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Қазақстан Республикасы Ұлттық ғылым академиясы «ҚР ҰҒА Хабарлары. Физикалық-математикалық сериясы» ғылыми журналының Web of Science-тің жаңаланған нусқасы Emerging Sources Citation Index-те индекстелуге қабылданғанын хабарлайды. Бұл индекстелу барысында Clarivate Analytics компаниясы журналды одан әрі the Science Citation Index Expanded, the Social Sciences Citation Index және the Arts & Humanities Citation Index-ке қабылдау мәселесін қарастыруда. Webof Science зерттеушілер, авторлар, баспашылар мен мекемелерге контент тереңдігі мен сапасын усынады. ҚР ҰҒА Хабарлары. Химия және технология сериясы Emerging Sources Citation Index-ке енуі біздің қоғамдастық үшін ең өзекті және беделді химиялық ғылымдар бойынша контентке адалдығымызды білдіреді.

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FICTITIOUS DOMAIN METHOD FOR THE NAVIER-STOKES EQUATIONS

Abstract. The paper is devoted to the study and mathematical justification of the fictitious domain method for the Navier-Stokes equations. The method of fictitious domain with continuation by the lowest coefficients for the Navier-Stokes equations is substantiated in the work. The theorem of unambiguous solvability of the initial boundary value problem and the existence and uniqueness of the generalized solution of the auxiliary problem for the Navier-Stokes equations in the case of two spatial variables is proved. A numerical implementation of the proposed method of fictitious domain with continuation by the lowest coefficients for the Navier-Stokes equations is carried out. Various graphical illustrations of the results of numerical simulation using the fictitious domain method for the Navier-Stokes equations are presented.

Keywords: Fictitious domain method, Navier-Stokes equation, solvability of the initial boundary value problem, existence and uniqueness of the generalized solution, numerical calculations.

Introduction. Now the development of high-performance computing has made it possible to solve a complex physical problem of continuous media which are of great practical importance. Numerical simulation of flow of fluid in curved boundaries was carried out in most of the work based on the model of a viscous incompressible fluid. Algorithms using the finite-difference scheme for the Navier-Stokes equation are widely used. For numerical modeling of multidimensional problems of the mathematical physics in complex domains are needed an effective, universal computational algorithms. One of universal, effective computational algorithms is a fictitious domain method.

There are now several methods for the numerical solution of a boundary-value problems in complex geometric domains such as the method of curvilinear grid and the fictitious domain method. Using the curvilinear grid method requires converting the equation to curved coordinates which is more complex than the original equations. Also, the diverse requirements imposed on difference finite-difference grids make constructing curvilinear grids a complex mathematical problem.

The method of fictitious domain is easy to use and easily implemented on a computer. However, its disadvantage is in the loss of accuracy due to the presence of a small parameter in the auxiliary equations which will lead to poor conditionality of the system of difference equations. Originally the fictitious domain method was proposed in the works of Saulyev V.K. [1-2], Rikvind V.Ya. [3]. Justification of the fictitious domain method for a nonlinear model of hydrodynamics is considered in [4-7]. The paper [8] is devoted to the construction, research, and application of the augmented domain method for the approximate solution of mathematical physics problems in irregular domains. A distinctive feature and advantage of the augmented domain method is the absence of a small parameter in the auxiliary problem that affects the numerical implementation of the equations. The main idea of the method is that the original irregular domain Ω is replaced by another, simpler region Q such that $\Omega \subset Q$ in this new domain, the original differential problem is replaced by a certain variational problem with a bounded operator. The equivalence of these problems is proved and a method for minimizing the resulting functional is proposed.

In [9-10], we consider a variant of the method of fictitious domain associated with the modification of nonlinear terms in a fictitious subdomain. The model problem shows the effectiveness of using such a modification. The proposed version of the method is used to solve the problem of an arbitrary domain and to set a boundary condition for the pressure. In [11], the method of fictitious domains with continuation by lowest coefficients for the model of the boundary layer of the atmosphere is mathematically justified. This model describes the distribution of pollutants in the atmosphere. To account for the terrain that affects the distribution of pollutants in the atmosphere, the method of fictitious domains is used. In [12], an approximate method based on the fictitious domain method is proposed for solving the two – dimensional Navier-Stokes equations in irregular domains. A computational finite – difference algorithm for solving the auxiliary problem of the fictitious domain method is developed. The results of numerical simulation of two-dimensional Navier-Stokes equations by the method of fictitious domain with continuation by the lowest coefficient are presented. For this task, a parallel algorithm was developed using the CUDA architecture, which was tested on different grid dimensions. In [13-14], an original computational method was developed for the numerical solution of an elliptic equation with strongly varying coefficients. Equations of this type are obtained at the second stage when solving the Navier-Stokes equation and the boundary layer of the atmosphere in regions of complex form by the method of fictitious domain. The proposed method is based on a special variable substitution that leads a problem with discontinuous coefficients of the second kind to a problem with discontinuous coefficients of the first kind. An iterative process with two parameters and considering the ratio of the coefficients of the equation in the subdomains is constructed. A theorem is proved for estimating the convergence rate of the developed iterative process. A computational algorithm is developed, and numerical calculations are performed to illustrate the effectiveness of the proposed method. In [15-16], we consider a family of domain separation methods based on the explicit use of the Lagrange multiplier defined on the actual boundary. The technique proposed by the author associated with genuine boundary conditions is common for modeling inviscid incompressible flows. In this method, the set differential problem sets the optimal control problem with a saddle point, and for the numerical implementation, the iterative conjugate gradient method is used.

Statement of the problem. In this part, let us consider the justification of the fictitious domain method for the Navier-Stokes equations with continuation by the lowest coefficients. In the domain $Q_T^0 = \Omega_0 \times (0,T), S_T^0 = S^0 \times (0,T)$, we consider the initial boundary value problem for the Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \vartheta_i \vec{v}_{x_i} + \nabla p = \mu \Delta \vec{v} + f(x,t) \\ div \vec{v} = 0 \end{cases} \tag{1}$$

$$\vec{v}\big|_{t=0} = a(x), \;\; \vec{v}\big|_{S_T^0} = 0 \tag{2}$$

Ω_0 is defined as follows:

$$\Omega_0 = \{(x_1, x_2, x_3), 0 \le x_1 \le X, 0 \le x_2 \le Y, \delta(x_1, x_2) \le x_3 \le Z\} \tag{3}$$

In the work of Ladyzhenskaya O. A. [17], a review is made to justify the initial boundary value problem of the form (1), (2). The unique solvability of the boundary value problem of the form (1)-(2) with two spatial variables is proved. The problem (1)-(2) will be solved by the method of splitting by physical parameters, using the fictitious domain method with continuation by lowest coefficients to account the internal subdomain.

Let us add the domain Ω_0 to the rectangular parallelepiped $\Omega = \Omega_0 \cup \Omega_1$ and consider the following auxiliary problem in $Q_T = \Omega \times (0,T)$:

$$\frac{\partial v^{\varepsilon}}{\partial t} + \vartheta_i^{\varepsilon} \vec{v}_{x_i}^{\varepsilon} + \nabla p^{\varepsilon} = \mu \Delta \vec{v}^{\varepsilon} - \frac{\xi(x)}{\varepsilon} \cdot \vec{v}^{\varepsilon} + f^{\varepsilon}(x,t), \qquad (4)$$

$$div \vec{v}^{\varepsilon} = 0,$$

where

$$\xi(\vec{x}) = \begin{cases} 1, \vec{x} \in \Omega_1, \\ 0, \vec{x} \in \Omega_0, \end{cases} \quad f^{\varepsilon}(x,t) = \begin{cases} f(\vec{x},t), \ (\vec{x},t) \in Q_T^{\circ} \\ 0, (x,t) \in Q_T / Q_T^{\circ} \end{cases} \qquad (5)$$

Here, the complement of the main domain of Ω_0 to the rectangular parallelepiped Ω is denoted by Ω_1 The boundary of the domain Ω_0 at $x_3 = \delta(x_1, x_2)$ has become the interface between the sub-domains Ω_0, Ω_1 and the following matching conditions must be met on it:

$$\left[\vec{v}^{\varepsilon}\right]_{X_3=\delta(x_1,x_2)} = 0, \ \left[\left(\mu \nabla \vec{v}^{\varepsilon} - \delta \cdot p^{\varepsilon}\right) \cdot \vec{n}\right]_{x_3=\delta(x_1,x_2)} = 0. \qquad (6)$$

where $\vec{n}$ is the normal vector to boundary $x_3 = \delta(x_1, x_2)$.

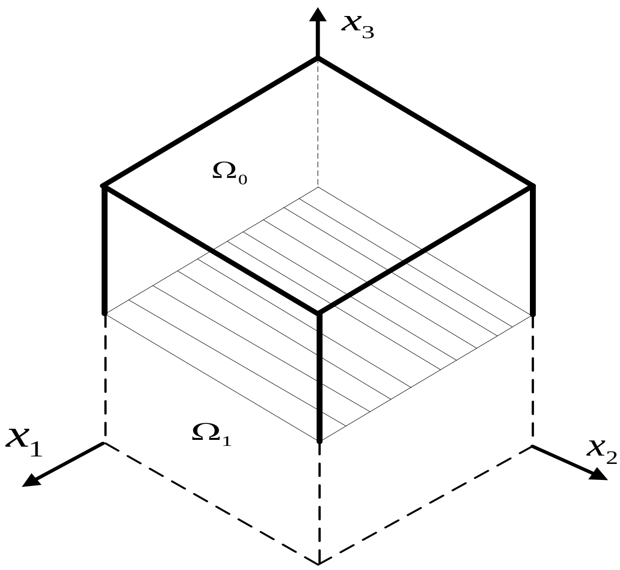

Figure 1. Problem determination domain

The following initial boundary conditions are set on the boundary of the domain Ω:

$$\vec{v}^{\varepsilon}\Big|_{t=0} = a^{\varepsilon}(x), \quad \vec{v}^{\varepsilon}\Big|_{S_T} = 0 \qquad (7)$$

where $a^{\varepsilon}(x) = \begin{cases} a(x), x \in \Omega_0 \\ 0, x \in \Omega_1 \end{cases}$

Study of the problem. For further investigation of the problem (4)-(7), we use the spaces $L_m(\Omega), L_m(Q_T), L_{q,r}(Q_T)$, and we denote the scalar product and the corresponding norms:

$$(u,\vartheta)=\int_{\Omega}u_{x_i}\cdot\vartheta_{x_i}\,dx=\int_{\Omega}u_i\vartheta_i dx,\quad \|u\|_{m,\Omega}=\left(\int_{\Omega}|u|^m dx\right)^{1/m},$$

$$\|u\|_{m,Q_T}=\left[\int_0^T\left(\int_{\Omega}|u|^m dx\right)dt\right]^{1/m}\quad \|u\|_{q,r,Q_T}=\left[\int_0^T\left(\int_{\Omega}|u|^q dx\right)^{r/q}dt\right]^{1/r}$$

where $Q_T=\Omega\times(0,T)$.

By $M(Q_T)$ let us denote the following set of the function:

$$M(Q_T)=\left\{\vec v\in C^{\infty}(Q_T), div\vec v=0, \vec v\big|_{S_T}=0\right\}$$

where $C^{\infty}(Q_T)$ is the set of infinite differentiable functions in the domain Q_T. Closures of a set $M(Q_T)$ in norm $L_2(Q_T), W_2^1(Q_T), W_2^2(Q_T)$ accordingly we denote as $V_0(Q_T), V_1(Q_T), V_2(Q_T)$.

Multiplying equation (4) scalar by $\vec v^{\varepsilon}$ to Q_T we get the energy balance equation:

$$\int_0^t\left(\int_{\Omega}v^{\varepsilon}\cdot\frac{dv^{\varepsilon}}{dt}dx\right)dt+\int_0^t\left(\int_{\Omega}\vec v^{\varepsilon}\cdot\nabla p^{\varepsilon}dx\right)dt-\mu\int_0^t\left(\int_{\Omega}\Delta\vec v^{\varepsilon}\cdot\vec v^{\varepsilon}dx\right)dt+$$
$$\int_0^t\left(\int_{\Omega}\frac{\xi(x)}{\varepsilon}\cdot(\vec v^{\varepsilon})^2dx\right)dt-\int_0^t\left(\int_{\Omega}f\cdot\vec v^{\varepsilon}dx\right)dt=0, \tag{8}$$

With the help of boundary conditions and integration in parts, we have:

$$\frac{1}{2}\int_0^t\frac{d}{dt}\int_{\Omega}(v^{\varepsilon})^2dxdt-\int_0^t\left(\int_{\Omega}p^{\varepsilon}\cdot div\vec v^{\varepsilon}dx\right)dt+$$
$$+\mu\int_0^t\|v_x^{\varepsilon}\|^2dt+\int_0^t\left(\int_{\Omega}\left(\sqrt{\frac{\xi(x)}{\varepsilon}}\vec v^{\varepsilon}\right)dx\right)dt-\int_0^t\left(\int_{\Omega}f\cdot\vec v^{\varepsilon}dx\right)dt=0 \tag{9}$$

Then taking into account $div\vec v^{\varepsilon}=0$, we get

$$\frac{1}{2}\left(\|\vec v^{\varepsilon}\|^2_{t=T}-\|\vec v^{\varepsilon}\|^2_{t=0}\right)+\mu\int_0^T\|\vec v_x^{\varepsilon}\|^2dt+$$
$$+\int_0^T\left(\int_{\Omega}\left(\sqrt{\frac{\xi(x)}{\varepsilon}}\vec v^{\varepsilon}\right)^2dx\right)dt-\int_0^T\left(\int_{\Omega}f^{\varepsilon}\cdot\vec v^{\varepsilon}dx\right)dt=0 \tag{10}$$

Using the Hölder's inequality and the initial condition we have

$$\|\vec v^{\varepsilon}(x,T)\|^2+2\mu\|\vec v_x^{\varepsilon}\|^2_{2,Q_T}+\frac{2\cdot\xi(x)}{\sqrt{\varepsilon}}\|\vec v^{\varepsilon}\|^2_{2,Q_T}\le\|a^{\varepsilon}(x)\|^2+2\cdot\|f^{\varepsilon}\|_{2,Q_T}\cdot\|\vec v^{\varepsilon}\|_{2,Q_T} \tag{11}$$

To obtain a further estimate, an additional solution estimation inequality is needed. Let us multiply (4) in scalar by $\vec v^{\varepsilon}$ to $L_2(\Omega)$ so we get

$$\frac{1}{2}\frac{d}{dt}\|v^{\varepsilon}\|^2+\mu\|v_x^{\varepsilon}\|^2=(f^{\varepsilon},\ v^{\varepsilon}) \tag{12}$$

From equation (12) by virtue of positivity $\mu\|v_x^\varepsilon\|^2$, we have

$$\|v^\varepsilon\|\frac{d\|v^\varepsilon\|}{dt}\le\|f^\varepsilon\|\cdot\|v^\varepsilon\| \tag{13}$$

this inequality holds if $\|V^\varepsilon(x,t)\|=0$ or

$$\frac{d}{dt}\|v^\varepsilon(x,t)\|\le\|f^\varepsilon(x,t)\| \tag{14}$$

Since the continuous function $\|v(x,t)\|$ of the argument t, integrating (14) by we have:

$$\|v^\varepsilon(x,t)\|\le\|v^\varepsilon(x,0)\|+\int_0^t\|f(x,t)\|dt \tag{15}$$

or

$$\|v^\varepsilon(x,t)\|\le\|a^\varepsilon(x)\|+\int_0^t\|f^\varepsilon(x,t)\|dt \tag{16}$$

From (13) and (16), we get

$$\|\vec v^\varepsilon(x,T)\|^2+2\mu\|\vec v_x^\varepsilon\|_{2,Q_T}^2+\frac{2\cdot\xi(x)}{\varepsilon}\cdot\|\vec v^\varepsilon\|_{2,Q_T}^2\le\|a^\varepsilon(x)\|^2+2\left(\|a^\varepsilon(x)\|+\int_0^T\|f^\varepsilon\|dt\right)\cdot\|f^\varepsilon\|_{2,Q_T} \tag{17}$$

Definition. The generalized solution of the problem (4)-(7) is a function $v^\varepsilon(x,t)\in V_2(Q_T)\cap L_{q,r}(Q_T)$ that satisfies the integral identity

$$\int_{Q_T}\left[-v^\varepsilon\cdot\phi_t+\mu v_x^\varepsilon\cdot\phi_x-\vartheta_i^\varepsilon\cdot v^\varepsilon\cdot\phi_{x_i}\right]dxdt+\frac{1}{\varepsilon}\int_{Q_t}\xi(x)\cdot v^\varepsilon\phi dxdt+\int_\Omega v^\varepsilon\phi\Big|_{t=t}dx-\int_\Omega a\phi\Big|_{t=0}dx=\int_{Q_t}f\phi dxdt \tag{18}$$

for $\forall\phi\in\overset{\circ}{W}{}_2^{1,1}(Q_T)$

Theorem 1. Let $v^\varepsilon(x,t)\in V_2(Q_T)\cap L_{q,r}(Q_T)$, $f(x,t)\in V_2(Q_T)\cap L_{q,r}(Q_T)$. Then there is a unique generalized solution to problem (4)-(7) and an estimate for the solution exists (17).

Proof. Omitting the parameter ε assuming that there are two solutions v' and v'' for their difference $u=v'-v''$ from the identity (18) we get

$$\int_{Q_T}\left[-u\cdot\phi_t+\mu u_x\cdot\phi_x-(\vartheta_i'u+u_iv'')\phi_{x_i}\right]dxdt+\frac{1}{\varepsilon}\int_{Q_t}\xi(x)u\phi dxdt+\int_\Omega u\phi\Big|_{t=t}dx=0 \tag{19}$$

Reasoning similarly as in [17] $\vec u$ can be considered as the generalized solution from the $L_2(Q_T)$ problem

$$u_t-\mu\Delta u=-gradq-\frac{\xi(x)}{\varepsilon}u-\frac{\partial\vec f^{\,i}}{\partial x_i} \tag{20}$$

$$divu=0,\ u\big|_{S_T}=0,\ u\big|_{t=0}=0,$$

where $\vec f^{\,k}\equiv\vartheta_i'\vec u+u_i\cdot\vec v''$, $i=1,2,3.$ are considered as free terms.

In [17], the uniqueness theorem for $\vec{u}(x,t) \in V_2(Q_T) \cap L_{q,r}(Q_T)$ is proved with q and r satisfying the conditions

$$\frac{1}{r} + \frac{n}{2q} \le \frac{1}{2},\ r \in [2,\infty),\ q \in (n,\infty] \tag{21}$$

or

$$q > n,\ r = \infty \tag{22}$$

After scalar multiplication (20), we have

$$\frac{1}{2}\|u(x,t)\|^2 + \mu\int_0^t \|u_x\|^2 dt + \frac{1}{\varepsilon}\int_0^t\int_\Omega \xi(x)u^2(x)dxdt = \int_{Q_t} f^k u_{x_i}dxdt = \int_{Q_t} u_i \vec{v}'' u_{x_i}dxdt \tag{23}$$

the right part od which is

$$\int_{Q_t} \vec{f}^k \vec{u}_{x_i}dxdt = \int_{Q_t} u_i \vec{v}'' \vec{u}_{x_i}dxdt \tag{24}$$

in the class of functions satisfying conditions (21) and (22), it is estimated using the Hölder's inequality in a special way [17].

$$\left|\int_{Q_t} u_i \cdot \vec{v}'' \vec{u}_{x_i}dxdt\right| \le \left(\int_{Q_t}\vec{u}_x^2 dxdt\right)^{1/2} \cdot \left(\int_{Q_t}\vec{v}''^2 \cdot \vec{u}^2 dxdt\right)^{1/2} \le \|\vec{u}_x\|_{2,Q_t} \cdot \left[\int_0^t \left(\int_\Omega |\vec{v}''|^{2\cdot\frac{q}{2}}dx\right)^{2/q} \cdot \left(\int_\Omega |u|^{2\cdot\frac{q}{q-2}}dx\right)^{\frac{q-2}{q}} dt\right]^{1/2} \le$$

$$\le \|\vec{u}_x\|_{2,Q_t}\left[\int_0^t\left(\int_\Omega |\vec{v}''|^q dx\right)^{\frac{2}{q}\cdot\frac{r}{2}} dt\right]^{1/r} \cdot \left[\int_0^t\left(\int_\Omega |u|^{\frac{2q}{q-2}}dx\right)^{\frac{q-2}{2q}\cdot\frac{2r}{r-2}} dt\right]^{\frac{r-2}{r}} = \|\vec{u}_x\|_{2,Q_t} \cdot \|\vec{v}''\|_{q,r,Q_t} \cdot \|\vec{u}\|_{\frac{2q}{q-2},\frac{2r}{r-2},Q_t} \tag{25}$$

Then using the inequality

$$\|\vec{u}_x\|_{2,Q_t} \cdot \|\vec{u}\|_{\bar{q},\bar{r},Q_t} \le 0{,}5\beta \cdot \left(\sup_t \|\vec{u}(x,t)\|_{2,\Omega} + \|\vec{u}_x\|_{2,Q_t}\right)^2 = \frac{\beta}{2}\left|u\right|^2_{Q_t},$$

where $\bar{q} = \dfrac{2q}{q-2}, \bar{r} = \dfrac{2r}{r-2}, \left|\vec{u}\right| = \sup_t \|\vec{u}(x,t)\| + \|\vec{u}_x\|_{2,Q_t}$.

from (25) we get

$$\left|\vec{u}\right|_{Q_t} \le \left(1 + (2\mu)^{-1/2} + (\varepsilon/2)^{1/2}\right)\beta^{1/2}\|\vec{v}''\|^{1/2}_{q,r,Q_t}\left|\vec{u}\right|_{Q_t} \tag{26}$$

from which it follows that $\vec{u} \equiv 0$ for t satisfying the requirement

$$\left(1 + (2\mu)^{-1/2} + (\varepsilon/2)^{1/2}\right)\beta^{1/2}\|\vec{v}''\|^{1/2}_{q,r,Q_t} < 1.$$

The theorem is proved.

Numerical calculations. To perform a methodical calculation, let us consider a numerical solution of the auxiliary problem (4)-(5) with the following initial and boundary conditions:

$$\tau = 0\ ,\ u|_{x=0} = u^0, v = 0.\ x = 0\ ,\ u = u^0, v = 0.\ x = X\ ,\ \frac{\partial u}{\partial x} = 0, v = 0.\ y = 0 = Y\ ,\ u = 0, v = 0$$

As the domain under consideration, let us take a channel with an obstacle and with solid boundaries (Figure 2). Domain Ω_1 is the fictitious domain, Ω is the physical domain.

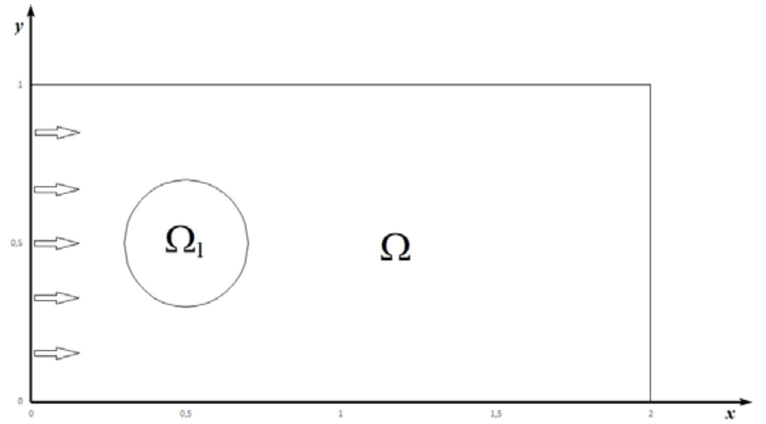

Figure 2. Schematic representation of the domain under consideration

For the numerical implementation of the auxiliary problem (4)-(5) let us the finite difference method and the splitting by physical parameters scheme [18]. The solution domain is covered by the so-called MAC-grid [19], in which the pressure determination nodes are located inside a rectangular grid cell, and the velocity component determination nodes are located on its faces. When approximating convective terms, a sign-sensitive scheme is used. This provides an approximation with second-order accuracy in space and first-order accuracy in time.

Let us assume that at some time $t^n = n\tau$ (τ is time step, n is number of steps) the velocity field is known as $\vec{v}^n$. Then the scheme for determining unknown functions at a time $t^{n+1} = (n+1)\tau$ can be represented as a three-stage splitting scheme:

I-stage

$$\frac{\vec{v}^{n+1/2} - \vec{v}^n}{\tau} = -(\vec{v}^n \nabla)\vec{v}^{n+1/2} - \nabla p^n + \mu^\varepsilon \Delta \vec{v}^{n+1/2} - \frac{\xi(x)}{\varepsilon}\vec{v}^{n+1/2} \tag{27}$$

II-stage

$$\Delta p^{n+1} - \Delta p^n = \frac{\nabla \vec{v}^{n+1/2}}{\tau} \tag{28}$$

III-stage

$$\frac{\vec{v}^{n+1} - \vec{v}^{n+1/2}}{\tau} = -\nabla p^{n+1} + \nabla p^n \tag{29}$$

where $\xi(x) = \begin{cases} 0, & (x,y) \in \Omega \\ 1, & (x,y) \in \Omega_1 \end{cases}$ $f_{\ddot{u}} = \begin{cases} f(\vec{x},t), & (x,y) \in \Omega \\ 0, & (x,y) \in \Omega_1 \end{cases}$

In the numerical implementation, the values and zero values of the tangent component of the fluid flow velocity at the inlet and outlet to the computational domain were set. At the “solid: boundaries, the pressure values are given as linear functions and the tangent component of the velocity is zero.

The following physical interpretation of the given splitting scheme is proposed. At the first stage, it is assumed that the transfer of the amount of motion (momentum of a unit of mass) is carried out only by convention and diffusion. The velocity field obtained in this way does not satisfy the incompressibility condition. In this paper, in contrast to the classical version of the method of splitting by physical parameters [18], an implicit scheme is used at the first stage in the form of (27). This approach was substantiated and numerically implemented in [20]. Iterative schemes for solving auxiliary grid Navier-Stokes equations have been proposed. This approach improves the conditionality number of the matrix of a system of linear equations and accelerates the convergence of the solution.

At the second stage, the pressure field is found according to the intermediate velocity field, considering the condition of solenoidality of the velocity vector from the solution of the Poisson equation. At the second stage, numerical methods for solving grid elliptic equations with Dirichlet boundary conditions for pressure are used.

At stage III, it is assumed that the transfer is carried out only due to the pressure gradient (convection and diffusion do not occure).

In this case, tracking the actual boundary of the source area is necessary to set the coefficients of the auxiliary problem. To ensure a smooth change in the viscosity coefficient of the auxiliary problem, bicubic spline interpolation of two-dimensional function is used. This function, interpolated in a small neighborhood of the boundary region, provides a smooth change that will increase computational accuracy and reliability. A 50x20 grid and the following dimensionless parameter values are used in the numerical implementation: $0 < x_i < 2, 0 < y_i < 1,\ \tau = 0.001,\ \mu = 1/\mathrm{Re} = 0.001,\ \varepsilon = 10^{-5}$.

The upper and lower solid boundary curves are described by the equations

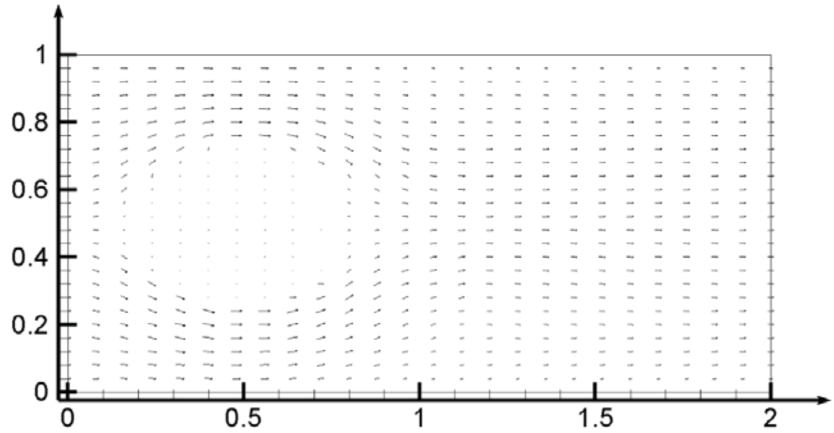

Figure 3. Velocity vector fields

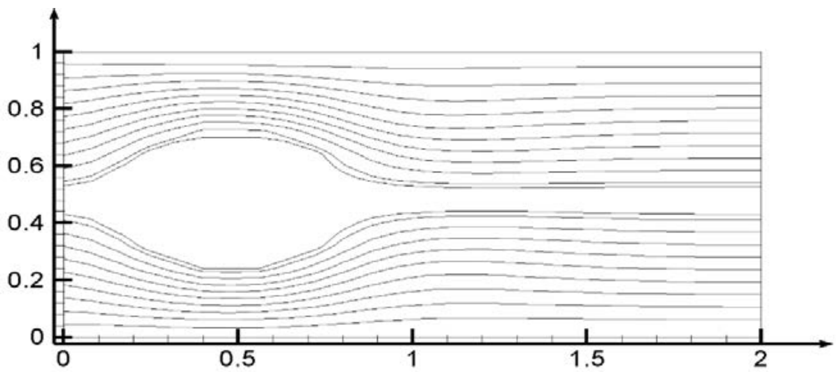

Figure 4. Isolines of the stream function

Conclusion. The fictitious domain method for the Navier-Stokes equations is studied and mathematically justified in the work. The theorem of unambiguous solvability of the initial boundary value problem and the existence and uniqueness of the generalized solution of the auxiliary problem for the Navier-Stokes equations in the case of two spatial variables are proved. A numerical implementation of the proposed method of fictitious domain with continuation by the lowest coefficients for the Navier-Stokes equations is carried out. Various graphical illustrations of the results of numerical simulation using the fictitious domain method for the Navier-Stokes equations are presented.

The isolines of the stream function reveal the current flowing around smooth bodies. The boundary layer of the flow that is directly adjacent to the streamlined body is slowing down harder than the layers that are distanced from the boundary. This leads to tangential friction reduction down to zero on the streamlined surface, then separation of the flow from the surface and a return flow behind the separation line occurrence (Figure 3-4).

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Аңдатпа. Бұл жұмыс Навье-Стокс теңдеулеріне арналған жалған аймақтар әдісін зерттеуге және математикалық негіздеуге арналған. Жұмыста Навье-Стокс теңдеулері үшін кіші коэффициенттерді жалғастыра отырып, жалған аймақтардың әдісі негізделген. Екі кеңістіктік айнымалылар жағдайында Навье-Стокс теңдеулері үшін көмекші мәселенің жалпыланған шешімінің болуы мен бірегейлігі бастапқы шекаралық мәселенің біржақты шешілуінің теоремасы дәлелденді. Навье-Стокс теңдеулері үшін кіші коэффициенттер бойынша жалғастықпен жалған облыстардың ұсынылған әдісін сандық іске асыру жүргізілді. Навье-Стокс теңдеулері үшін жалған аймақтар әдісін қолдану кезінде сандық модельдеу нәтижелерінің әртүрлі графикалық суреттері ұсынылған.

Түйін сөздер: Жалған аймақ әдісі, Навье-Стокс теңдеуі, бастапқы шекті мәселенің шешілуі, жалпыланған шешімнің бар болуы және жалғыздығы, сандық есептеулер.

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Аннотация. Данная работа посвящена изучению и математическому обоснованию метода фиктивных областей для уравнений Навье-Стокса. В работе обоснован метод фиктивных областей с продолжением по младшим коэффициентам для уравнений Навье-Стокса. Доказана теорема однозначной разрешимости начально-краевой задачи и существования и единственности обобщенного решения вспомогательной задачи для уравнений Навье-Стокса в случае двух пространственных переменных. Проведена численная реализация предлагаемого метода фиктивных областей с продолжением по младшим коэффициентам для уравнений Навье-Стокса. Представлены различные графические иллюстрации результатов численного моделирования при применении метода фиктивных областей для уравнений Навье-Стокса.

Ключевые слова: метод фиктивных областей, уравнение Навье-Стокса, разрешимость начально-краевой задачи, существование и единственность обобщенного решения, численные расчеты.

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