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AN ALGORITHM FOR SOLVING A BOUNDARY VALUE PROBLEM FOR ESSENTIALLY LOADED DIFFERENTIAL EQUATIONS

Abstract. A linear boundary value problem for essentially loaded differential equations is considered. Using the properties of essentially loaded differential we reduce the considering problem to a two-point boundary value problem for loaded differential equations. This problem is investigated by parameterization method. We offer algorithm for solving to boundary value problem for the system of loaded differential equations. This algorithm includes of the numerical solving of the Cauchy problems for system of the ordinary differential equations and solving of the linear system of algebraic equations. For numerical solving of the Cauchy problem we apply the Runge–Kutta method of 4th order. The proposed numerical implementation is illustrated by example.

Key words: essentially loaded differential equation, numerically approximate method, algorithm.
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Introduction. The mathematical description of various dynamic control processes in which the future flow of processes depends not only on the present, but also is substantially determined by the history of the process, is carried out using ordinary differential equations with different types of memory, also called equations with aftereffect or loaded differential equations [1]. Loaded differential equations are used to solve problems of long-term prediction and control of the groundwater level and soil moisture [2-4]. Various problems for loaded differential equations and methods of finding their solutions considered in [1, 6-16].

In the present paper, a linear boundary value problem for essentially loaded differential equations is investigated. The significance is that the loading member of the equation appear in the form of derivative of solution at loaded point of the interval, i.e. the order of the loaded term is equal to the order of the differential part of the equation. Presence of derivative of solution in loaded point has a strong influence on the properties of equations.

Statement of problem.

We consider a linear boundary value problem for essentially loaded differential equations

$$\frac{dx}{dt} = A(t)x + \sum_{j=0}^{m} M_j(t)x(\theta_j) + K(t)\dot{x}(\theta_0) + f(t), t \in (0,T), \quad (1)$$

$$Bx(0) + Cx(T) = d, d \in R^n, x \in R^n, \quad (2)$$

where the $(n \times n)$-matrices $A(t)$, $K(t)$, $M_j(t)$ $(j = 0,1,\dots,m)$, and n-vector-function $f(t)$ are continuous on $[0,T]$, B and C are constant $(n \times n)$- matrices, d is constant n-vector, and $0 = \theta_0 < \theta_1 < \dots < \theta_m < \theta_{m+1} = T$, $\|x\| = \max_i |x_i|$.

Let $C([0,T], R^n)$ denote the space of continuous on $[0,T]$ functions $x(t)$ with norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$.

A solution to problem (1), (2) is a continuously differentiable on $(0,T)$ function $x(t) \in C([0,T],R^n)$ satisfying the essentially loaded differential equations (1) and boundary condition (2).

By setting $t = \theta_0$ in equation (1), we get

$$[I - K(\theta_0)]\dot{x}(\theta_0) = A(\theta_0)x(\theta_0) + \sum_{j=0}^{m} M_j(\theta_0)x(\theta_j) + f(\theta_0). \quad (3)$$

Assume that the matrix $[I - K(\theta_0)]$ is invertible. We obtain

$$\dot{x}(\theta_0) = [I - K(\theta_0)]^{-1}\{A(\theta_0)x(\theta_0) + \sum_{j=0}^{m} M_j(\theta_0)x(\theta_j) + f(\theta_0)\}. \quad (4)$$

We consider the following linear boundary value problem for loaded differential equations

$$\frac{dx}{dt} = A(t)x + \sum_{i=0}^{m} D_i(t)x(\theta_i) + F(t), t \in (0,T), \quad (5)$$

$$Bx(0) + Cx(T) = d, d \in R^n, x \in R^n, \quad (6)$$

where
$$D_0(t) = M_0(t) + K(t)[I - K(\theta_0)]^{-1}\{A(\theta_0) + M_0(\theta_0)\}$$

$$D_j(t) = M_j(t) + K(t)[I - K(\theta_0)]^{-1}M_j(\theta_0), j = \overline{1,m},$$

$$F(t) = K(t)[I - K(\theta_0)]^{-1}f(\theta_0) + f(t).$$

Let us consider an example showing that loads influences significantly to the property of boundary value problem. Consider the following Cauchy problem for the loaded differential equation:

$$\frac{dx}{dt} = -\alpha x(0.4) + f(t), t \in [0,1], \quad (7)$$

$$x(0) = 1. \quad (8)$$

Solving the problem (7), (8) we get

$$x(t) - 1 = -\alpha x(0.4)t + \int_0^t f(\tau)d\tau.$$

The value of $x(0.4)$ satisfies the following equation:

$$\left(1 + \frac{2}{5}\alpha\right)x(0.4) = 1 + \int_0^{0.4} f(\tau)d\tau. \quad (9)$$

But if we take $\alpha = -\frac{5}{2}, f(t) = 1$ then the equation (9) does not hold and the Cauchy problem (7), (8) is not solved. At the same time, the Cauchy problem for a linear system of ordinary differential equations (without loading) always has a unique solution.

On $[0,1]$ we consider a periodic boundary value problem for an ordinary differential equation

$$\frac{dx}{dt} = t, x(0) = x(1), t \in [0,1].$$

The General solution of the differential equation has the form: $x(t) = \frac{t^2}{2} + C$. Substituting the General solution in the boundary conditions for determining C, we obtain the relation: $C = C + \frac{1}{2}$. Since there is no such number C, the problem has no solution.

Now, adding the load at the point $t = 0.5$ to the right side of the differential equation we obtain the following periodic boundary value problem for a loaded differential equation

$$\frac{dx}{dt} = t + x(0.5), x(0) = x(1), t \in [0,1],$$

and the solution of this problem has the form $x(t) = -\frac{3}{8} + \frac{t^2}{2} - \frac{t}{2}$.

Scheme of parametrization method.

We use the approach offered in [16-21] to solve the boundary value problem (5), (6). This approach based on the algorithms of the parameterization method and numerical methods for solving Cauchy problems.

Let us now investigate boundary value problem (5), (6) by the parametrization method. The interval $[0,T]$ is divided into subintervals by loading points:

$$[0,T) = \cup_{r=1}^{m+1}[\theta_{r-1},\theta_r).$$

Introduce $C([0,T],\theta_m,R^{n(m+1)})$ as a spase of systems of functions $x[t] = (x_1(t),x_2(t),\dots,x_{m+1}(t))$, where $x_r: [\theta_{r-1},\theta_r) \to R^n$ are continuous on $[\theta_{r-1},\theta_r)$ and have finite left-sided limits $\lim_{t\to t_r-0} x_r(t)$ for all $r = 1:(m+1)$, with norm $\|x[\cdot]\|_2 = \max_{r=\overline{1,m+1}} \sup_{t\in[\theta_{r-1},\theta_r)} \|x_r(t)\|$.

Let $x_r(t)$ be the restriction of the function $x(t)$ to the r −th interval $[\theta_{r-1},\theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1},\theta_r)$, $r = 1:(m+1)$. Then we reduce problem (5), (6) to the equivalent multipoint boundary value problem

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{i=0}^{m} D_i(t)x_{i+1}(\theta_i) + \mathrm{F(t)}, t \in [\theta_{r-1},\theta_r), r = 1:(m+1), \quad (10)$$

$$Bx_1(0) + C\lim_{t\to T-0} x_{m+1}(t) = d, \quad (11)$$

$$\lim_{t\to\theta_s-0} x_s(t) = x_{s+1}(\theta_s), s = 1:m, \quad (12)$$

where (12) are conditions for matching the solution at the interior points of partition.

The solution of problem (10) - (12) is a system of functions $x^*[t] = (x_1^*(t),x_2^*(t),\dots,x_{m+1}^*(t)) \in C([0,T],\theta_m,R^{n(m+1)})$, where the functions $x_r^*(t), r = \overline{1,m+1}$, are continuously differentiable on $[\theta_{r-1},\theta_r)$, which satisfies system (10) and conditions (11), (12).

Problems (5), (6) and (10)-(12) are equivalent. If a system of functions $\tilde{x}[t] = (\tilde{x}_1(t),\tilde{x}_2(t),\dots,\tilde{x}_{m+1}(t)) \in C([0,T],\theta_m,R^{n(m+1)})$ is a solution of problem (10)-(12), then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \tilde{x}_r(t)$, $t \in [\theta_{r-1},\theta_r), r = 1:(m+1)$, $\tilde{x}(T) = \lim_{t\to T-0} \tilde{x}_{m+1}(t)$ is a solution of the original problem (5), (6). Conversely, if $x(t)$ is a solution of problem (5), (6), then the system of functions $x[t] = (x_1(t),x_2(t),\dots,x_{m+1}(t))$, where $x_r(t) = x(t)$, $t \in [\theta_{r-1},\theta_r), r = 1:(m+1)$, and $\lim_{t\to T-0} x_{m+1}(t) = x(T)$, is a solution of problem (10)-(12).

Introducing the additional parameters $\lambda_r = x_r(\theta_{r-1}), r = 1:(m+1)$, and performing a replacement of the function $u_r(t) = x_r(t) - \lambda_r$ on each r-th interval $[\theta_{r-1},\theta_r), r = 1:(m+1)$, we obtain the boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)[u_r + \lambda_r] + \sum_{i=0}^{m} D_i(t)\lambda_{i+1} + \mathrm{F(t)}, \quad (13)$$
$$t \in [\theta_{r-1},\theta_r), r = 1:(m+1),$$

$$u_r(\theta_{r-1}) = 0, r = 1:(m+1), \quad (14)$$

$$B\lambda_1 + C\lambda_{m+1} + C\lim_{t\to T-0} u_{m+1}(t) = d, \quad (15)$$

$$\lambda_s + \lim_{t\to\theta_s-0} u_s(t) = \lambda_{s+1}, s = 1:m. \quad (16)$$

A pair $(u^*[t],\lambda^*)$ with elements $u^*[t] = (u_1^*(t),u_2^*(t),\dots,u_{m+1}^*(t)) \in C([0,T],\theta_m,R^{n(m+1)})$, $\lambda^* = (\lambda_1^*,\lambda_2^*,\dots,\lambda_{m+1}^*) \in R^{n(m+1)}$ is said to be a solution to problem (13)-(16) if the functions $u_r^*(t), r = 1:(m+1)$, are continuously differentiable on $[\theta_{r-1},\theta_r)$ and satisfy (13) and additional conditions (15), (16) with $\lambda_j = \lambda_j^*, j = 1:(m+1)$, and initial conditions (14).

Problems (5), (6) and (13)-(16) are equivalent. If the $x^*(t)$ is a solution of problem (5), (6), then the pair $(u^*[t],\lambda^*)$, where $u^*[t] = (x^*(t) - x^*(\theta_0), x^*(t) - x^*(\theta_1),\dots,x^*(t) - x^*(\theta_m))$, and $\lambda^* = (x^*(\theta_0),x^*(\theta_1),\dots,x^*(\theta_m))$, is a solution of problem (13)-(16). Conversely, if a pair $(\tilde{u}[t],\tilde{\lambda})$ with elements $\tilde{u}[t] = (\tilde{u}_1(t),\tilde{u}_2(t),\dots,\tilde{u}_{m+1}(t)) \in C([0,T],\theta_m,R^{n(m+1)})$, $\tilde{\lambda} = (\tilde{\lambda}_1,\tilde{\lambda}_2,\dots,\tilde{\lambda}_{m+1}) \in R^{n(m+1)}$, is a solution of (13)-(16) , then the function $\tilde{x}(t)$ defined by the equalities $\tilde{x}(t) = \tilde{u}(t) + \tilde{\lambda}_r, t \in [\theta_{r-1},\theta_r), \mathrm{r} = 1:(m+1)$, will be the solution of the original problem (5), (6).

Using the fundamental matrix $X_r(t)$ of differential equation $\frac{dx}{dt} = A(t)x$ on $t \in [\theta_{r-1},\theta_r), \mathrm{r} = 1:(m+1)$, we reduce the Cauchy problem for the system of ordinary differential equations with parameters (13), (14) to the equivalent system of integral equations

$$u_r(t) = X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r + X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\sum_{i=0}^{m} D_i(t)\lambda_{i+1}\,d\tau +$$
$$+X_r(t)\int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau\,, t \in [\theta_{r-1}, \theta_r), \mathrm{r} = 1:(m+1). \quad (17)$$

Substituting the corresponding right-hand sides of (17) into the conditions (15), (16), we obtain a system of linear algebraic equations with respect to the parameters $\lambda_r, r = 1:(m+1)$

$$B\lambda_1 + C\lambda_{m+1} + CX_{m+1}(T) \int_{\theta_m}^{T} X_{m+1}^{-1}(\tau)\left\{A(\tau)\lambda_{m+1} + \sum_{i=0}^{m} D_i(\tau)\lambda_{i+1}\right\}d\tau =$$
$$= d - CX_{m+1}(T)\int_{\theta_m}^{T} X_{m+1}^{-1}(\tau)F(\tau)d\tau, \quad (18)$$

$$\lambda_s + X_s(\theta_s) \int_{\theta_{s-1}}^{\theta_s} X_s^{-1}(\tau)A(\tau)d\tau\,\lambda_s + X_s(\theta_s) \int_{\theta_{s-1}}^{\theta_s} X_s^{-1}(\tau)\sum_{i=0}^{m} D_i(\tau)\lambda_{i+1}\,d\tau -$$
$$-\lambda_{s+1} = -X_s(\theta_s)\int_{\theta_{s-1}}^{\theta_s} X_s^{-1}(\tau)F(\tau)d\tau\,, s = 1:m. \quad (19)$$

We denote the matrix corresponding to the left side of the system of equations (18), (19) by $Q_*(\theta)$ and write the system in the form

$$Q_*(\theta)\lambda = F_*(\theta), \lambda \in R^{n(m+1)}\,, \quad (20)$$

where

$$F_*(\theta) = \Big(d - CX_{m+1}(T)\int_{\theta_m}^{T} X_{m+1}^{-1}(\tau)F(\tau)d\tau\,, -X_1(\theta_1)\int_{\theta_0}^{\theta_1} X_1^{-1}(\tau)F(\tau)d\tau\,, \dots,$$
$$-X_m(\theta_m)\int_{\theta_{m-1}}^{\theta_m} X_m^{-1}(\tau)F(\tau)d\tau\Big)'.$$

It is not difficult to establish that the solvability of the boundary value problem (5), (6) is equivalent to the solvability of the system (20). The solution of the system (20) is a vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{m+1}^*) \in R^{n(m+1)}$ consists of the values of the solutions of the original problem (5), (6) in the initial points of subintervals, i.e. $\lambda_r^* = x^*(\theta_{r-1}), \mathrm{r} = 1:(m+1)$.

Further we consider the Cauchy problems for ordinary differential equations on subintervals

$$\frac{dz}{dt} = A(t)z + P(t),$$
$$z(\theta_{r-1}) = 0, t \in [\theta_{r-1}, \theta_r], \mathrm{r} = 1:(m+1), \quad (21)$$

where $P(t)$ is either $(n \times n)$ matrix, or n vector, both continuous on $[\theta_{r-1}, \theta_r], \mathrm{r} = 1:(m+1)$. Consequently, solution to problem (21) is a square matrix or a vector of dimension n. Denote by $a(P,t)$ the solution to the Cauchy problem (21). Obviously,

$$a(P,t) = X_r(t) \int_{\theta_{r-1}}^{t} X_r^{-1}(\tau)\mathrm{P}(\tau)d\tau\,, t \in [\theta_{r-1}, \theta_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (21) on the r-th interval.

An algorithm for solving problem (1), (2).

We offer the following numerical implementation of algorithm based on the Runge–Kutta method of 4th order.

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \dots < \theta_m < \theta_{m+1} = T$. Divide each r-th interval $[\theta_{r-1}, \theta_r], \mathrm{r} = 1:(m+1)$, into N_r parts with step $h_r = (\theta_r - \theta_{r-1})/N_r$. Assume on each interval $[\theta_{r-1}, \theta_r]$ the variable $\hat{\theta}$ takes its discrete values: $\hat{\theta} = \theta_{r-1}$, $\hat{\theta} = \theta_{r-1} + h_r, \dots, \hat{\theta} = \theta_{r-1} + (N_r -$

$1)h_r, \hat{\theta} = \theta_r$, and denote by $\{\theta_{r-1}, \theta_r\}$ the set of such points.

2. Solving the Cauchy problems for ordinary differential equations

$$\frac{dz}{dt} = A(t)z + A(t), z(\theta_{r-1}) = 0, t \in [\theta_{r-1}, \theta_r],$$

$$\frac{dz}{dt} = A(t)z + D_i(t), z(\theta_{r-1}) = 0, t \in [\theta_{r-1}, \theta_r], \mathrm{i} = 0: m,$$

$$\frac{dz}{dt} = A(t)z + F(t), z(\theta_{r-1}) = 0, t \in [\theta_{r-1}, \theta_r], \mathrm{r} = 1: (m+1),$$

by using again the Runge–Kutta method of 4th order, we find the values of $(n \times n)$ matrices $a_r(\mathrm{A}, \hat{\theta}), a_r(D_i, \hat{\theta}), \mathrm{i} = 0: m$, and n vector $a_r(f, \hat{\theta})$ on $\{\theta_{r-1}, \theta_r\}, \mathrm{r} = 1: (m+1)$

3. Construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\tilde{h}}(\theta)\lambda = -F_*^{\tilde{h}}(\theta), \lambda \in R^{n(m+1)}, \tag{22}$$

Solving the system (22), we find $\lambda^{\tilde{h}}$. As noted above, the elements of $\lambda^{\tilde{h}}=(\lambda_1^{\tilde{h}}, \lambda_2^{\tilde{h}}, \ldots, \lambda_{m+1}^{\tilde{h}})$ are the values of approximate solution to problem (5), (6) in the starting points of subintervals: $x^{\tilde{h}_r}(\theta_{r-1}) = \lambda_r^{\tilde{h}}, \mathrm{r} = 1: (m+1)$.

4. To define the values of approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A(t)x + \sum_{i=0}^{m} D_i(t)\lambda_{r+1}^{\tilde{h}} + F(t),$$

$$x(\theta_{r-1}) = \lambda_r^{\tilde{h}}, t \in [\theta_{r-1}, \theta_r], \mathrm{r} = 1: (m+1).$$

And the solutions to Cauchy problems are found by the Runge–Kutta method of 4th order. Thus, the algorithm allows us to find the numerical solution to the problem (5), (6).

We can see that the solution of boundary value problem (5), (6) also is the solution of boundary value problem (1), (2), when the matrix $[I - K(\theta_0)]$ is invertible.

To illustrate the proposed approach for the numerical solving linear two-point boundary value problem for essentially loaded differential equations (1), (2) on the basis of parameterization method, let us consider the following example.

Example. We consider a linear boundary value problem for essentially loaded differential equations

$$\frac{dx}{dt} = A(t)x + \sum_{i=0}^{3} M_i(t)x(\theta_i) + K(t)\dot{x}(\theta_0) + f(t), t \in (0,1), \tag{23}$$

$$Bx(0) + Cx(T) = d, d \in R^2, x \in R^2. \tag{24}$$

Here $\theta_0 = 0, \theta_1 = \frac{1}{4}, \theta_2 = \frac{1}{2}, \theta_3 = \frac{3}{4}, \theta_4 = T = 1, A(t) = \begin{pmatrix} t+2 & t^3 \\ t^2 & t-1 \end{pmatrix}$,

$$M_0(t) = \begin{pmatrix} 6 & t \\ t^2 & 4t \end{pmatrix}, M_1(t) = \begin{pmatrix} 3t & 5 \\ t & t^2 \end{pmatrix}, M_2(t) = \begin{pmatrix} t^2 & t+3 \\ 0 & 11 \end{pmatrix},$$

$$M_3(t) = \begin{pmatrix} 9 & t-3 \\ t^2+5 & t^3 \end{pmatrix}, K(t) = \begin{pmatrix} t^2+3 & 5t \\ 2 & t-3 \end{pmatrix}, B = \begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 9 & 8 \\ -9 & 3 \end{pmatrix}, d = \begin{pmatrix} 18 \\ -43 \end{pmatrix}, f(t) = \begin{pmatrix} -t^5 + t^4 - \frac{37t^2}{8} + \frac{601t}{64} - \frac{1179}{64} \\ -t^5 - \frac{17t^3}{16} + \frac{177t^2}{64} + \frac{719t}{64} + \frac{57}{64} \end{pmatrix}.$$

We find $\dot{x}(0)$ from (23):

$$[I - K(0)]\dot{x}(0) = A(0)x(0) + \sum_{i=0}^{3} M_i(0)x(\theta_i) + f(0).$$

The matrix $[I-K(0)]$ is invertible. Then

$$\dot{x}(0)=\begin{pmatrix}-0.5 & 0\\ -0.25 & 0.25\end{pmatrix}\left\{A(0)x(0)+\sum_{i=0}^{m}M_i(0)x(\theta_i)+f(0)\right\}.$$

We consider a linear boundary value problem for loaded differential equations

$$\frac{dx}{dt}=A(t)x+\sum_{i=0}^{3}D_i(t)x(\theta_i)+F(t), t\in(0,1),$$

$$Bx(0)+Cx(T)=d, d\in R^2, x\in R^2,$$

where

$$D_0(t)=M_0(t)+K(t)[I-K(0)]^{-1}\{A(0)+M_0(0)\}=\begin{pmatrix}-4t^2-10t-6 & -\frac{t}{4}\\ t^2-2t-2 & \frac{15t}{4}+\frac{3}{4}\end{pmatrix}$$

$$D_1(t)=M_1(t)+K(t)[I-K(0)]^{-1}M_1(0)=\begin{pmatrix}3t & -\frac{5}{4}(2t^2+5t+2)\\ t & t^2-\frac{5t}{4}-\frac{5}{4}\end{pmatrix}$$

$$D_2(t)=M_2(t)+K(t)[I-K(0)]^{-1}M_2(0)=\begin{pmatrix}t^2 & 11t-\frac{2t^2}{3}-\frac{3}{2}\\ 0 & 2t+2\end{pmatrix}$$

$$D_3(t)=M_3(t)+K(t)[I-K(0)]^{-1}M_3(0)=\begin{pmatrix}-\frac{9t^2}{2}-5t-\frac{9}{2} & \frac{3t^2}{2}+\frac{19t}{4}+\frac{3}{2}\\ t^2-t-1 & t^3+\frac{3t}{4}+\frac{3}{4}\end{pmatrix}$$

$$F(t)=K(t)[I-K(0)]^{-1}f(0)+f(t)=\begin{pmatrix}-t^5+t^4+\frac{587t^2}{128}+\frac{1073t}{32}+\frac{1179}{128}\\ -t^5-\frac{17t^3}{16}+\frac{177t^2}{64}+\frac{257t}{16}+\frac{309}{64}\end{pmatrix}.$$

We use the numerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals and evaluating definite integrals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals [0, 0.25], [0.25, 0.5] , [0.5, 0.75] , [0.75, 1] with step h = 0.025.

Solving the system of equations (22), we obtain the numerical values of the parameters

$\lambda_1^{\tilde{h}}=\begin{pmatrix}-0.000000041\\ -2.00000002\end{pmatrix}, \lambda_2^{\tilde{h}}=\begin{pmatrix}0.765624981\\ -2.437500012\end{pmatrix},$

$\lambda_3^{\tilde{h}}=\begin{pmatrix}1.625000009\\ -2.750000008\end{pmatrix}, \lambda_4^{\tilde{h}}=\begin{pmatrix}2.671875034\\ -2.937500004\end{pmatrix}.$

We find the numerical solutions at the other points of the subintervals using Runge-Kutta method of the 4-th order to the following Cauchy problems

$$\frac{d\tilde{x}_r}{dt}=A(t)\tilde{x}_r+\sum_{i=0}^{3}D_i(t)\lambda_{r+1}^{\tilde{h}}+F(t),$$

$$x(\theta_{r-1})=\lambda_r^{\tilde{h}}, t\in[\theta_{r-1},\theta_r], \mathrm{r}=1:4.$$

Exact solution of the problem (23), (24) is $x^*(t)=\begin{pmatrix}t^3+3t\\ t^2-3t-2\end{pmatrix}.$

The results of calculations of numerical solutions at the partition points are presented in the following table:

t	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	t	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$
0	-0.000000041	-2.00000002	0.5	1.625000009	-2.750000008
0.025	0.075015586	-2.049375019	0.525	1.719703137	-2.774375007
0.05	0.150124963	-2.097500018	0.55	1.816375014	-2.797500007
0.075	0.22542184	-2.144375017	0.575	1.915109392	-2.819375007
0.1	0.300999967	-2.190000016	0.6	2.01600002	-2.840000006
0.125	0.376953094	-2.234375015	0.625	2.119140648	-2.859375006
0.15	0.453374971	-2.277500014	0.65	2.224625025	-2.877500006
0.175	0.530359349	-2.319375014	0.675	2.332546903	-2.894375005
0.2	0.607999976	-2.360000013	0.7	2.44300003	-2.910000005
0.225	0.686390603	-2.399375012	0.725	2.556078157	-2.924375005
0.25	0.765624981	-2.437500012	0.75	2.671875034	-2.937500004
0.275	0.845796858	-2.474375011	0.775	2.79048441	-2.949375004
0.3	0.926999986	-2.510000011	0.8	2.912000037	-2.960000004
0.325	1.009328114	-2.54437501	0.825	3.036515662	-2.969375004
0.35	1.092874991	-2.57750001	0.85	3.164125037	-2.977500004
0.375	1.177734369	-2.60937501	0.875	3.294921912	-2.984375005
0.4	1.263999997	-2.640000009	0.9	3.429000036	-2.990000005
0.425	1.351765625	-2.669375009	0.925	3.566453159	-2.994375006
0.45	1.441125003	-2.697500008	0.95	3.707375031	-2.997500007
0.475	1.532171881	-2.724375008	0.975	3.851859402	-2.999375009
0.5	1.625000009	-2.750000008	1	4.000000021	-3.000000011

For the difference of the corresponding values of the exact and constructed solutions of the problem the following estimate is true:

$$\max_{j=\overline{0,40}} \left\| x^*(t_j) - \tilde{x}(t_j) \right\| < 0.00000004.$$

Conclusion. In this work, we propose a numerical implementation of parametrization method for finding solutions to linear two-point boundary value problem for system of essentially loaded differential equations. Using the parametrization method, we reduce the considered problem to the equivalent boundary value problem with parameters. The example illustrating the numerical algorithms of parametrization method are provided.

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ЕЛЕУЛІ ТҮРДЕ ЖҮКТЕЛГЕН ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН ШЕТТІК ЕСЕПТІ ШЕШУ АЛГОРИТМІ

Аннотация. Елеулі түрде жүктелген дифференциалдық теңдеулер үшін сызықтық шеттік есеп қарастырылады. Біз қарастырылып отырған есепті елеулі түрде жүктелген дифференциалдық теңдеу қасиеттерін пайдалана отырып жүктелген дифференциалдық теңдеу үшін екі нүктелі шеткі есептерге келтіреміз. Аталған есеп параметрлеу әдісі арқылы зерттеледі. Жүктелген дифференциалдық теңдеулер жүйесі үшін шеттік есептің шешімін табудың алгоритмі ұсынылады. Бұл алгоритм жәй дифференциалдық теңдеулер жүйесі үшін Коши есептерін сандық шешуді және алгебралық теңдеулер жүйесін шешуді қамтиды. Коши есептерін сандық түрде шешу үшін төртінші ретті Рунге-Куттаның әдісі қолданылады. Ұсынылып отырған сандық жүзеге асырылу мысалмен көрнектеледі.

Түйін сөздер: елеулі түрде жүктелген дифференциалдық теңдеу, сандық жуықталған әдіс, алгоритм.

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АЛГОРИТМ РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ДЛЯ СУЩЕСТВЕННО НАГРУЖЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Аннотация. Рассматривается линейная краевая задача для существенно нагруженных дифференциальных уравнений. Используя свойства существенно нагруженного дифференциального уравнения, мы сводим рассматриваемую задачу к двухточечной краевой задаче для нагруженных дифференциальных уравнений. Данная задача исследуется методом параметризации. Предлагается алгоритм нахождения решения краевой задачи для системы нагруженных дифференциальных уравнений. Данный алгоритм включает численное решение задач Коши для системы обыкновенных дифференциальных уравнений и решение линейной системы алгебраических уравнений. Для численного решения задачи Коши применяется метод Рунге-Кутта четвертого порядка. Предлагаемая численная реализация иллюстрируется примером.

Ключевые слова: существенно нагруженное дифференциальное уравнение, численно приближенный метод, алгоритм.

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