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QUASICLASSICAL LIMIT OF THE SCHRÖDINGER-MAXWELL-BLOCH EQUATIONS

Abstract. The study of integrable equations is one of the most important aspects of modern mathematical and theoretical physics. Currently, there are a large number of nonlinear integrable equations that have a physical application. The concept of nonlinear integrable equations is closely related to solitons. An object being in a nonlinear medium that maintains its shape at moving, as well as when interacting with its own kind, is called a soliton or a solitary wave. In many physical processes, nonlinearity is closely related to the concept of dispersion. Soliton solutions have dispersionless properties. Connection with the fact that the nonlinear component of the equation compensates for the dispersion term.

In addition to integrable nonlinear differential equations, there is also an important class of integrable partial differential equations (PDEs), so-called the integrable equations of hydrodynamic type or dispersionless (quasiclassical) equations [1-13]. Nonlinear dispersionless equations arise as a dispersionless (quasiclassical) limit of known integrable equations. In recent years, the study of dispersionless systems has become of great importance, since they arise as a result of the analysis of various problems, such as physics, mathematics, and applied mathematics, from the theory of quantum fields and strings to the theory of conformal mappings on the complex plane. Well-known classical methods of the theory of intrinsic systems are used to study dispersionless equations.

In this paper, we present the quasicalassical limit of the system of (1+1)-dimensional Schrödinger-Maxwell-Bloch (NLS-MB) equations. The system of the NLS-MB equations is one of the classic examples of the theory of nonlinear integrable equations. The NLS-MB equations describe the propagation of optical solitons in fibers with resonance and doped with erbium. And we will also show the integrability of the quasiclassical limit of the NLS-MB using the obtained Lax representation.

Key words. Dispersionless integrable system, quasiclassical limit, Schrödinger-Maxwell-Bloch equations, Lax pair.

Introduction

The study of the integration of nonlinear equations and systems dominates one of the main places in theoretical and mathematical physics. Such equations have physical applications that multiply interest in similar studies. At the present time there are a lot of nonlinear integrable equations describing different phenomena in different fields of physics.

The system of the (1+1)-dimensional Schrödinger-Maxwell-Bloch equations obtained by A.I. Maimistov and E.A. Manykin [14] and were studied by different scientists [15-17].

The studied system of the (1+1)-dimensional Schrödinger-Maxwell-Bloch equations reads as

$$iq_t + q_{xx} + 2|q|^2 q - 2ip = 0, \tag{1}$$

$$p_x - 2i\omega_0 p - 2\eta q = 0, \tag{2}$$

$$\eta_x + q\bar{p} + \bar{q}p = 0, \tag{3}$$

where x and t are the normalized distance and time, respectively; $q(x,t)$ is the slowly varying envelope axial field, $p(x,t)$ is the measure of the polarization of the resonant medium, $\eta(x,t)$ represents the extent of the population inversion, ω_0 is a constant corresponding to the frequency. q, p are complex variable functions, and η is a real variable function, ω_0 is real constant. q_t, q_{xx}, p_x and η_x are partial derivatives with respect to variables x and t, i is an imaginary unit, $\bar{q}$ and $\bar{p}$ are complex conjugates of q and p quantities, respectively.

The system of the (1+1)-dimensional Schrödinger-Maxwell-Bloch equations is completely integrable by the inverse scattering transformation (IST) [18]. It is known that the IST makes use of the Lax equations to solve such systems. Lax pair for the system of the (1+1)-dimensional Schrödinger-Maxwell-Bloch equations (1)-(3) has the form

$$\Phi_x = U\Phi, \tag{4}$$

$$\Phi_t = V\Phi, \tag{5}$$

where the matrices U and V have the form

$$U = -\lambda\sigma_3 + U_0, \tag{6}$$

$$V = -2\sigma_3\lambda^2 + 2\lambda U_0 + V_0 + \frac{1}{\lambda + \omega_0} V_{-1}. \tag{7}$$

Here U_0, V_0, V_{-1} depend on q, p, η functions and U_0, V_0, V_{-1} are 2×2 matrices

$$U_0 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

and λ is the complex eigenvalue parameter (constant).

Quasiclassical limit of the system of (1+1)-dimentional Schrödinger-Maxwell-Bloch equations. To find the quasiclassical limit of the given system, we use the following change of variables $x \to \varepsilon x,\ t \to \varepsilon t$, where ε is a constant [13, 19-21]. Then, the partial derivatives will change as

$$\frac{\partial}{\partial t} \to \varepsilon \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x} \to \varepsilon \frac{\partial}{\partial x}.$$

Taking into account the change of variables, the equations (1)-(3) has the next form

$$i\varepsilon q_t + \varepsilon^2 q_{xx} + 2|q|^2 q - 2ip = 0, \tag{9}$$

$$\varepsilon p_x - 2q\eta - 2i\omega_0 p = 0, \tag{10}$$

$$\varepsilon \eta_x + q\bar{p} + \bar{q}p = 0. \tag{11}$$

Now we enter the scale transformation in the forms of q, p and η

$$q = \sqrt{u}e^{\frac{i}{\varepsilon}S}, \quad p = i\sqrt{uw}e^{\frac{i}{\varepsilon}S}, \quad \eta = \sqrt{1-uw}, \tag{12}$$

where $u = u(x,t),\ w = w(x,t)$ are real functions, p and η are related as $\eta^2 + |p|^2 = 1$. Further, to calculate all the terms of the nonlinear system (9)-(11), we differentiate equation (12) with respect to the variables x and t.

$$q_t=\left(\frac{u_t}{2\sqrt{u}}+\frac{i}{\varepsilon}S_t\sqrt{u}\right)e^{\frac{i}{\varepsilon}S},\ q_x=\left(\frac{u_x}{2\sqrt{u}}+\frac{i}{\varepsilon}S_x\sqrt{u}\right)e^{\frac{i}{\varepsilon}S}, \tag{13}$$

$$q_{xx}=\left\{\left(\left(\frac{u_x}{2\sqrt{u}}\right)_x-\frac{S_x^2\sqrt{u}}{\varepsilon^2}\right)+\frac{i}{\varepsilon}\left(S_{xx}\sqrt{u}+\frac{S_xu_x}{\sqrt{u}}\right)\right\}e^{\frac{i}{\varepsilon}S}, \tag{14}$$

$$p_x=\left(i\left(\sqrt{uw}\right)_x-\frac{S_x}{\varepsilon}\sqrt{uw}\right)e^{\frac{i}{\varepsilon}S},\ \eta_x=-\frac{(uw)_x}{2\sqrt{1-uw}}. \tag{15}$$

Substituting formula (13)-(15) into system (9)-(11) and collecting the coefficients of different powers of ε we get the system of equation

$$u_t+2(\upsilon u)_x=0, \tag{16}$$

$$\upsilon_t+\left(\upsilon^2-2u-2\sqrt{w}\right)_x=0, \tag{17}$$

$$w_x-\frac{\upsilon_x w}{2\omega_0-\upsilon}=0, \tag{18}$$

where $\upsilon=S_x$. Thus, system of equations (16)-(18) is the quasiclassical (or dispersionless) limit of the system of the (1+1)-dimensional Schrödinger-Maxwell-Bloch equations.

Lax pair of the system of the (1+1)-dimensional dispersionless Schrödinger-Maxwell-Bloch equations. To construct Lax pair of the system of the dispersionless (1+1)-dimentional Schrödinger-Maxwell-Bloch equations (16)-(18), at first we note that the function Φ in Lax pair (4)-(5) can be written as $\Phi=(\psi_1,\psi_2)^T$. Let's consider the system of differential equations with respect to variable x:

$$\psi_{1x}=i\lambda\psi_1+q\psi_2, \tag{19}$$

$$\psi_{2x}=-\bar{q}\psi_1+i\lambda\psi_2, \tag{20}$$

and variable t, respectively.

$$\psi_{1t}=\left(-2i\lambda^2+i|q|^2+\frac{i}{\lambda+\omega_0}\eta\right)\psi_1+\left(2\lambda q+iq_x-\frac{i}{\lambda+\omega_0}p\right)\psi_2, \tag{21}$$

$$\psi_{2t}=\left(-2\lambda\bar{q}+i\bar{q}_x-\frac{i}{\lambda+\omega_0}\bar{p}\right)\psi_1+\left(2i\lambda^2-i|q|^2-\frac{i}{\lambda+\omega_0}\eta\right)\psi_2. \tag{22}$$

Then using transformation similar to previous section

$$\psi_1=e^{\frac{i}{\varepsilon}[F+\lambda x]},\ \psi_2=\xi e^{\frac{i}{\varepsilon}[F+\lambda x-S]}, \tag{23}$$

where $S=\partial_x^{-1}v$ and F, $v(x,t)$, ξ are real functions. We obtain the equivalent relations to differential systems (19)-(22)

$$\varepsilon\psi_{1x} = -i\lambda\psi_1 + q\psi_2, \tag{24}$$

$$\varepsilon\psi_{2x} = -\bar{q}\psi_1 + i\lambda\psi_2, \tag{25}$$

$$\varepsilon\psi_{1t} = \left(-2i\lambda^2 + i|q|^2 + \frac{i}{\lambda+\omega_0}\eta\right)\psi_1 + \left(2\lambda q + i\varepsilon q_x - \frac{i}{\lambda+\omega_0}p\right)\psi_2, \tag{26}$$

$$\varepsilon\psi_{2t} = \left(-2\lambda\bar{q} + i\varepsilon\bar{q}_x - \frac{i}{\lambda+\omega_0}\bar{p}\right)\psi_1 + \left(2i\lambda^2 + i|q|^2 - \frac{i}{\lambda+\omega_0}\eta\right)\psi_2. \tag{27}$$

We differentiate (23) with respect to variables x and t:

$$\psi_{1x} = \frac{i}{\varepsilon}\left[F_x + \lambda\right]e^{\frac{i}{\varepsilon}[F+\lambda x]},\ \psi_{2x} = \left\{\xi_x + \frac{i\xi}{\varepsilon}\left[F_x + \lambda - S_x\right]\right\}e^{\frac{i}{\varepsilon}[F+\lambda x - S]}, \tag{28}$$

$$\psi_{1t} = \frac{i}{\varepsilon}F_t e^{\frac{i}{\varepsilon}[F+\lambda x]},\ \psi_{2t} = \left\{\xi_t + \frac{i\xi}{\varepsilon}\left[F_t - S_t\right]\right\}e^{\frac{i}{\varepsilon}[F+\lambda x - S]}. \tag{29}$$

Now by equating the expressions (19)-(22) and (28)-(29), and collecting the coefficients of different powers of ε we get the next equations

$$p + 2\lambda - \frac{u}{p-v} = 0, \tag{30}$$

$$p_t - \left(\frac{u(p+2\lambda-2v)}{p-v}\right)_x - \frac{1}{\lambda+\omega_0}\left(\sqrt{1-uw} - \frac{u\sqrt{w}}{p-v}\right)_x = 0, \tag{31}$$

where $p = F_x$. Finally, last equations (30) and (31) are the Lax pair of the system of the (1+1)-dimentional quasiclassical (dispersionless) Schrödinger-Maxwell-Bloch equations.

Conclusions. In this paper, we considered the system of (1+1)-dimentional Schrödinger-Maxwell-Bloch equations, which are integrable by IST method. The quasiclassical limit of the system of (1+1)-dimentional Schrödinger-Maxwell-Bloch equations were obtained using the scale transformation. Also we presented its Lax representation, which proves its integrability. The resulting model can be used to describe quantum-optical phenomena in the absence of dispersive properties of the medium.

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ШРЕДИНГЕР-МАКСВЕЛЛ-БЛОХ ТЕҢДЕУІНІҢ КВАЗИКЛАССИКАЛЫҚ ШЕГІ

Аннотация. Интегралданатын теңдеулерді зерттеу қазіргі математикалық және теориялық физиканың маңызды аспектілерінің бірі болып табылады. Қазіргі уақытта физикалық қолдануға ие сызықтық емес интегралданатын теңдеулер саны өте көп. Сызықтық емес интегралданатын теңдеулер ұғымы солитонмен

тығыз байланысты. Сызықтық ортада пайда болатын, қозғалу кезінде, сондай-ақ өзі тәріздес түрлермен әрекеттесу кезінде пішінін сақтайтын нысан солитон немесе оңашаланған толқын деп аталады. Көптеген физикалық процестерде сызықтық емес дисперсия ұғымымен тығыз байланысты. Солитондық ерітінділер дисперсиясыз қасиетке ие. Теңдеудің сызықтық емес компоненті дисперсия мүшесінің орнын толтыратындығына байланыс.

Интегралданатын сызықтық емес дифференциалдық теңдеулерден басқа, гидродинамикалық типтегі немесе дисперсиясыз теңдеулер деп аталатын интегралданатын дербес дифференциалдық теңдеулердің маңызды класы да бар. Сызықты емес дисперсиясыз теңдеулер белгілі интегралданатын теңдеулердің диспер-сиясыз (квазиклассикалық) шегі ретінде туындайды. Соңғы жылдары дисперсиясыз жүйелерді зерттеу үлкен маңызға ие болды, өйткені олар физика, математика және қолданбалы математика, кванттық өрістер мен шектер теориясынан бастап комплексті жазықтықтағы конформды кескіндер сияқты әр түрлі мәселелерді талдау нәтижесінде пайда болды. Дисперсиясыз теңдеулерді зерттеу үшін белгілі жүйелер теориясының классикалық әдістері қолданылады.

Бұл жұмыста біз (1+1) өлшемді Шредингер-Максвелл-Блох теңдеулер жүйесінің дисперсиясыз шегін ұсынамыз. Шредингер-Максвелл-Блох теңдеуі - сызықтық емес интегралданатын теңдеулер теориясының классикалық мысалдарының бірі. Шредингер-Максвелл-Блох теңдеуі оптикалық солитондардың резонансты және эрбиймен қосылатын талшықтарда таралуын сипаттайды. Алынған Лакс бейнесін пайдаланып, біз Шредингер-Максвелл-Блох теңдеулер жүйесінің квазиклассикалық (дисперсиясыз) шегі интегралдылығын көрсетеміз.

Түйін сөздер. Дисперсиясыз интегралданатын жүйе, квазиклассикалық шегі, Шредингер-Максвелл-Блох теңдеулері, Лакс жұбы.

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КВАЗИКЛАССИЧЕСКИЙ ПРЕДЕЛ УРАВНЕНИЙ ШРЕДИНГЕРА-МАКСВЕЛЛА-БЛОХА

Аннотация. Исследование интегрируемых уравнений является одним из важнейших аспектов современной математической и теоретической физики. В настоящее время существует большое количество нелинейных интегрируемые уравнений, которые имееют физическое приложение. Понятие нелинейных интегрируемых уравнений тесно связано с солитономи. Объект, возникающий в нелинейной среде, сохраняющий форму при движении, а также при взоимодействии с себе подобными, называется солитоном или уединенной волной. Во многих физических процессах нелинейность тесно связана с понятием дисперсий. Солитонные решения обладают бездисперсионным свойством, в связи с тем, что нелинейный компонент уравнения компенсирует дисперсионный член.

Помимо интегрируемых нелинейных дифференциальных уравнений существует также важный класс интегрируемых уравнений в частных производных, так называемые интегрируемые уравнения гидродинамического типа или бездисперсионные уравнения. Нелинейные бездисперсионные уравнения возникают как бездисперсионный (квазиклассический) предел известных интегрируемых уравнений. В последние годы большое значение приобретает изучение бездисперсионных систем, поскольку они возникают в результате анализа различных проблем, таких как физика, математика и прикладная математика, от теории квантовых полей и струн до теории конформных отображений на комплексной плоскости. Для изучения бездисперсионных уравнений используются хорошо известные классические методы теории интегрируемых систем.

В данной работе представлен бездисперсионный предел системы (1+1)-мерных уравнений Шредингера-Максвелла-Блоха. Уравнение Шредингера-Максвелла-Блоха является одним из классических примеров теории нелинейных интегрируемых уравнений. Уравнение Шредингера-Максвелла-Блоха описывает распространение оптических солитонов в волокнах с резонансными и легированными эрбием. Также покажем интегрируемость бездисперсионного предела УШМБ с помощью полученного представление Лакса.

Ключевые слова: бездисперсионная интегрируемая система, квазиклассический предел, уравнения Шредингера-Максвелла-Блоха, пара Лакса.

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