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SOME QUESTIONS ON EXTERNAL DEFINABILITY

Abstract. The article discusses the various approaches to the concept of external definability developed in o-minimal theories. An example of o-minimal theories shows how external constants help determine the existence of a solution in a model of a formula with external constants. The basic concepts are formulated with the help of which external definability is proved. A brief review of the results for dependent theories is given. In conclusion, sufficient conditions are formulated so that the NSOP theory has the some property of external definability. A brief explanation of the stated theorem is given.

Keywords: externally definable, neighborhood of tuple of the set in the type, non orthogonality of two types.

External definability. Let $\mathfrak{M}$ be elementary substructure of $\mathfrak{N}$. Let $\bar{\alpha} \in N\setminus M$ and $p := tp(\bar{\alpha}|M)$. Then for any formula $\psi(\bar{x},\bar{y})$ define the predicate $R_{(\psi,p)}(\bar{y})$ on the set M, $\vDash R_{(\psi,p)}(\bar{a})$ iff $\psi(\bar{x},\bar{a}) \in tp(\bar{\alpha}|M)$ iff $\mathfrak{N} \vDash \psi(\bar{\alpha},\bar{a})$. Denote by $\mathfrak{M}^+ = \langle M;\Sigma^+\rangle$, where $\Sigma^+ := \{R_{(\psi,p)}(\bar{y})|p \in S(M),\ \psi \in \Sigma\}$. It follows from definition that if a pair of models (M,N) is conservative pair (type of any tuple elements from N over M is definable), then the structure $\mathfrak{M}^+$ is the structure obtained from $\mathfrak{M}$ scolemisation of $\mathfrak{M}$. We will consider the simple cases when $\mathfrak{M}^+$ constructed from one 1-type for o-minimal theory from two approaches.

Let $\mathfrak{M}$ be a model of an arbitrary complete theory T of the signature Σ. We say that $\mathfrak{M}^+_p$ is expansion of $\mathfrak{M}$ by type $p \in S_1(M)$, if $\mathfrak{M}^+_p := \langle M;\Sigma^+_p\rangle$, where $\Sigma^+_p := \{R_{(\psi,p)}(\bar{y})|\psi \in \Sigma\}$.

We say that $\mathfrak{M}^+_p$ *admits uniformly representation of* Σ^+_p-formulas by Σ-formulas, if for any formula $\phi(\bar{y})$ of Σ^+_p there exists Σ-formula $K_\phi(\bar{y},\bar{z})$, there exists $\bar{\alpha} \in N\setminus M$ such that for any $\bar{a} \in M$ the following holds:

$$\mathfrak{M}^+_p \vDash \phi(\bar{a}) \Leftrightarrow \vDash K_\phi(\bar{a},\bar{\alpha}).$$

Approach of Macpherson-Marker-Steinhorn. In the paper [1] (preprint 1994 Macpherson-Marker-Steinhorn proved weak o-minimality of the expansion of an o-minimal structure by unary convex predicate, such that the predicate is traversed by a uniquely realizable 1-type. Following D. Marker [5], an uniquely realizable 1-type $p \in S_1(M)$ over model is that prime model over model and one realization of this 1-type p contains just this element from the set of realization of the type. An uniquely realizable 1-type has the next property: there is no definable function acting on the set of realizations of this 1-type p. Macpherson-Marker-Steinhorn considered at the same time two structures $\mathfrak{M}^+ = \langle M;\Sigma \cup \{U^1\}\rangle$ and $\mathfrak{N} = \langle N;\Sigma\rangle$, where $\mathfrak{N}$ is a model of an o-minimal theory of the signature Σ and a saturated elementary extension of $\mathfrak{M}$. They defined a new unary convex predicate U by using an element $\alpha \in N\setminus M$ from the set of realizations of an irrational $1-$type $p \in S_1(M)$ such that for every $a \in M$ the following holds:

$$\mathfrak{M}^+ \vDash U(a) \Leftrightarrow \mathfrak{N} \vDash a < \alpha.$$

By induction of construction of formulas $\phi(\bar{y})$ of the signature $\Sigma^+ = \Sigma \cup \{U^1\}$ there is a formula $K_\phi(\bar{y},\alpha)$ of the signature Σ such that for any $\bar{a} \in M$ the following holds:

for any $n, k, m < \omega$,

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) < QV_{p_n}(\alpha_{n+k+1}, \alpha_{n+k+2}, \dots, \alpha_{n+k+m}),$$

and consequently, all these sets have empty intersection. D. Marker proved that for any set A in o-minimal theory, for any $q, r \in S_1(A)$ if q and r are not weakly orthogonal than there is $A-$definable monotonic bijection from $q(\mathfrak{N})$ to $r(\mathfrak{N})$. Then for $r \in S_1(M)$, p_i non weakly orthogonal to r, $QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset$.

$$\mathfrak{M}^+ \vDash \phi(\bar{a}) \Leftrightarrow \mathfrak{N} \vDash K_\phi(\bar{a}, \alpha). \tag{1}$$

The crucial point in this construction was the case $\phi(\bar{y}) = \exists x \psi(x, \bar{y})$. They proposed

$$K_{\exists x \psi(x,\bar{y})}(\bar{y}, \alpha) := \exists z_1 \exists z_2 \exists x (z_1 < \alpha < z_2 \wedge \forall z (z_1 < z < z_2 \to K_{\psi(x,\bar{y})}(x, \bar{y}, z)).$$

Since the 1-type $p \in S_1(M)$ is uniquely realizable, two convex to right and to left from α $M\alpha-1$-formulas have solutions out of $p(\mathfrak{N})$. Thus for any $\bar{a} \in M$, if $\mathfrak{N} \vDash K_{\exists x \psi(x,\bar{y})}(\bar{a}, \alpha)$, then for some $b_1, b_2 \in M$,

$$\mathfrak{N} \vDash \exists x (b_1 < \alpha < b_2 \wedge \forall z (b_1 < z < b_2 \to K_{\psi(x,\bar{y})}(x, \bar{a}, z)).$$

This means that in an elementary submodel of $\mathfrak{N}$ the part of the last formula holds on $\mathfrak{M} \vDash \exists x \forall z (b_1 < z < b_2 \to K_{\psi(x,\bar{y})}(x, \bar{a}, z))$. Then there is an element $c \in M$ such that $\mathfrak{M} \vDash \forall z (b_1 < z < b_2 \to K_{\psi(x,\bar{y})}(c, \bar{a}, z))$. So, $K_{\psi(x,\bar{y})}(c, \bar{a}, z) \in p$.

Thus, any Σ^+-M-1-formula $\phi(x, \bar{a})$ has the set of its realizations, $\phi(\mathfrak{M}^+, \bar{a}) = K_\phi(\mathfrak{N}, \bar{a}) \cap M$, being a finite union of convex sets because $K_\phi(\mathfrak{N}, \bar{a})$ is a finite union of intervals and points. The elementary theory of $\mathfrak{M}^+$ is weakly o-minimal since the number of convex sets is bounded and consequently does not depend on parameters.

Approach of B.S. Baizhanov. For the case when $p \in S_1(M)$ is a non uniquely realizable type, B.S. Baizhanov proposed [2] (1995), on the base of theory of (non)orthogonality of 1-types and its classification made in [4], [5], [6], [8] (Pillay-Steinhorn, Marker, Mayer, Marker-Steinhorn, 1986–1994), to take the constants for $K_{\exists x \psi(x,\bar{y})}$ from an infinite indiscernible sequence $I = \langle \alpha_n \rangle_{n<\omega}$ over M and α_n from $p(\mathfrak{N})$. Taking into consideration that if $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n) \cap M = \emptyset$, then there is a finite number irrational cuts (1-types over M) such that for any such 1-type $r \in S_1(M)$, $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n)$ is a subset of

$$QV_r(\bar{\alpha}_n) := \{\beta \in r(\mathfrak{N}) \mid \text{there exists an } M\bar{\alpha}_n\text{-1-formula } \Theta(x, \bar{\alpha}_n), \text{ such that } \beta \in \Theta(\mathfrak{N}, \bar{\alpha}_n) \subset r(\mathfrak{N})\}.$$

The idea to use an indiscernible sequence consists from two parts.

B1. On the one hand, if for some $c \in M, \mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\alpha}_n)$, then for any $\bar{\gamma} = \langle \alpha_{i_0}, \dots, \alpha_{i_n} \rangle$ $(n < i_0 < \dots < i_n)$, $\mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\gamma})$, because $\bar{\alpha}_n$ and $\bar{\gamma}$ have the same type over over M.

B2. On the other hand, to find a sequence I such that for any $r \in S_1(M)$, for any $\bar{\gamma} = \langle \alpha_{i_0}, \dots, \alpha_{i_n} \rangle$ $(n < i_0 < \dots < i_n)$, $QV_r(\bar{\alpha}_n) \cap QV_r(\bar{\gamma}) = \emptyset$.

For find the indiscernible sequence I define the properties A1-A3 that follow from the classification of 1-types and theory non orthogonality of 1-types over sets in o-minimal theories.

A1. [5] (Marker 1986). *Let $q, r \in S_1(A)$, and let type $q(x) \cup r(y)$ be non complete (q is non weakly orthogonal to r, Shelah, 1978). Then there is an A-definable monotonic bijection $g: q(\mathfrak{N}) \to r(\mathfrak{N})$ and consequently, q is irrational if and only if r is irrational;*

q is uniquely realizable if and only if r is uniquely realizable.

Recall that if $q \in S_1(A)$ is irrational then $q(\mathfrak{N})$ is a convex non-definable set without maximal and minimal elements.

A2. *If $q \in S_1(A)$ is irrational, then for any $\bar{\gamma}$, $QV_q(\bar{\gamma}) = V_q(\gamma)$, here*

$$V_q(\bar{\gamma}) := \{\beta \in q(\mathfrak{N}) \mid \exists \delta_1, \delta_2 \in q(\mathfrak{N}), \textit{ there exists an } A\bar{\gamma}\text{-1-formula } S(x, \bar{\gamma}), \delta_1 < S(\mathfrak{N}, \bar{\gamma}) < \delta_2, \beta \in S(\mathfrak{N}, \bar{\gamma})\}.$$

Indeed, the *quasi-neighborhood of* $\bar{\gamma}$ in q ($QV_q(\bar{\gamma})$) is the union of $A\bar{\gamma}$-definable sets, and any such definable set is a subset of $q(\mathfrak{N})$, a convex non-definable set without minimal and maximal elements. The last means that such a definable set is a subset of $V_q(\bar{\gamma})$ (*neighborhood of* $\bar{\gamma}$[n q). This explains equality of two convex sets.

A3. *If* $q \in S_1(A)$ *is irrational and non uniquely realizable, then for any* $\bar{\gamma} \in N$, if $QV_q(\bar{\gamma}) \neq \emptyset$ *then the 1-types* $q(x) \cup \{x < QV_q(\bar{\gamma})\}$ *and* $q(x) \cup \{QV_q(\bar{\gamma}) < x\}$ *are irrational and non uniquely realizable.*

By $A2$ and theorem of compactness there exist $\delta_1, \delta_2 \in q(\mathfrak{N})$ such that $\delta_1 < V_q(\bar{\gamma}) < \delta_2$ and because q is non uniquely realizable i.e. there is A-definable monotonic bijection $f: q(\mathfrak{N}) \to q(\mathfrak{N})$, $V_q(\mathfrak{N})$ can not have minimal and maximal element. Taking in consideration that for irrational q, $q(\mathfrak{N})$ is a convex non-definable set without maximal and minimal elements, $r_1 := tp(\delta_1|A\bar{\gamma})$ and $r_2 = tp(\delta_2|A\bar{\gamma})$ are irrational and since $f(V_q(\bar{\gamma})) = V_q(\bar{\gamma})$, f acts on $r_1(\mathfrak{N})$ and $r_2(\mathfrak{N})$. The last means r_1 and r_2 are non uniquely realizable.

Let $p_n(x) := p(x) \cup (QV_p(\bar{\alpha}_{n-1}) < x)$. Then by A2, A3 p_n is irrational, non uniquely realizable and finitely satisfiable in M since right sides of p and p_n coincide

For any $n, k, m < \omega$,

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) < QV_{p_n}(\alpha_{n+k+1}, \alpha_{n+k+2}, \dots, \alpha_{n+k+m}), \tag{2}$$

and consequently, all these sets have empty intersection.

The proof of (2) is done by induction on m. Assume (2) for m. Denote by $r_1(x) = p_n \cup (x < QV_{p_n}(x)(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}))$ and $r_2(y) = p_n(y) \cup (QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}) < y)$, Suppose that

$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \cap QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}, \alpha_{n+k+m+1}) \neq \emptyset$. Since the first set does not change, there exists $M\bar{\alpha}_n\alpha_{n+k+1} \dots \alpha_{n+k+m+1}$-formula $L(x, \alpha_{n+k+m+1})$ such $L(\mathfrak{N}, \alpha_{n+k+m+1}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N})$. Let β be end point of one of interval of formula L, then because p_n is non uniquely realizable then $\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N})$. Since $\beta \vDash r_1$ and $\alpha_{n+k+m+1} \vDash r_2$ by A1 there exists $M\bar{\alpha}_n\alpha_{n+k+1} \dots, \alpha_{n+k+m}$-definable monotonic function $f: r_2(\mathfrak{N}) \to r_1(\mathfrak{N})$ such that $f(\alpha_{n+k+m+1}) = \beta$. On other hand $\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k})$ and consequently, there is $M\bar{\alpha}_{n+k} - 1$-formula $H(x)$ such that $\beta \in H(\mathfrak{N}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r_1(\mathfrak{N})$. Then $\alpha_{n+k+m+1}$ belongs to $M\bar{\alpha}_{n+k+m}$-definable set $f^{-1}(H(\mathfrak{N})) \subset r_2(\mathfrak{N})$. This means $\alpha_{n+k+m+1} \in QV_p(\bar{\alpha}_{n+k+m})$. Contradiction.

It follow from (2) and A1 that for any $r \in S_1(M)$, if for any $i < n$, $p_i \perp^w r$ and $p_n \not\perp^w r$ then $QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset$.

$$QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset \tag{3}$$

Suppose for the formula $\psi(x, \bar{y})$ of signature Σ^+ corresponding formula of signature Σ is $K_{\psi(x,\bar{y})}(x, \bar{y}, \bar{\alpha}_n)$. Thus for any formula $K_\psi(x, \bar{a}, \bar{\alpha}_n)$ to have the solution in M it is sufficient to write the formula

$$K_{\exists x\psi(x,\bar{y})}(\bar{y}, \bar{\alpha}_{2n+1}) := \exists x(K_\psi(x, \bar{y}, \bar{\alpha}_n) \wedge$$
$$\wedge_{i \le n} K_\psi(x, \bar{y}, \bar{\alpha}_{n-i}, \alpha_{(n-i)+n+1}, \alpha_{(n-i)+n+2} \dots, \alpha_{2n+1})).$$

B.S. Baizhanov in 1996 obtained a classification of 1-types over a subset of a model of weakly o-minimal theory and solved the problem of expanding a model of weakly-o-minimal theory by a unary convex predicate in the preprint "Classifications of 1-types in weakly o-minimal theories and its applications" and submitted in the JSL, that revised version published in [9](2001). Ye.Baisalov and B. Poizat [10] (preprint 1996) in the paper on "beautiful" pairs of models of o-minimal theories proved the elimination of quantify $\exists x \in M$. It is difficult to say that the approach in [10] is alternative to approach elaborated in [2], because they used the same principles B1-B2 from [2].

We say that $\mathfrak{M}^+$ *the expansion by all externally definable subsets admits quantifier elimination*, if for any formula $\phi(\bar{y})$ of Σ^+ there exists Σ-formula $K_\phi(\bar{y}, \bar{z})$, there exists $\bar{\alpha} \in N \backslash M$ such that for any $\bar{a} \in M$ the following holds:

$$\mathfrak{M}^+ \vDash \phi(\bar{a}) \Leftrightarrow \vDash K_\phi(\bar{a}, \bar{\alpha}).$$

Approach of Shelah. In his paper, S. Shelah [11] (2004) considered a model of NIP theory and proved that the expansion by all externally definable subsets admits quantifier elimination and thereby is NIP. The key problem here is eliminating quantifier "there exists in the submodel". In his proof in the way of contradiction Shelah used an indiscernible sequence $\langle \bar{b}_n : n < \omega \rangle$ in order to show that if eliminating quantifier "there exists x in the submodel" $\varphi(x, \bar{a})$ fails, then $\varphi(\alpha, \bar{b}_n)$ holds iff n is even, for some α, which implies the independence property, for a contradiction.

V.V. Verbovskiy [12] (preprint 2005) found a somewhat simplified account of Shelah's proof, namely by using noting of a finitely realizable type. A. Pillay [13] (preprint 2006) gave two re-proofs of Shelah's theorem, the first going through quantifier-free heirs of quantifier-free types and the second through quantifier-free coheirs of quantifier-free types.

The analysis of approaches shows that the using the theory of orthogonality we can control the set of realizations of one-types. If we consider the complete theory satisfies A2, it gives the possibility to construct the indiscenible sequence satisfied the condition B2. Notice that the of indiscernible sequence constructed by mathematical induction satisfies the condition of finite realizability of an one-type of new element over model and beginning of sequence. The generalization of the approach for o-minimal model in case of non uniquely realizable one-type by introduction of generalization of the notions of (quasi)-neighborhood and almost (non)-orthogonality of two types gives the possibility to formulate the next

Theorem 1 Let T be a complete NSOP theory such that for any set A the following holds:

1) For any $p \in S_1(A)$, for any $\bar{\gamma}$, $QV_p(\bar{\gamma}) = V_p(\bar{\gamma})$

2) For any $p, q \in S_1(A)$ the following holds. If $p \not\perp^a q$, then $q \not\perp^a p$.

Then for model of the theory T the expansion of this model by one-type admits uniformly representation of Σ_p^+-formulas by Σ-formulas.

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НЕКОТОРЫЕ ВОПРОСЫ О ВНЕШНЕЙ ОПРЕДЕЛИМОСТИ

Аннотация. В статье рассматриваются различные подходы к концепции внешней определимости, разработанные в о-минимальных теориях. Пример о-минимальных теорий показывает, как внешние константы помогают определить существование решения в модели формулы с внешними константами. Сформулированы основные понятия, с помощью которых доказывается внешняя определимость. Дается краткий обзор результатов для зависимых теорий. В заключение сформулированы достаточные условия, так что теория обладает свойством внешней определимости. Дано краткое объяснение изложенной теоремы.

Ключевые слова: внешне определяемая, окрестность кортежа множества в типе, неортогональность двух типов

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СЫРТҚЫ ЕРЕЖЕЛЕРГЕ ҚАТЫСТЫ САУАЛДАР

Аннотация. Мақалада о-минималды теорияларда жасалған сыртқы анықтама тұжырымдамасына әртүрлі көзқарастар қарастырылған. О-минималды теориялардың мысалы сыртқы тұрақтылар формула моделінде ерітіндінің болуын анықтауға көмектесетінін көрсетеді. Негізгі ұғымдар сыртқы анықталуы дәлелденген, тұжырымдалған. Тәуелді теориялардың нәтижелеріне қысқаша шолу келтірілген.

Қорытындылай келе, теория сыртқы анықталу қасиетіне ие болатындай жеткілікті жағдайлар жасалған. Көрсетілген теоремаға қысқаша түсініктеме беріледі.

Түйін сөздер: сыртқы анықталған, типтегі жиынның көршілес, екі түрге жатпайтындығы.

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