NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X
Volume 2, Number 336 (2021), 15 – 23 **https://doi.org/10.32014/2021.2518-1726.16**

UDK 517.951
MRNTI 27.31.15

B. D. Koshanov[1], A. Baiarystanov[2], M. Daurenkyzy[3], S. O. Turymbet[4]

[1]Institute of Mathematics and Mathematical modeling, Almaty, Kazakhstan;
[2]Gumilev Eurasian National University, Nursultan, Kazakhstan;
[1,3,4]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan.
E-mail: koshanov@list.ru, oskar_62@mail.ru, daurenova.meruert@mail.ru, salta_04kz@mail.ru

GREEN'S FUNCTIONS OF SOME BOUNDARY VALUE PROBLEMS FOR BYHARMONIC OPERATORS AND THEIR CORRECT CONSTRICTIONS

Abstract. In this paper, a constructive method is given for constructing the Green function of the Dirichlet problem for a biharmonic equation in a multidimensional ball.

The need to study boundary value problems for elliptic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, thermal conductivity, elasticity theory, and quantum physics. The distributions of the potential of the electrostatic field are described using the Poisson equation. When studying the vibrations of thin plates of small deflections, biharmonic equations arise.

There are various ways to construct the Green Function of the Dirichlet problem for the Poisson equation. For many types of domains, it is constructed explicitly. And for the Neumann problem in multidimensional domains, the construction of the Green function is an open problem. For the ball, the Green function of the internal and external Neumann problem is constructed explicitly only for the two-dimensional and three-dimensional cases.

Finding general correct boundary value problems for differential equations is always an urgent problem. The abstract theory of operator contraction and expansion originates from the work of John von Neumann, in which a method for constructing self-adjoint extensions of a symmetric operator was described and a theory of extension of symmetric operators with finite defect indices was developed in detail. Many problems for partial differential equations lead to operators with infinite defect indices.

In the early 80s of the last century, M.O. Otelbaev and his students built an abstract theory that allows us to describe all correct constrictions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator.

In this paper, the correct boundary value problems for the biharmonic operator are described using the Green's function.

Key words: biharmonic equations, Dirichlet problem, biharmonic operator, domain of operator definition, correct problems, correct operator constrictions.

1. Introduction. The distributions of the potential of the electrostatic field are described using the Poisson equation. When studying the vibrations of thin plates of small deflections, biharmonic equations arise [1,2].

There are various ways to construct the Green Function of the Dirichlet problem for the Poisson equation. For many types of domains, it is constructed explicitly. And for the Neumann problem in multidimensional domains, the construction of the Green function is an open problem. For the ball, the Green function of the internal and external Neumann problem is constructed explicitly only for the two-dimensional and three-dimensional cases. In the general case, for a multidimensional ball, the explicit form of the Green function of the Neumann and Robin problems for the Poisson equation is constructed recently in [3,4].

Note that recently there has been renewed interest in the explicit construction of Green's functions for classical problems. In [5-7], the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed explicitly. In [8], the Green harmonic functions of the Dirichlet, Neumann, and Robin problems are used to construct the Green functions of the biharmonic Dirichlet, Neumann, and Robin problems in a two-dimensional circle. Similar results in the class of inhomogeneous biharmonic and triharmonic functions in the sector were obtained in [9-12]. Note also that the construction of explicit Green functions of the Robin problem in a circle, when the parameter in the boundary condition is equal to one, is devoted to the work [13,14]. The results of these studies are based on the classical theory of integral representations for analytic, harmonic, and polyharmonic functions on the plane.

The solvability of various boundary value problems for a biharmonic equation in a multidimensional sphere is studied in [15-18].

The abstract theory of operator contraction and expansion originates from the work of John von Neumann [19], in which a method for constructing self-adjoint extensions of a symmetric operator was described and a theory of extension of symmetric operators with finite defect indices was developed in detail. Many problems for partial differential equations lead to operators with infinite defect indices.

M.I. Vishik [20, 21] considered extensions of the minimal operator, rejecting its symmetry, and described the areas of definition of the extension that have certain solvability properties. M.I. Vishik applied his results to the study of general boundary value problems for general elliptic differential equations of the second order. Then A.V. Bitsadze and A.A. Samarsky [22] found a correct problem that is not contained among the problems described by M.I. Vishik. This type of problem for ordinary differential equations was studied by A.A. Desin [23].

In the early 80s of the last century, M.O. Otelbaev and his students [24-26] constructed an abstract theory that allows us to describe all correct constrictions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. This theory was extended to the case of Banach spaces [27].

This paper is devoted to the construction of the Green Function of the Dirichlet problem for a biharmonic equation in a multidimensional ball and to the description of correct boundary value problems for the biharmonic operator.

Green's function of the Dirichlet, Neumann, and Robin problem for the Poisson equation in a multidimensional unit ball

In the n-dimensional ball $\Omega=\{x=(x_1,x_2,...,x_n)\in R^n : |x|<r\}$, we consider the Dirichlet problem for the biharmonic equation

$$\Delta^2 u(x)=f(x), x\in\Omega, \tag{1}$$

$$u(x)=\varphi_0(x),\ \frac{\partial u(x)}{\partial n_x}=\varphi_1(x),\ \ x\in\partial\Omega. \tag{2}$$

The classical solution $u(x)\in C^4(\Omega)\cap C^1(\overline{\Omega})$ of the Dirichlet problem (1), (2) exists, is unique, and is represented by the Green's function $G_{4,n}(x,y)$ in the following form [1]

$$u(x)=\int_\Omega G_{4,n}(x,y)f(y)dy+\int_{\partial\Omega}\left[\frac{\partial}{\partial n_y}G_{4,n}(x,y)\cdot\Delta_y\varphi_0(y)-G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y}\Delta_y\varphi_0(y)\right]dS_y+$$
$$\int_{\partial\Omega}\left[\frac{\partial}{\partial n_y}\Delta_y G_{4,n}(x,y)\cdot\varphi_1(y)-\Delta_y G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y}\varphi_1(y)\right]dS_y, \tag{3}$$

where $\frac{\partial}{\partial n_y}$ is the outer normal to the boundary of $\partial\Omega$.

The Green function of the Dirichlet problem (1), (2) is determined from the following theorem.

Theorem 1. a) If n is odd or n is fair and $n>4$, then the Green function of the Dirichlet problem (1), (2) is representable as

$$G_{4,n}(x,y)\Big/d_{4,n} = |x-y|^{4-n} - \left[\left|\frac{y}{r}\right|\left|x-\frac{y}{|y|^2}r^2\right|\right]^{4-n} + \frac{(4-n)}{2}r^2\left[\left|\frac{y}{r}\right|\left|x-\frac{y}{|y|^2}r^2\right|\right]^{2-n}\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right) \quad (4)$$

where are $d_{4,n} = \dfrac{\Gamma(n/2-2)\Gamma(n/2-2)}{8n\pi^{n/2}\Gamma(n/2)}$

b) In the case of $n=2$ and $n=4$, function $G_{4,n}(x,y)$ has the form

$$G_{4,2}(x,y)\Big/d_{4,2} = |x-y|^2\left(\ln|x-y|^2 - \ln\left[\left|\frac{y}{r}\right|\left|x-\frac{y}{|y|^2}r^2\right|\right]^2\right) + r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right),$$

$$d_{4,2} = -\frac{1}{32\pi},$$

$$G_{4,4}(x,y)\Big/d_{4,4} = \ln|x-y| - \ln\left[\left|\frac{y}{r}\right|\left|x-\frac{y}{|y|^2}r^2\right|\right] + 2\left[\left|\frac{y}{r}\right|\left|x-\frac{y}{|y|^2}r^2\right|\right]^{-2} r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right),$$

$$d_{4,4} = \frac{1}{32\pi^2}.$$

In the future, for convenience, we will only consider the cases n -odd or n -fair and $n>4$.

Lemma 1. [2] a) In the case of n -odd or n -fair and $n>4$ the function

$$\varepsilon_{4,n}(x,y) = d_{4,n}|x-y|^{4-n}$$

is the fundamental solution of equation (1);

b) **[6,7]** Functions

$$g^0_{4,n}(x,y) = d_{4,n}\left|\frac{y}{r}\right|^{4-n}\cdot\left|x-\frac{y}{|y|^2}r^2\right|^{4-n}, \quad g^1_{4,n}(x,y) = -\frac{d_{4,n}}{2}\left|\frac{y}{r}\right|^{2-n}\cdot\left|x-\frac{y}{|y|^2}r^2\right|^{2-n} \quad (5)$$

are biharmonic functions, i.e. satisfy the homogeneous biharmonic equation

$$\Delta_x^2 g^k_{4,n}(x,y) = 0, x\in\Omega,\ k=0,1.$$

It is easy to show that the functions

$$|x-y|^2 = X^2(x,y) = X^2,\ \left|\frac{y}{r}\right|^2\cdot\left|x-\frac{y}{|y|^2}r^2\right|^2 = Y^2(x,y) = Y^2,$$

$$\left(1-\left|\frac{y}{r}\right|^2\right)^k\left(1-\left|\frac{x}{r}\right|^2\right)r^2 = Z^2(x,y) = Z^2 \quad (6)$$

satisfies the identity

$$X^2 - Y^2 = -Z^2, x,y\in\Omega. \quad (7)$$

Proof of Theorem 1. a) Using equality (7) and decomposition of function $f(x)=(1-x)^{\alpha}, 0<x\le 1$, we decompose the fundamental solution $\varepsilon_{4,n}(x,y)$ into a series

$$\varepsilon_{4,n}(x,y)=X^{4-n}=Y^{4-n}\cdot\left(1-\frac{Z^2}{Y^2}\right)^{\frac{4-n}{2}}=Y^{4-n}-(2-\frac{n}{2})Y^{2-n}Z^2+$$

$$+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(1-\frac{n}{2})(-\frac{n}{2})...(2-\frac{n}{2}-k)Y^{2-n-2k}Z^{2k}.$$

Moving the two terms to the left side of the equality, we get the desired Green's function in the following form

$$G_{4,n}(x,y)=G^2_{4,n}(x,y)=G^{\infty}_{4,n}(x,y),$$

where are

$$G^2_{4,n}(x,y)=d_{4,n}\left[X^{4-n}-Y^{4-n}+\frac{(4-n)}{2}Y^{2-n}Z^2\right],$$

$$G^{\infty}_{4,n}(x,y)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}(1-\frac{n}{2})(-\frac{n}{2})...(2-\frac{n}{2}-k)Y^{2-n-2k}Z^{2k}.$$

Because

$$\left(X^2-Y^2\right)\Big|_{x\in\partial\Omega,y\in\Omega}=-Z^2\Big|_{x\in\partial\Omega,y\in\Omega}=-r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)\Bigg|_{x\in\partial\Omega,y\in\Omega}=0,$$

$$Z^2\Big|_{x\in\partial\Omega,y\in\Omega}=0,\ \frac{\partial Z^2}{\partial n_x}\Bigg|_{x\in\partial\Omega,y\in\Omega}=0$$

it is easy to show that function $G^{\infty}_{4,n}(x,y)$ satisfies the boundary condition

$$G^{\infty}_{4,n}(x,y)\Big|_{x\in\partial\Omega,y\in\Omega}=0,\ \frac{\partial G^{\infty}_{4,n}(x,y)}{\partial n_x}\Bigg|_{x\in\partial\Omega,y\in\Omega}=0.$$

According to Lemma 1 and Representation $G^2_{4,n}(x,y)$ we have

$$\Delta^2 G_{4,2}(x,y)=\Delta^2 G^2_{4,2}(x,y)=\delta(x-y),\ x,y\in\Omega.$$

Due to the uniqueness of the solution of the Dirichlet problem for the biharmonic equation, the Green function of the problem (1), (2) is (4):

$$G_{4,n}(x,y)=d_{4,n}\left[X^{4-n}-Y^{4-n}+\frac{(4-n)}{2}Y^{2-n}Z^2\right].$$

3. Correct constrictions and extensions of differential operators

In the early 80s of the last century, M. O. Otelbaev and his students [24-26] constructed an abstract theory that allows us to describe all correct constrictions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. Moreover, this theory was extended to the case of Banach spaces and it was possible to partially abandon the linearity of operators. Moreover, M. O. Otelbaev shows that the Bitsadze-Samarsky problem [22] is a correct narrowing of the corresponding maximal operator. We give a brief summary of this theory in the case of Hilbert spaces.

Let the Hilbert space H be a linear operator L with a domain of definition $D(L)$ and a domain of value $R(L)$. The kernel of operator L is the set $KerL=\{f\in D(L): Lf=0\}$.

Definition 1. A linear closed operator $\hat{L}$ in a Hilbert space H is called *maximal* if 3 $R(\hat{L}) = H$ and $Ker\hat{L} \neq \{0\}$.

Definition 2. A linear closed operator L_0 in a Hilbert space H is called *minimal* if $\overline{R(L_0)} \neq H$ and there is a bounded inverse operator L_0^{-1} by $R(L_0)$.

Definition 3. A linear closed operator L in a Hilbert space H is called *correct* if there is a bounded inverse operator L^{-1} defined on all H.

Definition 4. Operator L is called a *contraction* of operator L_1, and operator L_1 is called an extension of operator L, and briefly write $L \subset L_1$ if

1)$D(L) \in D(L_1)$,

2)$Lf = L_1 f, \forall f \in D(L)$.

Definition 5. The correct operator L in the Hilbert space H is called the correct contraction of the *maximum operator* $\hat{L}$ (the correct extension of the minimum operator L_0) if $L \subset \hat{L}$ $(L_0 \subset L)$.

Definition 6. A correct operator L in a Hilbert space H is called a *boundary-correct extension* if L is both a correct contraction of the maximum operator $\hat{L}$ and a correct extension of the minimum operator L_0, i.e., $L_0 \subset L \subset \hat{L}$.

Theorem 2 [24,25]. Let $\hat{L}$ be a maximal linear operator in a Hilbert space H, L − a known correct narrowing of operator $\hat{L}$ and K-an arbitrary linear operator bounded in H that satisfies the following condition

$$R(K) \subset Ker\hat{L}. \tag{8}$$

Then the operator L_K^{-1} defined by the formula

$$L_K^{-1} f = L^{-1} f + Kf, \forall f \in H, \tag{9}$$

is the inverse of some correct narrowing of L_K of the maximal operator $\hat{L}$, i.e. $L_K \subset \hat{L}$.

Conversely, if L_1 is some correct narrowing of the maximal operator $\hat{L}$, then there exists a linear operator K_1 bounded in H that satisfies condition (8), such that the equality holds

$$L_1^{-1} f = L^{-1} f + K_1 f, \forall f \in H.$$

As a rule, it is difficult to describe the kernel of the maximal operator. Therefore, often the following Theorem 3 is more effective than Theorem 2.

Theorem 3 [26]. Let $\hat{L}$ be the maximal operator, L_ϕ be the known correct constriction of $\hat{L}$, and K be the continuous operator acting from H to $D(\hat{L})$ be the domain of the definition of operator $\hat{L}$. Then operator L_K^{-1}, defined by the formula

$$L_K^{-1} f = L_\phi^{-1} f + (\mathrm{E} - L_\phi^{-1} \hat{L}) Kf \tag{10}$$

is the inverse of some correct narrowing $\hat{L}$, i. e. $L_K \subset \hat{L}$.

Conversely, any correct narrowing of operator $\hat{L}$ is represented as (10).

This theory will then be applied to the biharmonic operator.

4. Correct boundary value problems for a biharmonic operator in a multidimensional ball

In this paragraph $\Omega = \{x \in R^n : |x| < r\}$. On the set $D(\hat{L}) = W_2^4(\Omega)$, we define the maximum operator $\hat{L}$ by the formula $\hat{L}u \equiv \Delta_x^2 u(x), \forall\, u \in D(\hat{L})$.

By definition, $R(\hat{L}) = L_2(\Omega)$, and $Ker(\hat{L}) \neq \{0\}$ is not trivial.

In the previous section, we proved that the Dirichlet boundary value problem for the biharmonic equation

$$L_\phi u = \{u : \Delta_x^2 u(x) = f(x), x \in \Omega, u(x)\big|_{\partial\Omega} = 0, \frac{\partial u(x)}{\partial n_x}\bigg|_{\partial\Omega} = 0\}$$

has a unique solution of $u(x)$ and it is represented as

$$L_\phi^{-1} f = u(x) = \int_\Omega G_{4,n}^D(x,y) f(y) dy, \qquad (11)$$

where $G_{4,n}^D(x,y) \equiv G_{4,n}(x,y)$ is the Green function of the Dirichlet problem from (4). Thus, operator L_ϕ is invertible.

Further, based on the representation of the solution (11) of the Dirichlet problem, we give other correct boundary value problems for the inhomogeneous biharmonic equation. To do this, we apply Theorem 3 to describe the correct constrictions of the maximal operator $\hat{L}$.

Lemma 2. Using the explicit form of the Green function of the Dirichlet problem (4), for the biharmonic equation (1) and for any $h(x) \in W_2^4(\Omega)$, the representation is valid

$$(E - L_\phi^{-1}\hat{L})h(x) = \int_{\partial\Omega}\left[\frac{\partial}{\partial n_y} G_{4,n}(x,y)\cdot\Delta_y h(y) - G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y}\Delta_y h(y)\right] dS_y +$$

$$+\int_{\partial\Omega}\left[\frac{\partial}{\partial n_y}\Delta_y G_{4,n}(x,y)\cdot h(y) - \Delta_y G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y} h(y)\right]. \qquad (12)$$

Lemma 3. The Green function of the Dirichlet problem on the boundary of the domain has the following properties:

$$G_{4,n}\big|_{x\in\partial\Omega} = 0,\ \frac{\partial G_{4,n}}{\partial n_x}\bigg|_{x\in\partial\Omega} = 0,\ \Delta_y G_{4,n}\big|_{x\in\partial\Omega} = 0,\ \frac{\partial \Delta_y G_{4,n}}{\partial n_y}\bigg|_{x\in\partial\Omega} = \delta(x-y)\big|_{x\in\partial\Omega},$$

$$\frac{\partial^2 G_{4,n}}{\partial n_x \partial n_y}\bigg|_{x\in\partial\Omega} = 0,\ \frac{\partial^2 \Delta_y G_{4,n}}{\partial n_x \partial n_y}\bigg|_{x\in\partial\Omega} = 0, \frac{\partial \Delta_y G_{4,n}}{\partial n_x}\bigg|_{x\in\partial\Omega} = -\delta(x-y)\big|_{x\in\partial\Omega}.$$

The following statement is true, which allows us to describe the domain of the definition of the maximum operator $\hat{L}$ in terms of the Green's function $G_{4,n}(x,y)$.

Lemma 4. The domain of definition $D(\hat{L})$ of the maximal operator $\hat{L}$ has the representation

$$D(\hat{L}) = \{u : u(x) = \int_\Omega G_{4,n}(x,y)\cdot f(y) dy + \int_{\partial\Omega}\left[\frac{\partial}{\partial n_y} G_{4,n}(x,y)\cdot\Delta_y h(y) - G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y}\Delta_y h(y)\right] dS_y +$$

$$+\int_{\partial\Omega}\left[\frac{\partial}{\partial n_y}\Delta_y G_{4,n}(x,y)\cdot h(y) - \Delta_y G_{4,n}(x,y)\cdot\frac{\partial}{\partial n_y} h(y)\right] dS_y, \forall f \in L_2(\Omega), \forall h \in W_2^4(\Omega)\}.$$

In particular, if

$$h\big|_{y\in\partial\Omega} = 0,\ \frac{\partial h}{\partial n_y}\bigg|_{x\in\partial\Omega} = 0,\ \Delta_y h\big|_{x\in\partial\Omega} = 0,\ \frac{\partial \Delta_y h}{\partial n_y}\bigg|_{y\in\partial\Omega} = 0$$

then $D(\hat{L})$ coincides with the scope of definition $D(L_\phi)$ of operator L_ϕ.

Now the question arises: how to describe the definition areas of other possible correct constrictions of the maximum operator $\hat{L}$?

Let K be an operator that matches each function $f(x) \in L_2(\Omega)$ with a single function $h(x) \in W_2^4(\Omega)$, such that $\|Kf\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)}$, for the chosen operator K, we take the set $D(K) = \{u(x) \in D(\hat{L}) : h = Kf\}$. On the set $D(K)$, we define the operator $\hat{L}\big|_{D(K)} = L_K$.

From Theorem 3, it follows that L_K is a correct narrowing of the maximal operator $\hat{L}$. From Theorem 3, we obtain the following statement describing the operator L_K in terms of boundary conditions.

Theorem 4. Let K be an arbitrary continuous operator acting from $L_2(\Omega)$ to $D(\hat{L})$. Then the inhomogeneous operator equation $L_K u = f$ is equivalent to the following boundary value.

$$\Delta_x^2 u(x) = f(x), x \in \Omega, \tag{13}$$

$$u\big|_{x\in\partial\Omega} = (Kf)\big|_{x\in\partial\Omega}, \left.\frac{\partial u}{\partial n_x}\right|_{x\in\partial\Omega} = \left.\frac{\partial (Kf)}{\partial n_x}\right|_{x\in\partial\Omega}. \tag{14}$$

Note 1. If the invertible operator is on all $L_2(\Omega)$, then the boundary conditions in (14) can be written as

$$Ru\big|_{x\in\partial\Omega} = R(Kf)\big|_{x\in\partial\Omega}, R\left.\left(\frac{\partial u}{\partial n_x}\right)\right|_{x\in\partial\Omega} = R\left.\left(\frac{\partial Kf}{\partial n_x}\right)\right|_{x\in\partial\Omega}. \tag{15}$$

Therefore, to check the correctness of the boundary value problem, you need to try to convert the boundary conditions to the form (15).

Note 2. If the linear operator L-is the correct contraction of the maximum, then passing to the conjugates, we get the correct extensions of the minimum operator corresponding to the formally conjugate. This also leads to a class of "loaded" equations.

Note 3. Note that in Theorem 3, $K -$ nonlinear transformations can be taken as K.

Other applications of the results of M. Otelbaev in various sections of the theory of differential equations can be found in [27,28].

Б. Д. Қошанов[1], А. О. Байарыстанов[2], М. Дәуренқызы[3], С. О. Тұрымбет[4]

[1]Математика және математикалық моделдеу институты, Алматы, Қазақстан;
[2] Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұрсұлтан, Қазақстан;
[1,3,4]Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы, Қазақстан.

БИГАРМОНИКАЛЫҚ ОПЕРАТОРЛАР ҮШІН КЕЙБІР ШЕТТІК ЕСЕПТЕРДІҢ ГРИН ФУНКЦИЯЛАРЫ ЖӘНЕ ОЛАРДЫҢ ДҰРЫС ТАРЫЛУЛАРЫ

Аннотация. Бұл жұмыста көп өлшемді шарда бигармоникалық теңдеу үшін Дирихле есебінің Грин функциясын құрудың тиімді әдісі көрсетілген.

Эллиптикалық теңдеулер үшін шеттік есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы, кванттық физика процестерін теориялық зерттеуде көп-теген практикалық қосымшалармен тығыз байланысты. Электростатикалық өріс потенциалдарының таралуы Пуассон теңдеуімен, ал кіші иілімдердің жұқа тақталарының тербелістерінің таралуы бигармоникалық теңдеулермен сипатталады.

Пуассон теңдеуі үшін Дирихле есебінің Грин функциясын құрудың әртүрлі тәсілдері бар. Аудандардың көптеген түрлері үшін ол айқын түрде құрылған. Нейман есебі үшін көп өлшемді облыстарда Грин функциясын құру мәселесі қазіргі таңда ашық міндет болып табылады. Шар үшін Нейманның ішкі және сыртқы есебінің Грин функциясы тек екі өлшемді және үш өлшемді жағдайлар үшін айқын түрде құрылған.

Дифференциалдық теңдеулер үшін жалпы дұрыс шекаралық есептерді табу әрқашан өзекті мәселе болып табылады. Операторлардың тарылуы мен кеңеюінің абстрактілі теориясы Джон фон Нейманның жұмысынан бастау алады, онда ол симметриялық оператордың өз-өзіне түйіндес кеңейтулерін құру әдісі сипатталған және ақаудың ақырлы индекстері бар симметриялық операторларды кеңейту теориясы егжей-тегжейлі жасалған. Дербес туындылы дифференциалдық теңдеулерге арналған көптеген есептері ақаулары шексіз индексті операторларға алып келеді.

Өткен ғасырдың 80-ші жылдарының басында М. Өтелбаев және оның шәкірттері абстрактілі теория құрды. Бұл теорияның көмегімен белгілі бір максималды оператордың барлық дұрыс тарылуын сипаттауға болады, сондай-ақ белгілі бір минималды оператордың барлық дұрыс кеңеюін сипаттауға болады.

Бұл мақала операторлардың тарылуы мен кеңеюі теориясы қысқаша сипатталған және бигармоникалық операторлар үшін тиянақты шекаралық есептерді сипаттауға арналған.

Түйін сөздер: бигармоникалық теңдеулер, Дирихле есебі, бигармоникалық оператор, оператордың анықтау аймағы, дұрыс есептер, оператордың дұрыс тарылуы.

Б. Д. Кошанов[1], А. О. Байарыстанов[2], М. Дауренкызы[3], С. О. Турымбет[4]

[1]Институт математики и математического моделирования, Алматы, Казахстан;
[2]Евразийский национальный университет им. Л.Н. Гумилева, Нурсултан, Казахстан;
[1,3,4]Казахский национальный педагогический университет им. Абая, Алматы, Казахстан

ФУНКЦИИ ГРИНА НЕКОТОРЫХ КРАЕВЫХ ЗАДАЧ ДЛЯ БИГАРМОНИЧЕСКИХ ОПЕРАТОРОВ И ИХ КОРРЕКТНЫЕ СУЖЕНИЯ

Аннотация. В данной работе дан конструктивный способ построения функции Грина задачи Дирихле для бигармонического уравнения в многомерном шаре.

Необходимость исследования краевых задач для эллиптических уравнений продиктована с многочисленными практическими приложениями при теоретическом изучении процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики. Распределения потенциала электростатического поля описываются с помощью уравнения Пуассона. При исследовании колебаний тонких пластин малых прогибов возникают бигармонические уравнения.

Существуют различные способы построения функции Грина задачи Дирихле для уравнения Пуассона. Для многих видов областей она построена в явном виде. А для задачи Неймана в многомерных областях построение функции Грина является открытой задачей. Для шара функция Грина внутренней и внешней задачи Неймана построена в явном виде только для двумерном и трехмерном случаях.

Нахождение общих корректных краевых задач для дифференциальных уравнений всегда является актуальной задачей. Абстрактная теория сужения и расширения операторов берет свое начало с работы Джон фон Нейман, в которой был описан метод построения самосопряженных расширений симметрического оператора и подробно разработана теория расширения симметрических операторов с конечными индексами дефекта. Многие задачи для дифференциальных уравнений в частных производных приводят к операторам с бесконечными индексами дефекта.

В начале 80-х годов прошлого столетия М.О. Отелбаевым и его учениками была построена абстрактная теория, которая позволяет описать все корректные сужения некоторого максимального оператора и отдельно - все корректные расширения некоторого минимального оператора, независимо друг от друга, в терминах обратного оператора.

В настоящей работе с помощью построенного функции Грина описаны корректные краевые задачи для бигармонического оператора.

Ключевые слова: бигармонические уравнения, задача Дирихле, бигармонический оператор, область определения оператора, корректные задачи, корректные сужения оператора.

Information about authors:

Koshanov Bakhytbek Danebekovich, Institute of Mathematics and Mathematical Modeling, Chief scientific researcher, Abai Kazakh National Pedagogical University, Doctor of Physical and Mathematical Sciences, professor, koshanov@list.ru, https://orcid.org/0000-0002-0784-5183;

Baiarystanov Askar, L.N. Gumilev Eurasian National University, Professor of the Department of higher mathematics, oskar_62@mail.ru, https://orcid.org/0000-0002-5840-5401;

Dayrenkyzy Meruert, Abai Kazakh National Pedagogical University, student, daurenova.meruert@mail.ru, https://orcid.org/0000-0002-7241-9279;

Turymbet Saltanat Ospankyzy, Abai Kazakh National Pedagogical University, student, salta_04kz@mail.ru, https://orcid.org/0000-0003-3432-8840

REFERENCES

[1] Sobolev S. L. Introduction to the theory of cubature formulas. Moscow.: Nauka. 1974. 808 p.

[2] Vladimirov V. S. Equations of mathematical physics. Moscow: Nauka, 1981. 512 p.

[3] Sadybekov M.A., Torebek B.T., Turmetov B.Kh. Representation of Green's function of the Neumann problem for a multi-dimensional ball // Complex Variables and Elliptic Equation, 61:1 (2016) 104-123.

[4] Sadybekov M.A., Turmetov B.Kh., Torebek B.T. On an explicit form of the Green function of the Roben problem for the Laplace operator in a circle // Adv. Pure Appl. Math. 6:3 (2015) 163-172.

[5] Kalmenov T.Sh., Koshanov B.D., Nemchenko M.Y. Green function representation for the Dirichlet problem of the polyharmonic equation in a sphere // Complex Variables and Elliptic Equations, 53:2 (2008) 177-183. Doi: http://dx.doi.org/10.1080/17476930701671726

[6] Kalmenov T.Sh., Koshanov B.D. Representation for the Green's function of the Dirichlet problem for the polyharmonic equations in a ball // Siberian Mathematical Journal. 49:3 (2008) 423-428. http://dx.doi.org/10.1007/s11202-008-0042-8

[7] Kalmenov T. Sh., Suragan D. On a new method for constructing the Green function of the Dirichlet problem for a polyharmonic equation // Differential equations. 48:3 (2012) 435-438. DOI: http://dx.doi.org/10.1134/S0012266112030160

[8] Begehr H. Biharmonic Green functions // Le matematiche. 2006. Vol. LXI. P. 395-405.

[9] Wang Y., Ye L. Biharmonic Green function and biharmonic Neumann function in a sector // Complex Variables and Elliptic Equations. 58:1 (2013) 7-22.

[10] Wang Y. Tri-harmonic boundary value problems in a sector // Complex Variables and Elliptic Equations. 59:5 (2014) 732-749.

[11] Begehr H., Du J., Wang Y. A Dirichlet problem for polyharmonic functions // Ann. Math. Pura Appl. 187:4 (2008) 435-457.

[12] Begehr H., Vaitekhovich T. Harmonic boundary value problems in half disc and half ring // Funct. Approx. Comment. Math. 40:2 (2009) 251-282.

[13] Begehr H., Vaitekhovich V., Some harmonic Robin functions in the complex plane //Advances in Pure and Applied Mathematics. 1:1 (2010) 19-34.

[14] Begehr H., Vaitekhovich T. Modified harmonic Roben function // Complex Variables and Elliptic Equations. 58:4 (2013) 483-496.

[15] Koshanov B. D., Koshanova M. D. Dirichlet problem with Helder and Lp boundary data for polyharmonic functions in a unit ball / / Bulletin of the NAS RK, Series of Physics and Mathematics, 4 (2013) 35-41.

[16] Koshanov B. D., Edyl K. Green function of the Dirichle problem and polynomial solution of the Poisson equation for bigarmonic tenge on a wheel // Bulletin of the NAS RK, Series of Physics and Mathematics, 3 (2016) 102-121.

[17] Koshanov B.D., Nurikenova Zh.S. On the solvability of the generalized Dirichlet - Neumann problem for a higher order elliptic equation // Bulletin of the NAS RK, Series of Physics and Mathematics, 3 (2017) 125-131.

[18] Koshanov B.D., Koshanova G.D., Azimkhan G.E., Segizbayeva R.U. Solvability of boundary value problems with non-local conditions for multidimensional hyperbolic equations // Bulletin of the NAS RK, Series of Physics and Mathematics, 2:312 (2020) 116-125.

[19] J.von Neumann. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren // Math. Ann. 102 (1929) 49-131.

[20] Vishik M. I. On general boundary value problems for elliptic differential equations // Works of Matem. 3, (1952) 187-246.

[21] Vishik M. I. Boundary value problems for elliptic equations degenerating on the boundary of a domain // Math. Collection. 77:3 (1954) 1307-1311.

[22] Bitsadze A.V., Samarsky A. A. On some simplest generalizations of linear elliptic boundary value problems // Reports of the USSR Academy of Sciences 185:4 (1969) 739–740.

[23] Dezin A.A. Partial differential equations. Berlin etc.: Springer-Verlag, 1987.

[24] Kokebaev B. K., Otelbaev M., Shynybekov A. N. The theory of contraction and expansion operators. I // The news of the Kazakh SSR. Ser. Fiz.-Mat. 5 (1982) 24-27.

[25] Kokebaev B. K., Otelbaev M., Shynybekov A. N. The theory of contraction and expansion operators. II // The news of the Kazakh SSR. Ser. Fiz.-Mat. 1. (1983)24-27.

[26] Otelbaev M., Kokebaev B. K., Shynybekov A. N. On the issues of expanding and narrowing operators // Reports of the USSR Academy of Sciences. 6. (1983) 1307-1311.

[27] Oynarov R., Parasidi I. N. Correctly solvable extensions of operators with finite defects in a Banach space // The news of the Kazakh SSR. Ser. fiz-mat. 5 (1988) 35-44.

[28] Koshanov B.D., Otelbaev M.O. Correct Contractions stationary Navier-Stokes equations and boundary conditions for the setting pressure // AIP Conference Proceedings. 1759 (2016) http://dx.doi.org/10.1063/1.4959619