университеті, Шымкент, Казақстан;
[3]М.О.Ауезов атындағы Оңтүстік Қазақстан мемлекеттік университеті, Шымкент, Казақстан
[4]Халықаралық гуманитарлық-техникалық университет, Шымкент, Қазақстан

ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

Аннотация. Спектрәлдік анализдің кері есептері ретінде сызықтық операторды, оның спектрәлдік сыйпаттары арқылы қалпына келтіру есептері танылады. Бұл бағытта елеулі нәтиежеге, 1929 жылы В.А. Амбарцумян қол жеткізді. Ол келесі, теореманы дәлелдеді.

Мына,

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0, \ y'(\pi) = 0; \tag{1.2}$$

Штурм-Лиувилл есебінің меншікті мәндерін былай, $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ белгілейік, мұндағы $q(x)$ −дегеніміз, нақты әрі үзіліссіз функция.

Егер

$$\lambda_n = n^2 \ (n = 0,1,2, \dots) \text{ болса, онда } q(x) \equiv 0.$$

Амбарцумянның бұл еңбегінің маңыздылығына алғаш рет көңіл аударған швед математигі Борг еді. Штурм-Лиувиллдің (1.1) түріндегі өте маңызды байырғы операторының кері есебін спектрі арқылы жүйелі, әрі мақсатты түрде, алғаш рет зерттеген-де осы автордың өзі.

Борг бір спектрдің Штурм-Лиувилл операторын анықтауға жалпы жағдайда жетпейтінін,ал Амбарцумянның нәтиежесі сәтті бір кездейсоқтық екенін көрсетті.Дәл сол еңбегінде, Борг Штурм-Лиувилл операторын екі спектр арқылы (әртүрлі шекаралық шарттар бойынша) бірмәнді анықтауға болатынын көрсетті. Іле-шала, Боргтың еңбегінен соң, кері есептер теориясының маңызды есептерін Левинсон зерттеді, мысалы, егер $q(\pi - x) = q(x)$ болса, онда, Штурм-Лиувиллдің, мына,

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.3}$$

операторының, бір спектр арқылы, бірмәнді анықталатынын көрсетті.

Бұл еңбек Амбарцумян мен Левинсонның теоремаларын дамытуға арналған және өз бойында олардың теоремаларын алып жатыр.Бұл еңбекте потенциалы нақты үзіксіз әрі симметриялы,ал шекаралық шарттары әртарапты Штурм-Лиувилл операторын, бір спектр арқылы, бірмәнді анықтауға болатыны көрсетілді.Зерттеу әдісі бұрынғы әдістердің бәрінен өзгеше,және ол оператордың инвариантты іш кеңістіктерінің туындатқан симметриясына негізделген.

Түйін сөздер: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, әртарапты шекаралық шарттар, симметриялы потенциал, инвариантты ішкеңістіктері, дифференциәлдик операторлар, кері спектрәлді есептер.

А.Ш.Шалданбаев[1], Г.И. Бейсенова[2] ,
А.Ж.Бейсебаева[3] , А.А.Шалданбаева[4]

[1]Международный университет Silkway, Шымкент, Казахстан;
[2,4]Региональный социально-инновационный университет, Шымкент, Казахстан;
[3]Южно-Казахстанский Государственный университет им.М.Ауезова, Шымкент, Казахстан;
[4]Международный гуманитарно – технический университет, Шымкент, Казахстан

ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕРАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

Аннотация. Под обратными задачами спектрального анализа понимают задачи восстановления линейного оператора по тем или иным его спектральным характеристикам. Первый существенный результат в этом направлении был получен в 1929 году В.А. Амбарцумяном. Он доказал следующую теорему.

Обозначим через $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ собственные значения задачи Штурма-Лиувилля

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$

где $q(x) -$ действительная непрерывная функция. Если

$$\lambda_n = n^2 (n = 0,1,2, \dots) \text{ то } q(x) \equiv 0.$$

Первым из математиков, кто обратил внимание на важность этого результата Амбарцумяна, был шведский математик Борг. Он же выполнил первое систематическое исследование одной из важных обратных задач, а именно, обратной задачи для классического оператора Штурма-Лиувилля вида (1.1) по спектрам. Борг показал, что в общем случае один спектр оператора Штурм-Лиувилля его не определяет, так что результат Амбарцумяна является исключением из общего правила. В той же работе Борг показывает, что два спектра оператора Штурма-Лиувилля (при различных граничных условиях) однозначно его определяют. Точнее, Борг доказал следующую теорему.

Теорема Борга.

Пусть уравнения

$$-y'' + q(x)y = \lambda y,$$

$$-z'' + p(x)z = \lambda z,$$

имеют одинаковый спектр при краевых условиях

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases}$$

и при краевых условиях

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases}$$

Тогда $q(x) = p(x)$ почти всюду на отрезке $[0, \pi]$, если

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Вскоре после работы Борга важные исследования по теории обратных задач были выполнены Левинсоном, в частности, им доказано, что если $q(\pi - x) = q(x)$, то оператор Штурма-Лиувилля

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.3}$$

восстанавливается по одному спектру.

Обратные задачи для дифференциальных операторов с распадающимися краевыми условиями достаточно полно изучены. Более трудные обратные задачи для операторов Штурма-Лиувилля с неразделенными краевыми условиями также изучались. В частности, периодическая краевая задача рассматривалась в ряде работ. И. В. Станкевич предложил постановку обратной задачи и доказал соответствующую теорему единственности.

Настоящая работа посвящена обобщению теорем Амбарцумяна и Левинсона, в частности, наши результаты содержат в себе результаты этих авторов. В работе доказана теорема единственности, по одному спектру для оператора Штурма-Лиувилля с неразделенными краевыми условиями, вещественным

непрерывным и симметричным потенциалом. Метод исследования отличается от всех ранее известных методов и основан на внутренней симметрии оператора, порожденного инвариантными подпространствами.

Отметим, что рассматриваемый нами оператор является несамосопряженным, хотя потенциал вещественный и симметричный, этот момент играет существенную роль для нашего метода, ибо мы через оператора и его сопряженного строим пару операторов Борга. Другие авторы используют метод отображений Лейбензона.

Ключевые слова: оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства, дифференциальные операторы, обратные спектральные задачи.

Information about authors:

Shaldanbayev A.Sh., doctor of physical and mathematical Sciences, associate Professor, head of the center for mathematical modeling, «Silkway» International University, Shymkent; http://orcid.org/0000-0002-7577-8402;

Beissenova G.I., candidate of pedagogics. "Regional Social-Innovative University", Shymkent https://orcid.org/0000-0003-0545-0652;

Beisebayeva A.Zh., master of science in natural Sciences. M.Auezov South Kazakhstan State University, Shymkent; https://orcid.org/0000-0003-4839-9156;

Shaldanbayeva A.A., master of science in natural Sciences. International humanitarian and technical University, Shymkent https://orcid.org/0000-0003-2667-3097;

REFERENCES

[1] Marchenko V. A. Sturm– Liouville operators and their applications. Birkh¨auser**, 1986.**

[2] Levitan B. M. Inverse Sturm– Liouville problems.Utrecht, VNU Sci. Press, **1987**.

[3] P¨oschel J., Trubowitz E. Inverse Spectral Theory.New York, Academic Press, **1987**.

[4] Freiling G., Yurko V. A. Inverse Sturm– Liouville Problems and their Applications. New York, NOVA Science Publ., **2001**.

[5] Yurko V. A. Method of Spectral Mappings inthe Inverse Problem Theory. Inverse and Ill-posed Problems Series. Utrecht, VSP, **2002**.

[6] Stankevich I. V. An inverse problem of spectral analysis for Hill’s equation. Soviet Math. Dokl., **1970**, vol. 11, pp. 582–586.

[7] Marchenko V. A., Ostrovskii I. V. A characterization of the spectrum of the Hill operator. Math. USSR-Sb., **1975**, vol. 26, no. 4, pp. 493–554. DOI:10.1070/SM1975v026n04ABEH002493.

[8] Yurko V. A. An inverse problem for second order differential operators with regular boundary conditions.Math. Notes, **1975**, vol. 18, no. 3–4, pp. 928–932. DOI: 10.1007/BF01153046.

[9] Plaksina O. A. Inverse problems of spectral analysis for the Sturm– Liouville operators with nonseparated boundary conditions. Math. USSRSb,**1988,** vol. 59, no. 1, pp. 1–23. DOI:10.1070/SM1988v059n01ABEH003121.

[10] Hald O.H. The Inverse Sturm--Liouville Problem with symmetric Potentials, Acta mathematica 1978,141:1,263 .

[11] Isaakson E. L., Trubowitz - Pure Appl. Math. **1983**. V. 36. N 6. P. 763-783.

[12] Martin Bohner., Hikmet Koyunbakanb.Inverse Problems for Sturm–Liouville Di_erence Equations. Filomat 30:5 (**2016**), 1297–1304 DOI 10.2298/FIL1605297B.

[13] *Münevver* Tuz.On inverse sturm- liouville problems with symmetric potentials., ITM Web of Conferences 13, 01035 (**2017**) DOI: 10.1051/itmconf/20171301035.

[14] Hambardzumyan V.A. Uber eine Frage der Eigenwert-theorie, Zsch.f.Physik, 53(**1929**), 690-695.

[15] Borg G. Eine Umbehrung der Sturm – Liovillschen Eigenwertaufgabe, Acta Math., 78, №2(**1946**), 1-96.

[16] Levinson N. The inverse Sturm – Lioville problem, Math.Tidsskr. B., **1949**, 25-30.

[17] Levitan B.M. On definition of the Sturm - Liouville operator by one and two spectra. Proceedings of USSR Academy of Sciences, ser. Math., 42. No1. **1964**.

[18]Akylbayev M.I., Beysebayeva A., Shaldanbayev A. Sh., On the Periodic Solution of the Goursat Problem for a Wave Equation of a Special Form with Variable Coefficients. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 1, Number 317 **(2018**), 34 – 50. ISSN 1991-346X.

[19]Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh., Beisebaeva A.Zh., The Spectral Decomposition of Cauchy Problem’s Solution for Laplace Equation. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 5, Number 321 (**2018**), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10

[20]T.Sh. Kal'menov, A.Sh. Shaldanbaev, On a criterion of solvability of the inverse problem of heat conduction. Journal of Inverse and Ill-posed Problems. 18:5, **(2010**). 471_492.