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MULTIPERIODIC SOLUTIONS OF LINEAR SYSTEMS INTEGRO-DIFFERENTIAL EQUATIONS WITH D_c-OPERATOR AND ε-PERIOD OF HEREDITARY

Abstract. The article explores the questions of the initial problem and the problem of multiperiodicity solutions of linear systems integro-differential equations with an operator of the form $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ and with finite hereditary period $\varepsilon = const > 0$ by variable τ that describe hereditary phenomena. Along with the equation of zeros of the special differentiation operator D_c are considered linear systems of homogeneous and inhomogeneous integro-differential equations, sufficient conditions are established for the unique solvability of the initial problems for them, both necessary and sufficient conditions of multiperiodic existence are obtained by (τ, t) with periods (θ, ω) of the solutions. The integral representations of multiperiodic solutions of linear inhomogeneous systems with the uniqueness property are determined 1) in the particular case when the corresponding homogeneous systems have exponential dichotomy and 2) in the general case when the homogeneous systems do not have multiperiodic solutions, except for the trivial one. The article proposes a research technique for solving problems that satisfy initial conditions and have the property of multiperiodicity with a given ε heredity period for linear systems of integro-differential equations with a special partial differential operator D_c. Multiperiodic solutions obtained along characteristics $t = t^0 + c\tau - c\tau^0$ with fixed (τ^0, t^0) are used as an application in the theory of quasiperiodic solutions of systems of integro-differential equations.

Key words: integro-differential equation, hereditary, fluctuation, multiperiodic solution.

1. Problem statement.

In this paper, we've researched the problem of the existence of (θ, ω)-periodic solutions $u(\tau, t)$ by $(\tau, t) = (\tau, t_1, \ldots, t_m) \in R \times R^m$ systems of

$$D_c u(\tau, t) = A(\tau, t)u(\tau, t) + \int_{\tau-\varepsilon}^{\tau} K(\tau, t, s, t - c\tau + cs)u(s, t - c\tau + cs)ds + f(\tau, t) \qquad (1.1)$$

with a differentiation operator D_c of the form

$$D_c = \partial/\partial\tau + \langle c, \partial/\partial t\rangle, \qquad (1.2)$$

that turns into the operator of the total derivative $d/d\tau$ along the characteristics $t = c\tau - cs + \sigma$ with initial data $(s, \sigma) \in R \times R^m$, where $R = (-\infty, +\infty)$, c is constant vector, $\partial/\partial t = (\partial/\partial t_1, \ldots, \partial/\partial t_m)$ is vector, $\langle c, \partial/\partial t\rangle$ is the scalar product of vectors, $A(\tau, t)$ and $K(\tau, t, s, \sigma)$ are given $n \times n$-matrices, $f(\tau, t)$ is n-vector-function, $(\theta, \omega) = (\theta, \omega_1, \ldots, \omega_m)$ is vector-period with rationally incommensurable coordinates, ε is positive constant.

The problem of this kind involves the research problems of hereditary vibrations in mechanics and electromagnetism. For example, if the oscillation phenomenon is hereditary in nature, then the equation of motion of the string at a known moment $m(\tau)$ is set by changing the angle of string torsion $\omega(\tau)$, subordinated to the ratio $m(\tau)-\mu\frac{d^2\omega(\tau)}{d\tau^2}=h\omega(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}\varphi(\tau,s)\omega(s)ds$, where μ and h are constants and ε is the hereditary period of the vibrational phenomenon. It is also known that the hereditary biological phenomenon "predator-prey" is related by the law of oscillations described by the system of integro-differential equations.

The given integro-differential equations are the mathematical model hereditary phenomena described by system

$$\frac{dx}{dt}=P(\tau)x(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}Q(\tau,s)x(s)ds+\psi(\tau). \qquad (1.3)$$

Where $x(\tau)$ is unknown n-vector-function; $P(\tau)$, $Q(\tau,s)$ are $n\times n$-matrices; $\psi(\tau)$ is n-vector-function; $\varepsilon>0$ is a constant. Since the process is oscillatory, as a rule, $P(\tau)$ and $\psi(\tau)$are almost periodic by τ, and the kernel $Q(\tau,s)$ has the property of diagonal periodicity by $(\tau,s)\in R\times R$. In particular, the indicated input data of system (1.3) are quasiperiodic by $\tau\in R$ with a frequency basis $\nu_0=\theta^{-1},\nu_1=\omega_1^{-1},\ldots,\nu_m=\omega_m^{-1}$, then in the theory of fluctuations, the question of the existence of quasiperiodic solutions $x(\tau)$ of system (1.3), with a changed frequency basis $\tilde{\nu}_0=\theta^{-1},\tilde{\nu}_1=c_1\omega_1^{-1},\ldots,\tilde{\nu}_m=c_m\omega_m^{-1}$, is of great importance and we set $\varepsilon<\theta=\omega_0<\omega_1<\ldots<\omega_m$. An important role in solving this problem is played by the well-known theorem of G. Bohr on the deep connection between quasiperiodic functions and periodic functions of many variables. According to this theorem, matrix-vector functions are defined $A=A(\tau,t)$, $K=K(\tau,t,s,\sigma)$, $\sigma=t-c\tau+cs$, $f=f(\tau,t)$, $u=u(\tau,t)$ with properties of $A\big|_{t=c\tau}=P(\tau)$, $K\big|_{t=c\tau}=Q(\tau,s)$, $f\big|_{t=c\tau}=\psi(\tau)$, $u\big|_{t=c\tau}=x(\tau)$ and the operator $d/d\tau$ is replaced by a differentiation operator D_c of the form (1.2).

Thus, the problem of quasiperiodic fluctuations in systems (1.5) becomes equivalent to the problem on the existence of (θ,ω)-periodic by (τ,t) solutions $u(\tau,t)$ of the system partial integro-differential equations of the form (1.1) with differentiation operator (1.2).

The above problems on string vibrations and fluctuations in the numbers of two species living together associated with the task indicate the relevance of the latter, in terms of its applicability in life. Along with this, it is worth paying special attention to the fact that the methods of researching multiperiodic solutions of integro-differential equations and systems of such partial differential equations belong to a poorly studied section of mathematics. Therefore, the development of methods of the theory of multiperiodic solutions of partial differential integro-differential equations is of special scientific interest.

In the present work are investigated to obtain conditions for the existence of multiperiodic solutions of linear systems integro-differential equations with a given differentiation operator D_c. To achieve this goal, the initial problems for the considered systems of equations are solved from the beginning, the necessary and sufficient conditions for the existence of multiperiodic solutions of linear systems are established, integral structures of solutions linear systems are determined.

The theoretical basis of this research is based on the work of several authors. As noted above, taking into account the hereditary nature of various processes of physics, mechanics, and biology leads to the consideration of integro-differential equations [1–3], especially to the research of problems for them related to the theory of periodic fluctuations [2]. If the heredity of the phenomenon is limited to a finite period ε of time τ, then the hereditary effect is specified by the integral operator with variable limits from $\tau-\varepsilon$ to τ. Integro-differential equations describing phenomena with such hereditary effects are considered in [3]. The various processes of hereditary continuum mechanics are described by partial integro-differential equations, the study of which began with the works [1]. The work of many authors is devoted to finding effective signs of solvability and the construction of constructive methods for researching problems for systems of differential equations, we note only [2,4,5]. The research of multi-frequency oscillations led to the concept of multidimensional time. In this connection, of the theory solutions of partial differential equations that are periodic in multidimensional [6–13]. In [6], an approach is implemented where quasiperiodic solutions of ordinary differential equations are studied with a transition to the study of multiperiodic solutions of partial differential equations. This method was developed in [7–11] with its extension to the solution of a number of oscillation problems in systems of integro-differential equations.

In this research, it is examined for the first time that the problem of the existence multiperiodic solutions of systems integro-differential equations with a special differentiation operator D_c, describing hereditary processes with a finite period ε of hereditary time τ. In solving this problem, we encountered the problems associated with the multidimensionality of time; not developed general theory of such systems; determination of structures and integral representations of solutions of linear systems equations; extending the results of the linear case to the nonlinear case; the smoothness of the solutions integral equations equivalent to the problems under consideration, etc. These barriers to solving problems have been overcome due to the spread and development of the methods of works [12-13] used to solve similar problems for systems of differential equations.

2. Zeros of the differentiation operator and its multiperiodicity

By the zero of the operator D_c we mean a smooth function $u = u(\tau, t)$ satisfying the equation of $D_c u = 0$. The linear function y is a general solution of the characteristic equation with the initial data (τ^0, t^0), its integral is the zero of the operator D_c satisfying condition $h(\tau^0, \tau, t)\big|_{\tau=\tau^0} = t$. Note that if $\psi(t)$ is an any smoothness function $e = (1, \ldots, 1)$, by $t \in R^m$, then

$$u(\tau^0, \tau, t) = \psi(h(\tau^0, \tau, t)) \tag{2.1}$$

is the zero of the operator D_c satisfying condition $u\big|_{\tau=\tau^0} = \psi(t)$. Since the $\psi(t)$ is arbitrary in the class $C_t^{(e)}(R^m)$, relation (2.1) is a general formula of the zeros.

We give the properties of the characteristics of the operator D_c:

$$h(s + \theta, \tau + \theta, t) = h(s, \tau, t), \tag{2.2}$$

$$h(s, \tau + \theta, t) = h(s, \tau, t) - c\theta, \tag{2.3}$$

$$h(s, \tau, t + q\omega) = h(s, \tau, t) + q\omega,\ q\omega = (q_1\omega_1, \ldots, q_m\omega_m),\ q \in Z^m. \tag{2.4}$$

If $u(\tau, t)$ is the zero (θ, ω)-periodic, then the $u\big|_{\tau=\tau^0} = u^0(t)$ is ω-periodic by t:

$$u^0(t + q\omega) = u^0(t) \in C_t^{(e)}(R^m), q \in Z^m. \tag{2.5}$$

Therefore, (2.5) is a necessary condition for the (θ, ω)-periodicity of zero $u(\tau, t) \in C_{\tau,t}^{(1,e)}(R \times R^m)$. Suppose that for zero $u(\tau, t)$ is satisfied (2.5). From (2.1):

$$u(\tau, t) = u^0(h(\tau^0, \tau, t)). \tag{2.6}$$

Based on (2.3), zero $u(\tau, t)$ is θ-periodic by τ if $u^0(h(\tau^0, \tau + \theta, t)) = \ = u^0(h(\tau^0, \tau, t) - c\theta)$. This takes place if there is a vector $q^0 \in Z^m$ and

$$c\theta + q^0\omega = 0. \tag{2.7}$$

By virtue of (2.2), the zeros $u(\tau^0, \tau, t)$ form (2.1) have the property of diagonal θ-periodicity by (τ^0, τ). The proof follows from (2.2) and (2.1).

Theorem 2.1. *1) If condition (2.7) is not satisfied, then only constants are the* (θ, ω)*-periodic zeros and it does not have multiperiodic variables zeros. 2) If condition (2.7) is satisfied, then any zero of the operator* D_c *with an initial function of the form (2.5) is* (θ, ω)*-periodic, in particular, it can be any constant. 3) Zero of the form (2.1) has the property of diagonal* θ*-periodicity by* (τ^0, τ)*, and from its* θ*-periodicity zeros by* τ *follows its* θ*-periodicity by* τ^0.

3. Linear homogeneous equations and its multiperiodic solutions.

We consider the initial problem for a linear homogeneous system

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds \tag{3.1}$$

$$u(\tau,t)\big|_{\tau=\tau^0} = u^0(t) \in C_t^{(e)}(R^m) \tag{3.1^0}$$

under assumptions of

$$A(\tau+\theta, t+q\omega) = A(\tau,t) \in C_{(\tau,t)}^{(0,2e)}(R\times R^m),\ q\in Z^m, \tag{3.2}$$

$$K(\tau+\theta, t+q\omega, s,\sigma) = K(\tau,t,s+\theta,\sigma+q\omega) = K(\tau,t,s,\sigma) \in$$
$$\in C_{\tau,t,s,\sigma}^{(0,2e,0,2e)}(R\times R^m \times R\times R^m),\ q\in Z^m. \tag{3.3}$$

From [17-19] with (3.2), using the method of successive approximations, we can construct a matricant $W(\tau^0,\tau,t)$ of the system $D_c w(\tau,t) = A(\tau,t)w(\tau,t)$, and

$$D_c W(\tau^0,\tau,t) = A(\tau,t)W(\tau^0,\tau,t),\ W(\tau^0,\tau^0,t) = E, \tag{3.4}$$

$$D_c W^{-1}(\tau^0,\tau,t) = -W^{-1}(\tau^0,\tau,t)A(\tau,t), \tag{3.5}$$

$$W(\tau^0+\theta, \tau+\theta, t+q\omega) = W(\tau^0,\tau,t),\ q\in Z^m. \tag{3.6}$$

Then, using the replacement of $u(\tau,t) = W(\tau^0,\tau,t)v(\tau,t)$ system (3.1) is reduced to the form of integro-differential equation

$$D_c v(\tau,t) = \int_{\tau-\varepsilon}^{\tau} Q(\tau^0,\tau,t,s,h(s,\tau,t))v(s,h(s,\tau,t))ds \tag{3.7}$$

with the kernel $Q(\tau^0,\tau,t,s,\sigma) = W^{-1}(\tau^0,\tau,t)K(\tau,t,s,\sigma)W(\tau^0,s,\sigma)$. By virtue of (2.2)-(2.4), (3.3) and (3.4)-(3.6), $Q(\tau^0,\tau,t,s,\sigma)$ has the properties:

$$Q(\tau^0+\theta, \tau+\theta, t+q\omega, s+\theta, h(s+\theta,\tau+\theta,t+q\omega)) = Q(\tau^0,\tau,t,s,h(s,\tau,t)) =$$
$$= Q(\tau^0,\tau,t,s,\sigma) \in C_{\tau^0,\tau,t,s,\sigma}^{(1,1,e,1,e)}(R\times R\times R^m\times R\times R^m),\ q\in Z^m$$

Further, under condition (3.3), integrating along the characteristics: $\tau=\eta$, $t = h(\eta,\tau,t)$, using the group property of the characteristic, from equation (3.7):

$$V(s,\tau,t) = E + \int_s^{\tau} d\eta \int_{\eta-\varepsilon}^{\eta} Q(s,\eta,h(\eta,\tau,t),\xi,h(\xi,\tau,t))V(s,\xi,h(\xi,\tau,t))d\xi, \tag{3.8}$$

Obviously, by virtue of (3.8) and multiperiodicity, we have

$$D_c V(s,\tau,t) = \int_{\tau-\varepsilon}^{\tau} Q(s,\tau,t,\xi,h(\xi,\tau,t))V(\xi,h(\xi,\tau,t))d\xi, \tag{3.9}$$

$$V(s,s,t) = E. \tag{3.9^0}$$

The A, K, ε are such that matrix $V(s,\tau,t)$ is invertible. The matrix $U(s,\tau,t)= =W(s,\tau,t)V(s,\tau,t)$: $D_c U(s,\tau,t)=A(\tau,t)U(s,\tau,t)+\int\limits_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi$,

$$U(s,s,t)=E, \tag{3.10}$$

$$U(s+\theta,\tau+\theta,t+q\omega)=U(s,\tau,t)\in C^{(1,1,e)}_{s,\tau,t}(R\times R\times R^m),\ q\in Z^m. \tag{3.11}$$

Properties (3.10)-(3.11) are consequences of (3.4)-(3.6), (3.9)-(3.9^0). The matrix $U(s,\tau,t)$ is called the resolving operator of system (3.1).

Theorem 3.1. *Let conditions (3.2)-(3.3) are satisfied. Then the solution $u(\tau^0,\tau,t)$ of the problem (3.1)-(3.1^0) is uniquely determined by the relation*

$$u(\tau^0,\tau,t)=U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t)). \tag{3.12}$$

Proof.◀By condition (3.1^0), (2.5), (2.6) $u^0(h(\tau^0,\tau,t))$, is the zero of the operator D_c. Taking into account the group property of the characteristic, (3.10) it is shown that (3.12) satisfies the system (3.1).▶

Theorem 3.2. *Let the conditions of theorem 3.1 are satisfied. The solution $u(\tau,t)$ of system (3.1) is (θ,ω)-periodic, it is necessary, that its initial function $u(0,t)=u^0(t)$ at $\tau=0$ should be ω-periodic continuously differentiable:*

$$u^0(t+q\omega)=u^0(t)\in C^{e/\omega}_t(R^m),\ q\in Z^m. \tag{3.13}$$

Proof.◀Indeed, for $\tau^0=0$, from the solution (3.12) we have

$$u(\tau,t)=U(0,\tau,t)u^0(h(0,\tau,t)), \tag{3.14}$$

and it is (θ,ω)-periodic by (τ,t), in particular

$$u(\tau,t+q\omega)=u(\tau,t), q\in Z^m. \tag{3.15}$$

Using (3.14), (3.15), (2.4), (3.11) and (3.10), we find. $u^0(t+q\omega)=u^0(t)$. The smoothness of $u^0(t)$ follows from smoothness of solution $u(\tau,t)$ of system (3.1).

Theorem 3.3. *In order for the solution $u(\tau,t)$ of system (3.1) for being ω-periodic by $t\in R^m$ under the conditions of theorem 3.2, it is necessary and sufficient for condition (3.13) be satisfied by the initial function $u^0(t)$ for $\tau=0$.*

Proof.◀ Necessity follows from Theorem 3.2. For sufficiency, to show relation (3.15) follows from condition (3.13).▶

Theorem 3.4. *In order for the solution $u(\tau,t)$ to be θ-periodic by $\tau\in R$ under the conditions of theorem 3.3, it is necessary and sufficient that the initial function $u^0(t)$ a ω-periodic solution of the linear ω-periodic by t functional difference system with difference $\rho=c\theta$ by t*

$$U(0,\theta,t)u^0(t-c\theta)=u^0(t) \tag{3.16}$$

◀*The necessary and sufficient condition (3.16) follows from (3.14), (2.3), (3.10)* .▶

Theorem 3.5. *In order for the solution $u(\tau,t)$ to be (θ,ω)-periodic solution of (3.1) generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the operator D_c under the conditions of theorem 3.4, it is necessary and sufficient that the $u_0(\tau,t)=v(t)$ be an eigenvector of the monodromy matrix $U(0,\theta,t)=V(t)$: $[V(t)-E]v(t)=0$.*

The necessary and sufficient condition (3.16) follows from Theorems 2.1 and 3.4.

We assume that the operator $U\left(\tau^0, \tau, t\right)$ of system (3.1) satisfies condition

$$\left|U\left(s, \tau, t\right)\right| \le a\, e^{-\alpha(\tau - s)},\ a \ge 1,\ \alpha > 0,\ \tau \ge s. \qquad (3.17)$$

Theorem 3.6. *In order for the system of integro-differential equations (3.1) has no multiperiodic solutions, except for the zero one under the conditions of theorem 3.4, the fulfillment of condition (3.17) is sufficient.*

Note that theorem 3.6 is valid if condition (3.17) is replaced by condition $\left|U\left(s, \tau, t\right)\right| \le a\, e^{\alpha(\tau - s)}$, $a \ge 1$, $\alpha > 0$, $\tau \le s$. The resolving operator $U(s, \tau, t)$ is represented as:

$$U(s, \tau, t) = U_-(s, \tau, t) + U_+(s, \tau, t), \qquad (3.18)$$

$$D_c U_\mp(s, \tau, t) = A(\tau, t) U_\mp(s, \tau, t) + \int_{\tau-\varepsilon}^{\tau} K\left(\tau, t, \xi, h(\xi, \tau, t)\right) U_\mp\left(s, \xi, h(\xi, \tau, t)\right) d\xi\,, \qquad (3.19)$$

and satisfying conditions

$$\left|U_-\left(s, \tau, t\right)\right| \le a\, e^{-\alpha(\tau - s)},\ \tau \ge s,\ a \ge 1 \text{ and } \alpha > 0, \qquad (3.20)$$

$$\left|U_+\left(s, \tau, t\right)\right| \le a\, e^{\alpha(\tau - s)},\ \tau \le s,\ a \ge 1 \text{ and } \alpha > 0. \qquad (3.21)$$

Under conditions (3.18)-(3.21), they say that the resolving operator $U(s, \tau, t)$ has the property of exponential dichotomy.

Theorem 3.7. *Let conditions (3.2), (3.3), and (3.18)-(3.21) be satisfied. Then system (3.1) has no multiperiodic solutions, except for the trivial one.*

4. Linear inhomogeneous equations and its multiperiodic solutions

We consider the system of integro-differential equations

$$D_c u(\tau, t) = A(\tau, t) u(\tau, t) + \int_{\tau-\varepsilon}^{\tau} K\left(\tau, t, \xi, h(\xi, \tau, t)\right) u\left(\xi, h(\xi, \tau, t)\right) d\xi + f(\tau, t), \qquad (4.1)$$

where the $f(\tau, t)$ is given n-vector-function possessing property

$$f(\tau + \theta, t + q\omega) = f(\tau, t) \in C_{\tau,t}^{(0,e)}\left(R \times R^m\right),\ q \in Z^m. \qquad (4.2)$$

Find a solution to system (4.1) satisfying the initial condition

$$u\big|_{\tau=\tau^0} = u^0(t) \in C_t^{(e)}\left(R^m\right). \qquad (4.1^0)$$

We seek a particular solution $u^*\left(\tau^0, \tau, t\right)$ of system (4.1) with zero initial condition

$$u^*\left(\tau^0, \tau, t\right)\big|_{\tau=\tau_0} = 0. \qquad (4.1^*)$$

with an unknown n-vector function $v(\tau, t) \in C_{\tau,t}^{(0,e)}\left(R \times R^m\right)$ in the form

$$u^*\left(\tau^0, \tau, t\right) = \int_{\tau^0}^{\tau} U(s, \tau, t) v\left(s, h(s, \tau, t)\right) ds. \qquad (4.3)$$

Acting by the operator D_c on vector-function (4.3), considering $v(\tau, t)$, we have

$$D_c u^*\left(\tau^0, \tau, t\right) = A(\tau, t) u^*\left(\tau^0, \tau, t\right) + \int_{\tau-\varepsilon}^{\tau} K\left(\tau, t, \xi, h(\xi, \tau, t)\right) u^*\left(\tau^0, \xi, h(\xi, \tau, t)\right) d\xi + v(\tau, t). \qquad (4.5)$$

Substituting (4.3) and (4.5) into (4.1) we obtain that $v(\tau, t) = f(\tau, t)$. Then

$$u^*\left(\tau^0,\tau,t\right)=\int_{\tau^0}^{\tau}U(s,\tau,t)f(s,h(s,\tau,t))ds. \qquad (4.6)$$

The solution (4.6) satisfies condition (4.1^*). The general Cauchy solution of system (4.1) with initial condition (4.1^0) has the form

$$u\left(\tau^0,\tau,t\right)=U\left(\tau^0,\tau,t\right)u^0\left(h\left(\tau^0,\tau,t\right)\right)+u^*\left(\tau^0,\tau,t\right). \qquad (4.7)$$

Theorem 4.1. *Under conditions (3.2), (3.3) and (4.2), the initial problem (4.1) - (4.1^0) has the unique solution in the form (4.6)-(4.7).*

◄ For the proof we use (4.6) and (4.7).►

Under the conditions of Theorem 3.7, the existence of multiperiodic solutions of system (4.1) is investigated by the method of Green's functions, exponential dichotomy of

$$G(s,\tau,t)=\begin{cases}U_-(s,\tau,t), & \tau\ge s\\ -U_+(s,\tau,t), & \tau<s\end{cases}.$$

Theorem 4.2. *Suppose that the conditions of theorem 4.1 are satisfied and the matrix $A(\tau,t)$ with kernel $K(\tau,t,s,\sigma)$ are such that the system (3.1) has the property of exponential dichotomy, expressed by the relation (3.18)-(3.21). Then system (4.1) has the unique (θ,ω)-periodic solution $u^*(\tau,t)=\int_{-\infty}^{+\infty}G(s,\tau,t)f(s,h(s,\tau,t))ds$, satisfying estimate $\|u^*\|\le a/\alpha\|f\|$.*

◄ Under the conditions of Theorem 3.1, the properties of the Green's function, the proof of the theorem is carried out. Exponential dichotomy (3.1) ensures the uniqueness of a multiperiodic solution to system (4.1).►

Lemma 4.1. *Let the homogeneous linear system (3.1) under conditions (3.2), (3.3) and (4.2) have no (θ,ω)-periodic solutions except zero. Then the corresponding inhomogeneous linear system (4.1) can have at most one (θ,ω)-periodic solution.*

Finding the (θ,ω)-periodic solution $u(\tau,t)$ of system (4.1) among the solutions with initial conditions, it is shown that it is defined as

$$u(\tau,t)=\left[U^{-1}(0,\tau+\theta,t)-U^{-1}(0,\tau,t)\right]^{-1}\int_{\tau}^{\tau+\theta}\tilde{U}_\theta(s,\tau,t)f_\theta(s,h(s,\tau,t))ds. \qquad (4.8)$$

Theorem 4.3. *Suppose that conditions (3.2), (3.3), (4.2) are satisfied and the linear homogeneous system (3.1) has no (θ,ω)-periodic solutions, except for the trivial one. Then the system of inhomogeneous linear integro-differential equations (4.1) has the unique (θ,ω)-periodic solution $u(\tau,t)$ of the form (4.8).*

Note that uniqueness follows from Lemma 4.1.

We note that the research problems of the considered systems can be studied along characteristics with fixed initial data. From the proved theorems as a corollary, we have statements about the existence of solutions to the initial problems for ordinary integro-differential equations and about the existence of their quasiperiodic solutions in the sense of Bohr generated by multiperiodic solutions of the original systems.

Conclusion.

In this article proposes the method for (research) researching solutions of problems that satisfy the initial conditions and have the property of multiperiodicity with given periods for systems of integro-differential equations with a special D_c operator in partial differential, ε hereditary effect and the linear integral operator. This technique is a generalization of methods and solutions of similar problems for systems of partial differential equations with the operator D_c. The problems under consideration in this formulation are researched for the first time. The relevance of the main problem is substantiated. The solutions of all the subtasks analyzed to achieve the goal are formulated as theorems with proofs. Scientific novelties include the multi-periodicity theorems of zeros of the operator D_c; about solutions to initial problems for all considered of systems; about necessary as well as sufficient conditions for the

existence of multiperiodic solutions of both homogeneous and inhomogeneous systems, the integral representations of solutions systems in cases: exponential dichotomy and the absence of non-trivial multiperiodic solutions. We note that the consequences deduced by examining the results obtained along the characteristics refer to their applications in the theory of quasiperiodic solutions of systems ordinary integro-differential equations. The technique that developed here is quite applicable to the research of problems of hereditary-string vibrations and the “predator-prey” given in delivered part of work, which can be attributed to examples of applied aspect.

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D_c-ОПЕРАТОРЛЫ ЖӘНЕ ε -ЭРЕДИТАРЛЫҚ ПЕРИОДТЫ СЫЗЫҚТЫ ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІ

Аннотация. Мақалада $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$ операторлы, $c - const$ және тұқым қуалаушылық сипаттағы құбылыстарды сипаттайтын τ уақыты бойынша ε ақырлы эредитарлық периодты сызықты интегралды-дифференциалдық теңдеулер жүйесінің көппериодты шешімдері жөніндегі есептер мен бастапқы есеп мәселелері зерттеледі. Арнайы D_c дифференциалдау операторының нөлдерінің теңдеуімен қатар, сызықты біртекті және біртекті емес интегралды-дифференциалдық теңдеулер жүйесі қарастырылды, олар үшін бастапқы есептердің бірмәнді шешілімділігінің жеткілікті шарттары анықталған, (τ, t) бойынша (θ, ω) периодты, көппериодты шешімдердің бар болуының қажетті де, жеткілікті де шарттары алынған. Жалғыздық шартына ие сызықты біртекті емес жүйенің көппериодты шешімдерінің интегралдық өрнектері, 1) дербес жағдайда, яки теңдеуге сәйкес біртекті жүйелер экспоненциалды дихотомиялық қасиетке ие болғанда және 2) жалпы жағдайда, біртекті жүйелердің нөлден басқа көппериодты шешімдері болмағанда айқындалды. Мақалада дербес туындылы арнайы D_c дифференциалдау операторлы сызықты интегралды-дифференциалдық теңдеулер жүйесі үшін берілген ε эредитарлық периодты көппериодтылық қасиетіне ие және бастапқы шарттарға қанағаттандыратын есептерді шешудің зерттеу әдістемесі ұсынылған. (τ^0, t^0) бекітілген $t = t^0 + c\tau - c\tau^0$ характеристикалар бойында алынған көппериодты шешімдер қарапайым интегралды-дифференциалдық теңдеулер жүйесінің квазипериодты шешімдер теориясында қолданбалы түрде пайдаланылады.

Түйін сөздер: интегралды-дифференциалдық теңдеу, эредитарлық, флуктуация, көппериодты шешім.

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МНОГОПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ЛИНЕЙНЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С D_c- ОПЕРАТОРОМ И ε -ПЕРИОДОМ ЭРЕДИТАРНОСТИ

Аннотация. В статье исследуются вопросы начальной задачи и задачи о многопериодичности решений линейных систем интегро-дифференциальных уравнений с оператором вида $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c - const$ и конечным периодом эредитарности $\varepsilon = const > 0$ по времени τ, которые описывают явления наследственного характера. Наряду с уравнением нулей специального оператора дифференцирования D_c рассмотрены линейные системы однородных и неоднородных интегро-дифференциальных уравнений, для них установлены достаточные условия однозначной разрешимости начальных задач, получены как необходимые, так и достаточные условия существования многопериодических по (τ, t) с периодами (θ, ω) решений. Определены интегральные представления многопериодических решений линейных неоднородных систем, обладающих свойством единственности, 1) в частном случае, когда соответствующие однородные системы обладают экспоненциальной дихотомичностью и 2) в общем случае, когда однородные системы не имеют многопериодических решений, кроме тривиального. В статье предложена методика исследования решения задач, удовлетворяющих начальным условиям и обладающих свойством многопериодичности с заданным $\varepsilon -$ периодом эредитарности для линейных систем интегро-дифференциальных уравнений со специальным оператором дифференцирования D_c в частных производных. Многопериодические решения, полученные вдоль характеристик $t = t^0 + c\tau - c\tau^0$ с фиксированной (τ^0, t^0), применяются в виде приложения в теории квазипериодических решений систем обыкновенных интегро-дифференциальных уравнений.

Ключевые слова: интегро-дифференциальное уравнение, эредитарность, флуктуация, многопериодическое решение.

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