NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2019.2518-1726.73**
Volume 6, Number 328 (2019), 52 – 62

UDC 517.9

A.Sh. Shaldanbayev[1] , A.A. Shaldanbayeva[2] , A.Zh. Beisebayeva[3], B.A. Shaldanbay[4]

[1]Silkway International University, Shymkent, Kazakhstan;
[2,4]Regional social-innovative University, Shymkent, Kazakhstan;
[3]South Kazakhstan State University M.O.Auezov, Shymkent, Kazakhstan
shaldanbaev51@mail.ru, altima_a@mail.ru, akbope_a@mail.ru, baglan.shaldanbayev@bk.ru

INVERSE PROBLEM OF STURM-LIOUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

Abstract: In this paper, we prove uniqueness theorem, by one spectrum, for a Sturm-Liouville operator with non-separated boundary value conditions and a real continuous and symmetric potential. The research method differs from all previously known methods and is based on internal symmetry of the operator generated by invariant subspaces.
Keywords: Sturm-Liouville operator, spectrum, inverse Sturm-Liouville problem, Borg theorem, Ambartsumyan theorem, Levinson theorem, non-separated boundary value conditions, symmetric potential, invariant subspaces, differential operators, inverse spectral problems.

1. Introduction

We study the following inverse spectral problem for the Sturm-Liouville operator:

$$Ly := y'' + q(x)y,\ x \in (0, 1),$$

on a finite interval (0, 1) with non-separated boundary value conditions. Inverse problems consist in restoring the coefficients of differential operators by their spectral characteristics. Such problems often arise in mathematics and its applications.

Inverse problems for differential operators with decaying boundary value conditions have been thoroughly studied (see monographs [1–5] and references). More difficult inverse problems for Sturm – Liouville operators with non-decaying boundary value conditions were studied in [6–17] and other works. In particular, periodic boundary-value problem was considered in [6, 7, 9, 14]. I. V. Stankevich [6] proposed formulation of the inverse problem and proved the corresponding uniqueness theorem. V. A. Marchenko and I. V. Ostrovsky [7] characterized spectrum of a periodic boundary-value problem in terms of a special conformal mapping. The conditions proposed in [7] are difficult to verify. Another method, used in [9], made it possible to obtain necessary and sufficient conditions for solvability of the inverse problem in the periodic case that are more convenient to verify. Similar results were obtained in [9], and for another type of boundary conditions, namely

$$y'(0) - ay(0) + by(\pi) = y'(\pi) + dy(\pi) - by(0) = 0.$$

Later similar results were obtained in [12, 13]. In the paper [18], the case when the potential q is symmetric with respect to the middle of interval, i.e., $q(x) = q(\pi - x)$ a.e. on $(0, \pi)$, was studied, and for this case a solution of the inverse spectral problem was constructed and a spectrum was given. The symmetric case requires nontrivial changes in the method and allows us to specify less spectral information than in the general case. Some results for the symmetric case were obtained in [10] and [17] - [24].

The inverse problems of spectral analysis are understood as problems of reconstructing a linear operator from one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Ambardzumyan [25]. He proved the following theorem.

By $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ we denote eigenvalues of the following Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0,\ y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2\ (n = 0,1,2,\dots)\ \text{то}\ q(x) \equiv 0.$$

The first mathematician who drew attention to importance of this Ambardzumyan result was the Swedish mathematician Borg. He performed the first systematic research of one of important inverse problems, namely, the inverse problem for the classical Sturm – Liouville operator of the form (1.1) by the spectra [26]. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the Ambartsumyan result is an exception to the general rule. In the same paper [26], Borg showed that two spectra of the Sturm – Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Borg Theorem.

Let the equations

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$
$$-z'' + p(x)z = \lambda z, \tag{1.3}$$

have the same spectrum under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases} \tag{1.4}$$

under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases} \tag{1.4$'$}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0, \pi]$, if

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Soon after the Borg work, important studies on the theory of inverse problems were carried out by Levinson [27], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm – Liouville operator

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.5}$$

is reconstructed by one spectrum.

A number of B.M. Levitan works [28, 29] are devoted to reconstruction of the Sturm – Liouville operator by one and two spectra.

This work is devoted to a generalization of the theorems of Ambartsumian [25] and Levinson [27], in particular, our results contain the results of these authors. Research method of this work appeared under influence of [30] - [32], and differs from all previously known methods.

1. Research Method.

Idea of this work is very simple. Having studied in detail contents of [1, 3], we realized that both of these operators have an invariant subspace. If for the linear operator L, we have the formulas

$$LP = PL^*, \qquad QL = L^*Q,$$

where P, Q are orthogonal projectors, satisfying the condition $P + Q = I$, then the operators L and L^* have invariant subspaces, sometimes restriction of these operators to these invariant subspaces, under certain conditions, form a Borg pair.

2. Research Results.

In the Hilbert space $H = L^2(0,\pi)$ we consider the Sturm – Liouville operator.

$$Ly = -y'' + q(x)y,\ x \in (0,\pi); \tag{3.1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{3.2}$$

where $q(x)$ is a continuous complex function, a_{ij} $(i = 1,2;\ j = 1,2,3,4)$ are arbitrary complex coefficients, and by Δ_{ij} $(i = 1,2;\ j = 1,2,3,4)$ we denote minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Suppose that $\Delta_{13} \neq 0$, then the Sturm – Liouville operator (3.1) – (3.2) has the following form

$$Ly = -y'' + q(x)y,\ x \in (0,\pi); \tag{3.1}$$

$$\begin{cases} \Delta_{13}y(0) - \Delta_{32}y'(0) - \Delta_{34}y'(\pi) = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0, \end{cases} \tag{3.3}$$

and its conjugate operator L^+ has the form

$$L^+z = -z'' + \overline{q(x)}z,\ x \in (0,\pi); \tag{3.1$^+$}$$

$$\begin{cases} \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0. \end{cases} \tag{3.3$^+$}$$

Let P and Q be orthogonal projectors, defined by the formulas

$$Pu(x) = \frac{u(x)+u(\pi-x)}{2},\ Qv(x) = \frac{v(x)-v(\pi-x)}{2} \tag{3.4}$$

The main result of this paper is the following theorem.

Theorem 3.1. If $\Delta_{13} \neq 0$, and

1) $LP = PL^+$; (3.5)
2) $QL = L^+Q$; (3.6)
3) $\Delta_{12} = -\Delta_{34}$; (3.7)

then the Sturm – Liouville operator (3.1) – (3.3) is reconstructed by one spectrum.

3. Discussion.

In this section we prove the theorem and discuss the obtained results. The following Lemmas 4.1 and 4.2 can have independent values.

Lemma 4.1. If for a linear and discrete operator L, the following equalities hold:

1) $LP = PL^+$; (3.5)
2) $QL = L^+Q$; (3.6)
3) $P + Q = I$; (3.8)

where P, Q are orthogonal projectors, and I is unit operator, then all its eigenvalues are real.

Proof.

Let $LP = PL^*$, $QL = L^*Q$; then

$$(LP)^* = P^*L^* = PL^* = LP;$$

$$(QL)^* = L^*Q^* = L^*Q = QL;$$

i.e. operators LP and QL are selfadjoint, therefore their eigenvalues are real.

If $Ly = \lambda y,\ y \neq 0,$ then $QLy = \lambda Qy,\ L^+Qy = \lambda Qy,\ L^+Q(Qy) = \lambda Qy,\ QL(Qy) = \lambda Qy$ if $Qy \neq 0$, then λ is a real quantity; if $Qy = 0$, then $y = Py \neq 0$, and $LPy = \lambda Py,\ LP(Py) = \lambda Py$. Consequently, λ is again real quantity.

The following lemma shows that the spectrum $\sigma(L)$ of the operator L splits into two parts; therefore, the operator L, apparently, also splits into two parts. Furthermore, we will see that this is exactly what happens, and more precisely, these parts form a Borg pair under a certain condition.

Lemma 4.2. If L is a linear discrete operator, satisfying the conditions:

1) $LP = PL^{+}$; (3.5)
2) $QL = L^{+}Q$; (3.6)
3) $P + Q = I$; (3.8)

where P, Q are orthogonal projectors, and I is identity operator, then we have

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2). \tag{3.9}$$

where $L_1 = LP,\ L_2 = QL,\ \sigma(L)$ is a spectrum of the operator L.

Proof.

If $Ly = \lambda y,\ y \neq 0$, then $QLy = \lambda Qy,\ L^{+}Qy = \lambda Qy,\ L^{+}Q(Qy) = \lambda Qy,\ L_2 Qy = \lambda Qy$. If $Qy \neq 0$, then $\lambda \in \sigma(L_2)$. If $Qy = 0$, then $y = Py \neq 0$ and $LPy = \lambda Py,\ LP(Py) = \lambda Py,\ L_1 Py = \lambda Py$. Consequently, $\lambda \in \sigma(L_1)$.

Hence, $\sigma(L) \subset \sigma(L_1) \cup \sigma(L_2)$.

If $\lambda \neq 0$, and $\lambda \in \sigma(L_1) \cup \sigma(L_2)$, then

a) If $\lambda \in \sigma(L_1)$, then $\exists u \neq 0$, such that $u \in H_1,\ L_1 u = \lambda u,\ LPu = \lambda u, \rightarrow Lu = \lambda u$. Consequently, $\lambda \in \sigma(L)$.

b) If $\lambda \in \sigma(L_2)$, then $\exists v \in H_2,\ v \neq 0$ such that $L_2 v = \lambda v,\ QLv = \lambda v,\ L^{+}Qv = \lambda v,\ L^{+}v = \lambda v$. Thus, $\lambda \in \sigma(L^{+}) = \sigma(L)$.

c) If $0 \in \sigma(L_1) \cup \sigma(L_2)$, then if $0 \in \sigma(L_1)$, then $L_1 u = 0,\ u \in H_1,\ LPu = 0, \Rightarrow Lu = 0, \Rightarrow 0 \in \sigma(L)$. If $0 \in \sigma(L_2)$, then $L_2 v = 0,\ v \in H_2,\ QLv = 0, \Rightarrow L^{+}Qv = 0, L^{+}v = 0, \Rightarrow 0 \in \sigma(L^{+}) = \sigma(L)$.

The following two Lemmas 4.3 and 4.4 refine boundary conditions of the Sturm - Liouville operators with invariant subspaces.

Lemma 4.3. If

a) $\Delta_{13} \neq 0$;

b) $LP = PL^{+}$;

then the following formulas hold

1) $\Delta_{12} + \Delta_{32} = \Delta_{14} + \Delta_{34}$;

2) $\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}} = \overline{\left(\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}}\right)} = \frac{\Delta_{34}-\Delta_{32}}{\Delta_{13}}$;

3) $\overline{q(x)} = q(x),\ q(\pi - x) = q(x)$;

and the operators L and L^{+} have the following forms:

a) $Ly = -y'' + q(x)y,\ x \in (0, \pi)$;

$$\begin{cases} y(0) - y(\pi) - \dfrac{\Delta_{12} + \Delta_{32}}{\Delta_{13}}[y'(0) + y'(\pi)] = 0, \\ \Delta_{12} y'(0) + \Delta_{13} y(\pi) + \Delta_{14} y'(\pi) = 0. \end{cases}$$

b) $L^{+}z = -z'' + q(x)z,\ x \in (0, \pi)$;

$$\begin{cases} z(0) + z(\pi) + \dfrac{\overline{\Delta_{12}} - \overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0) + z'(\pi)] = 0, \\ \overline{\Delta_{13}} z(0) - \overline{\Delta_{32}} z'(0) - \overline{\Delta_{12}} z'(\pi) = 0. \end{cases}$$

Proof.

Assume that

$$LP = PL^{+}; \tag{3.5}$$

From the condition $z \in D(L^{+})$ it follows that $y = Pz \in D(L)$, therefore we have the following equalities:

$$\begin{cases} \Delta_{13}\dfrac{z(0)+z(\pi)}{2} - \Delta_{32}\dfrac{z'(0)-z'(\pi)}{2} - \Delta_{34}\dfrac{z'(\pi)-z'(0)}{2} = 0, \\ \Delta_{12}\dfrac{z'(0)-z'(\pi)}{2} + \Delta_{13}\dfrac{z(\pi)+z(0)}{2} + \Delta_{14}\dfrac{z'(\pi)-z'(0)}{2} = 0. \end{cases}$$

$$\begin{cases} \Delta_{13}\dfrac{z(0)+z(\pi)}{2} + (\Delta_{34}-\Delta_{32})\dfrac{z'(0)-z'(\pi)}{2} = 0, \\ \Delta_{13}\dfrac{z(0)+z(\pi)}{2} + (\Delta_{12}-\Delta_{14})\dfrac{z'(0)-z'(\pi)}{2} = 0. \end{cases}$$

From (3.5) it follows that $\Delta_{12}+\Delta_{32} = \Delta_{14}+\Delta_{34}$, then $\Delta_{34}-\Delta_{32} = \Delta_{12}-\Delta_{14}$, and two boundary conditions merge into one boundary condition. Hence,

$$\Delta_{13}\frac{z(0)+z(\pi)}{2} + (\Delta_{12}-\Delta_{14})\frac{z'(0)-z'(\pi)}{2} = 0. \tag{4.1}$$

Summing up the boundary conditions (3.3)$^{+}$, we get

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{34}}-\overline{\Delta_{32}})z'(0) + (\overline{\Delta_{14}}-\overline{\Delta_{12}})z'(\pi) = 0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(0) - (\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(\pi) = 0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{12}}-\overline{\Delta_{14}})[z'(0)-z'(\pi)] = 0. \tag{4.2}$$

From (4.1) and (4.2) we write the system of equations:

$$\Delta_{13}\frac{[z(0)+z(\pi)]}{2} + (\Delta_{12}-\Delta_{14})\frac{[z'(0)-z'(\pi)]}{2} = 0,$$

$$\overline{\Delta_{13}}\frac{[z(0)+z(\pi)]}{2} + (\overline{\Delta_{12}}-\overline{\Delta_{14}})\frac{[z'(0)-z'(\pi)]}{2} = 0;$$

This system has a nontrivial solution, therefore,

$$\begin{vmatrix} \Delta_{13} & \Delta_{12}-\Delta_{14} \\ \overline{\Delta_{13}} & \overline{\Delta_{12}}-\overline{\Delta_{14}} \end{vmatrix} = 0 \text{ или } \frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}} = \overline{\left(\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}}\right)}.$$

Further, subtracting the second boundary condition from the first condition (see 3.3), we obtain

$$\Delta_{13}[y(0)-y(\pi)] - (\Delta_{12}+\Delta_{32})y'(0) - (\Delta_{34}+\Delta_{14})y'(\pi) = 0,$$

$$\Delta_{13}[y(0)-y(\pi)] - (\Delta_{12}+\Delta_{32})[y'(0)+y'(\pi)] = 0,$$

$$y(0)-y(\pi) - \frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)] = 0$$

Now we study properties of the differential expression L. From the formula $LP = PL^{+}$, we get

$$LPz = L^{\circ}\frac{z(x)+z(\pi-x)}{2} = -\frac{z''(x)+z''(\pi-x)}{2} + q(x)\frac{z(x)+z(\pi-x)}{2};$$

$$PL^{+}z = P^{\circ}\left[-z'' + \overline{q(x)}z\right] = -\frac{z''(x)+z''(\pi-x)}{2} + $$

$$+\frac{\bar{q}(x)z(x)+\bar{q}(\pi-x)z(\pi-x)}{2};$$

$$q(x)z(x) - q(x)z(\pi-x) = \bar{q}(x)z(x) + \bar{q}(\pi-x)z(\pi-x),$$

$$\begin{cases} [q(x)-\bar{q}(x)]z(x)+[q(x)-\bar{q}(\pi-x)]z(\pi-x)=0, \\ [q(\pi-x)-\bar{q}(\pi-x)]z(\pi-x)+[q(\pi-x)-\bar{q}(x)]z(x)=0; \end{cases} \tag{4.3}$$

$$\Delta=\begin{vmatrix} q(x)-\bar{q}(x) & q(x)-\bar{q}(\pi-x) \\ q(\pi-x)-\bar{q}(x) & q(\pi-x)-\bar{q}(\pi-x) \end{vmatrix}=0;$$

$$[q(x)-\bar{q}(x)][q(\pi-x)-\bar{q}(\pi-x)]-$$

$$[q(x)-\bar{q}(\pi-x)][q(\pi-x)-\bar{q}(x)]=0;$$

$$q(x)q(\pi-x)-q(x)\bar{q}(\pi-x)-\bar{q}(x)q(\pi-x)+\bar{q}(x)\bar{q}(\pi-x)=$$

$$=q(x)q(\pi-x)-q(x)\bar{q}(x)-\bar{q}(\pi-x)q(\pi-x)+\bar{q}(\pi-x)\bar{q}(x);$$

$$q(x)\bar{q}(\pi-x)+\bar{q}(x)q(\pi-x)=q(x)\bar{q}(x)+\bar{q}(\pi-x)q(\pi-x),$$

$$q(x)[\bar{q}(\pi-x)-\bar{q}(x)]+q(\pi-x)[\bar{q}(x)-\bar{q}(\pi-x)]=0,$$

$$[\bar{q}(x)-\bar{q}(\pi-x)]\cdot[q(\pi-x)-q(x)]=0,$$

$$|q(x)-q(\pi-x)|^2=0, \ => \ q(x)=q(\pi-x).$$

Further, from (4.3) we get

$$[q(x)-\bar{q}(x)]z(x)+[q(x)-\bar{q}(x)]z(\pi-x)=0,$$

$$[q(x)-\bar{q}(x)][z(x)+z(\pi-x)]=0, => q(x)-\bar{q}(x)=0.$$

Lemma 4.4. If

a) $\Delta_{13}\neq 0$;

b) $QL=L^{+}Q$,

then

1) $\Delta_{12}+\Delta_{32}=\Delta_{14}+\Delta_{34}$;

2) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}\right)}=\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}=\frac{\Delta_{14}+\Delta_{34}}{\Delta_{13}}$;

3) $q(\pi-x)=q(x), \bar{q}(x)=q(x)$,

and the operators L and L^{+} have the form

4) $Ly=-y''+q(x)y, \ x\in(0,\pi)$;

$$\begin{cases} y(0)-y(\pi)-\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)]=0, \\ \Delta_{12}y'(0)+\Delta_{13}y(\pi)+\Delta_{14}y'(\pi)=0. \end{cases}$$

5) $L^{+}z=-z''+\bar{q}(x)z, \ x\in(0,\pi)$;

$$\begin{cases} z(0)+z(\pi)+\frac{\overline{\Delta_{12}}-\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0)-z'(\pi)]=0, \\ \overline{\Delta_{13}}z(0)-\overline{\Delta_{32}}z'(0)-\overline{\Delta_{12}}z'(\pi)=0. \end{cases}$$

Proof.

Suppose that the following equality holds:

$$QL=L^{+}Q$$

then the condition $y(x)\in D(L)$ implies that $z=Qy\in D(L^{+})$, therefore the following equalities hold:

$$z(x)=\frac{y(x)-y(\pi-x)}{2}, \qquad z'(x)=\frac{y'(x)+y'(\pi-x)}{2};$$

$$\begin{cases} \overline{\Delta_{13}}\frac{y(0)-y(\pi)}{2}-\overline{\Delta_{32}}\frac{y'(0)+y'(\pi)}{2}-\overline{\Delta_{12}}\frac{y'(\pi)+y'(0)}{2}=0, \\ \overline{\Delta_{34}}\frac{y'(0)+y'(\pi)}{2}+\overline{\Delta_{13}}\frac{y(\pi)-y(0)}{2}+\overline{\Delta_{14}}\frac{y'(\pi)+y'(0)}{2}=0; \end{cases}$$

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ -\overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}+(\overline{\Delta_{14}}+\overline{\Delta_{34}})\dfrac{y'(0)+y'(\pi)}{2}=0; \end{cases}$$

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{14}}+\overline{\Delta_{34}})\dfrac{y'(0)+y'(\pi)}{2}=0. \end{cases}$$

From $QL=L^{+}Q$ it follows that $\Delta_{12}+\Delta_{32}=\Delta_{14}+\Delta_{34}$, therefore there is only one boundary condition

$$\overline{\Delta_{13}}\frac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\frac{y'(0)+y'(\pi)}{2}=0. \tag{4.4}$$

Subtracting the second boundary condition from the first boundary condition in (3.3), we obtain

$$\Delta_{13}[y(0)-y(\pi)]-(\Delta_{12}+\Delta_{32})y'(0)-(\Delta_{14}+\Delta_{34})y'(\pi)=0,$$

$$\Delta_{13}\frac{[y(0)-y(\pi)]}{2}-(\Delta_{12}+\Delta_{32})\frac{[y'(0)+y'(\pi)]}{2}=0. \qquad (4.5)$$

Combining the boundary conditions (4.4) - (4.5), we have

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ \Delta_{13}\dfrac{[y(0)-y(\pi)]}{2}-(\Delta_{12}+\Delta_{32})\dfrac{[y'(0)+y'(\pi)]}{2}=0. \end{cases}$$

This system of equations has a nontrivial solution, therefore

$$\Delta=\begin{vmatrix} \overline{\Delta_{13}} & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) \\ \Delta_{13} & -(\Delta_{12}+\Delta_{32}) \end{vmatrix}=0,=>\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}\right)}=\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}.$$

Сложив граничных условий $(3.3)^{+}$, имеем

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{34}}-\overline{\Delta_{32}})z'(0)+(\overline{\Delta_{14}}-\overline{\Delta_{12}})z'(\pi)=0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(0)-(\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(\pi)=0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{12}}-\overline{\Delta_{14}})[z'(0)-z'(\pi)]=0.$$

Consequently, boundary conditions of the operators L and L^{+} have the following forms:

$$L:\begin{cases} y(0)-y(\pi)-\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)]=0, \\ \Delta_{12}y'(0)+\Delta_{13}y(\pi)+\Delta_{14}y'(\pi)=0; \end{cases}$$

$$L^{+}:\begin{cases} z(0)+z(\pi)+\frac{\overline{\Delta_{12}}-\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0)-z'(\pi)]=0, \\ \overline{\Delta_{13}}z(0)-\overline{\Delta_{32}}z'(0)-\overline{\Delta_{12}}z'(\pi)=0. \end{cases}$$

Further, from the formula $QL=L^{+}Q$, we get

$$QLy=Q^{\circ}[-y''+q(x)y]=-\frac{y''(x)-y''(\pi-x)}{2}+$$
$$+\frac{q(x)y(x)-q(\pi-x)y(\pi-x)}{2};$$

$$L^{+}Qy=L^{+}\left[\frac{y(x)-y(\pi-x)}{2}\right]=$$
$$=-\frac{y''(x)-y''(\pi-x)}{2}+\bar{q}(x)\frac{y(x)-y(\pi-x)}{2};$$

$$q(x)y(x)-q(\pi-x)y(\pi-x)=\bar{q}(x)y(x)-\bar{q}(x)y(\pi-x),$$
$$[q(x)-\bar{q}(x)]y(x)+[\bar{q}(x)-q(\pi-x)]y(\pi-x)=0, \tag{4.6}$$
$$[q(\pi-x)-\bar{q}(\pi-x)]y(\pi-x)+[\bar{q}(\pi-x)-q(x)]y(x)=0;$$
$$\Delta=\begin{vmatrix} q(x)-\bar{q}(x) & \bar{q}(x)-q(\pi-x) \\ \bar{q}(\pi-x)-q(x) & q(\pi-x)-\bar{q}(\pi-x) \end{vmatrix}=0,$$
$$[q(x)-\bar{q}(x)]\cdot[q(\pi-x)-\bar{q}(\pi-x)]-$$
$$-[\bar{q}(x)-q(\pi-x)][\bar{q}(\pi-x)-q(x)]=0,$$
$$q(x)q(\pi-x)-q(x)\bar{q}(\pi-x)-\bar{q}(x)q(\pi-x)+\bar{q}(x)\bar{q}(\pi-x)=$$
$$=\bar{q}(x)\bar{q}(\pi-x)-\bar{q}(x)q(x)-q(\pi-x)\bar{q}(\pi-x)+q(\pi-x)q(x),$$
$$q(x)\bar{q}(\pi-x)+\bar{q}(x)q(\pi-x)=\bar{q}(x)q(x)+q(\pi-x)\bar{q}(\pi-x),$$
$$q(x)[\bar{q}(\pi-x)-\bar{q}(x)]+q(\pi-x)[\bar{q}(x)-\bar{q}(\pi-x)]=0,$$
$$[\bar{q}(x)-\bar{q}(\pi-x)][q(\pi-x)-q(x)]=$$
$$=|q(x)-q(\pi-x)|^2=0,=>\ q(x)=q(\pi-x).$$

From (4.6) we have

$$[q(x)-\bar{q}(x)][y(x)-y(\pi-x)]=0,=>\ q(x)-\bar{q}(x)=0.$$

The previous Lemmas 4.3 and 4.4 yield the following theorem.

Theorem 4.1. If

a) $\Delta_{13}\neq 0$;

b) $LP=PL^{+}$;

c) $QL=L^{+}Q$,

then

1) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)}=\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}=\frac{\Delta_{14}+\Delta_{34}}{\Delta_{24}}$;

2) $\overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)}=\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}=\frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}$;

3) $q(\pi-x)=q(x), \bar{q}(x)=q(x)$;

and the operators L and L^{+} have the forms

4) $$Ly=-y''+q(x)y,\ x\in(0,\pi);$$
$$\begin{cases} y(0)-y(\pi)-\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)]=0, \\ \Delta_{12}y'(0)+\Delta_{13}y(\pi)+\Delta_{14}y'(\pi)=0. \end{cases}$$

5) $$L^{+}z=-z''+\bar{q}(x)z,\ x\in(0,\pi);$$
$$\begin{cases} z(0)+z(\pi)+\frac{\overline{\Delta_{12}}-\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0)-z'(\pi)]=0, \\ \overline{\Delta_{13}}z(0)-\overline{\Delta_{32}}z'(0)-\overline{\Delta_{12}}z'(\pi)=0. \end{cases}$$

Further from the formulas $LP=PL^{+}$ we note that the operator $L_1=LP$ acts in the subspace $H_1=PH$, where $H=L^2(0,\pi)$. Assuming

$$u(x)=Py(x)=\frac{y(x)+y(\pi-x)}{2},$$

we have

$$u'(x)=\frac{y'(x)-y'(\pi-x)}{2}.$$

Then Theorem 4.1 implies that

$$L_1u=-u''+q(x)u,\qquad x\in\left(0,\frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}u(0) + (\Delta_{12} - \Delta_{14})u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases} \tag{4.7}$$

If $y \in D(L)$, then $v(x) = Qy \in D(L^{+})$, and

$$QLy = L^{+}Qy = L^{+}QQy = L_2 v = L^{+}v = -v''(x) + \bar{q}(x)v = -v''(x) + q(x)v.$$

From $Qy \in D(L^{+})$ it follows that

$$\overline{\Delta_{13}}\frac{y(0) - y(\pi)}{2} - (\overline{\Delta_{12}} + \overline{\Delta_{32}})\frac{y'(0) + y'(\pi)}{2} = 0,$$

$$\overline{\Delta_{13}}v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{32}})v'(0) = 0,$$

$$v(0) - \frac{\overline{\Delta_{12} + \Delta_{32}}}{\overline{\Delta_{13}}}v'(0) = 0,$$

$$v(0) - \frac{\Delta_{12} + \Delta_{32}}{\Delta_{13}}v'(0) = 0,$$

$$\Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0.$$

Thus,

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{4.8}$$

Equating coefficients of the boundary conditions (4.7) and (4.8), we have

$$\Delta_{12} - \Delta_{14} = -(\Delta_{12} + \Delta_{32}), => \ \Delta_{12} = \Delta_{14} - \Delta_{12} - \Delta_{32} =$$

$$= -(\Delta_{12} + \Delta_{32} - \Delta_{14}) = -\Delta_{34}.$$

Then the operators L_1 and L_2 have the following forms

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}u(0) - (\Delta_{12} + \Delta_{32})u'(0) = 0, \\ u\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

$$L_2 v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

If spectrum of the operator L is known, then, by Lemma 4.2, proved earlier, spectra of the operators L_1 and L_2 are known. Then, by Borg theorem, the operator L_2 is uniquely defined on the interval $\left[0, \frac{\pi}{2}\right]$, and, due to parity and periodicity of the function $q(x)$, on the whole interval $[0, \pi]$.

ОӘЖ 517.9

А.Ш.Шалданбаев[1], А.А.Шалданбаева[2], А.Бейсебаева[3],Б.А.Шалданбай[4]

[1]Халықаралық Silkway университеті, Шымкент қ., Казақстан;
[2,4]Аймақтық әлеуметтік-инновациялық университеті, Шымкент қ., Казақстан;
[3]М.О.Ауезов атындағы Оңтүстік Қазақстан мемлекеттік университеті, Шымкент қ., Казақстан

ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

Аннотация. Бұл еңбекте потенциалы симметриялы, нақты әрі үзіксіз, ал шекаралық шарттары ажырамайтын Штурм-Лиувилл операторын бір спектр арқылы анықтауға болатыны көрсетілді. Зерттеу әдісі бұрынғы әдістердің ешбіріне ұқсамайды, және ол оператордың ішкі симметриясына негізделген, ал ол өз кезегінде инвариантты кеңістіктердің салдары.

Түйін сөздер: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, ажырамайтын шекаралық шарттар, симметриялы потенциал, инвариантты кеңістіктер.

УДК 517.9

А.Ш.Шалданбаев[1], А.А.Шалданбаева[2] , А.Ж.Бейсебаева[3] , Б.А.Шалданбай[4]

[1]Международный университет Silkway, г. Шымкент, Казахстан;
[2,4]Региональный социально-инновационный университет, г. Шымкент, Казахстан;
[3]Южно-Казахстанский Государственный университет им.М.Ауезова, г. Шымкент, Казахстан

ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕ РАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

Аннотация. В данной работе доказана теорема единственности, по одному спектру, для оператора Штурма-Лиувилля с не разделенными краевыми условиями и вещественным непрерывным и симметричным потенциалом. Метод исследования отличается от всех известных методов, и основан на внутреннюю симметрию оператора, порожденного инвариантными подпространствами.

Ключевые слова: Оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства.

Information about authors:

Shaldanbayev A.Sh. – doctor of physico-mathematical Sciences, associate Professor, head of the center for mathematical modeling, «Silkway» International University, Shymkent; http://orcid.org/0000-0002-7577-8402

Shaldanbayeva A.A. - "Regional Social-Innovative University", Shymkent; https://orcid.org/0000-0003-2667-3097

Beisebayeva A.Zh. - M.Auezov South Kazakhstan State University, Shymkent; https://orcid.org/0000-0003-4839-9156

Shaldanbay B.A. - "Regional Social-Innovative University", Shymkent; https://orcid.org/0000-0003-2323-0119

REFERENCES

[1] Marchenko V. A. Sturm – Liouville operators and their applications. Birkh¨auser, 1986.
[2] Levitan B. M. Inverse Sturm – Liouville problems.Utrecht, VNU Sci. Press, 1987.
[3] P¨oschel J., Trubowitz E. Inverse Spectral Theory.New York, Academic Press, 1987.
[4] Freiling G., Yurko V. A. Inverse Sturm – Liouville Problems and their Applications. New York, NOVA Science Publ., 2001.
[5] Yurko V. A. Method of Spectral Mappings inthe Inverse Problem Theory. Inverse and Ill-posed Problems Series. Utrecht, VSP, 2002.
[6] Stankevich I. V. An inverse problem of spectral analysis for Hill's equation. Soviet Math. Dokl., 1970, vol. 11, pp. 582–586.
[7] Marchenko V. A., Ostrovskii I. V. A characterization of the spectrum of the Hill operator. Math. USSR-Sb., 1975, vol. 26, no. 4, pp. 493–554. DOI:10.1070/SM1975v026n04ABEH002493.
[8] Yurko V. A. An inverse problem for second order differential operators with regular boundary conditions.Math. Notes, 1975, vol. 18, no. 3–4, pp. 928–932. DOI: 10.1007/BF01153046.

[9] Yurko V. A. On a periodic boundary value problem.Differ. Equations and Theory of Functions, Saratov, Saratov Univ. Press, 1981, pp. 109–115 (inRussian).

[10] Yurko V. A. On recovering differential operators with nonseparated boundary conditions. Study in Math. and Appl., Ufa, Bashkir Univ. Press, 1981,pp. 55–58 (in Russian).

[11] Plaksina O. A. Inverse problems of spectral analysis for the Sturm– Liouville operators with nonseparated boundary conditions. Math. USSRSb.,1988, vol. 59, no. 1, pp. 1–23. DOI:10.1070/SM1988v059n01ABEH003121.

[12] Guseinov I. M., Gasymov M. G., Nabiev I. M. An inverse problem for the Sturm– Liouville operator with nonseparable self-adjoint boundary conditions.Siberian Math. J., 1990, vol. 31, no. 6, pp. 910–918.

[13] Guseinov I. M., Nabiev I. M. Solution of a class of inverse boundary-value Sturm– Liouville problems.Sb. Math., 1995, vol. 186, no. 5, pp. 661–674. DOI:10.1070/SM1995v186n05ABEH000035.

[14] Kargaev P., Korotyaev E. The inverse problem for the Hill operator, a direct approach. Invent. Math., 1997, vol. 129, no. 3, pp. 567–593.

[15] Yurko V. A. On differential operators with nonseparated boundary conditions. Funct. Anal. Appl., 1994, vol. 28, no. 4, pp. 295–297. DOI: 10.1007/BF01076118.

[16] Yurko V. A. The inverse spectral problem for differential operators with nonseparated boundary conditions.J. Math. Analysis Appl., 2000, vol. 250,no. 1, pp. 266–289.

[17] Freiling G., Yurko V. A. On the stability of constructing a potential in the central symmetry case. Applicable Analysis, 2011, vol. 90, no. 12,pp. 1819–1828.

[18] V. A. Yurko.On the inverse periodic problem for centrally symmetric potentials. WPI. Sarat. UN-TA. New. ser. Ser. Mathematics. Mechanics. Informatics. 2016. Vol. 16, vol. 1,p. 68-75.

[19] Hald O.H. The Inverse Sturm--Liouville Problem with symmetric Potentials ,Acta mathematica 1978,141:1,263 .

[20] Isaakson E. L., Trubowitz - Pure Appl. Math. 1983. V. 36. N 6. P. 763-783.

[21] Isaakson E. L., McKean H. P., Trubowitz - Pure Appl. Math. 1984. V. 37. N 1. P. 1-12.

[22] Esin Inan Eskitascioglu, Mehmet Acil. An inverse Sturm-Liouville problem with a generalized symmetric potential., Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 41, pp. 17. ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu.

[23] Martin Bohner, Hikmet Koyunbakanb.Inverse Problems for Sturm–Liouville Di_erence Equations. Filomat 30:5 (2016), 1297–1304 DOI 10.2298/FIL1605297B.

[24] Münevver Tuz.On inverse Sturm- Liouville problems with symmetric potentials., ITM Web of Conferences **13**, 01035 (2017) DOI: 10.1051/itmconf/20171301035.

[25] Ambartsumyan V.A. Uber eine Frage der Eigenwert-theorie, Zsch.f.Physik, 53(1929), 690-695.

[26] Borg G. Eine Umbehrung der Sturm – Liovillschen Eigenwertaufgabe, Acta Math., 78, №2(1946), 1-96.

[27] Levinson N. The inverse Sturm – Lioville problem, Math.Tidsskr. B., 1949, 25-30.

[28] Levitan B.M. On definition of the Sturm - Liouville operator by two spectra. Proceedings of the USSR Academy of Sciences, ser. Math., 28 (1964), 63-78.

[29] Levitan B.M. On definition of the Sturm - Liouville operator by one and two spectra. Proceedings of the USSR Academy of Sciences, ser. Math., 42. No. 1. 1964.

[30] Akylbayev M.I., Beysebayeva A. and Shaldanbayev A. Sh., On the Periodic Solution of the Goursat Problem for a Wave Equation of a Special Form with Variable Coefficients. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 1, Number 317 (2018), 34 – 50.

[31] Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh. and Beisebaeva A.Zh., The Spectral Decomposition of Cauchy Problem's Solution for Laplace Equation. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 5, Number 321 (2018), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10

[32] T.Sh. Kal'menov and A.Sh. Shaldanbaev, On a criterion of solvability of the inverse problem of heat conduction. Journal of Inverse and Ill-posed Problems. 18:5, (2010). 471_492.