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D.Serikbaev[1], N. Tokmagambetov[2]

[1,2]Al–Farabi Kazakh National University, Almaty, Kazakhstan;
[2]Ghent University, Ghent, Belgium;
[1,2]Institute of Mathematics and Mathematical Modeling,Almaty, Kazakhstan
serykbaev.daurenbek@gmail.com, niyaz.tokmagambetov@gmail.com

AN INVERSE PROBLEM FOR THE PSEUDO-PARABOLIC EQUATION FOR A STURM-LIOUVILLE OPERATOR

Abstract. A class of inverse problems for restoring the right-hand side of the pseudo-parabolic equation for Sturm–Liouville operator is considered. The inverse problem is to be well-posed in the sense of Hadamard whenever an overdetermination condition of the final temperature is given. Mathematical statements involve inverse problems for the pseudo-parabolic equation in which, solving the equation, we have to find the unknown right-hand side depending only on the space variable. We prove the existence and uniqueness of classical solutions to the problem. The proof of the existence and uniqueness results of the solutions is carried out by using L-Fourier analysis. The mentioned results are presented as well as for the fractional time pseudo–parabolic equation.Inverse problem of identifying the right hand side function of pseudo-parabolic equation from the local overdetermination condition, which has important applications in various areas of applied science and engineering, alsosuch problems are modeled using common homogeneous left-invariant hypoelliptic operators on common graded Lie groups.

Key words: Pseudo-parabolic equation, Sturm-Liouville operator, fractional Caputo derivative, inverse problem, well-posedness.

1 Introduction

The study of inverse problems for pseudo-parabolic equations goes back to 1980s. The first result [3] refers to the inverse problems of determining a source function f in the pseudo-parabolic equation

$$(u + \mathcal{L}_1 u)_t + \mathcal{L}_2 u = f, \tag{1}$$

with linear operators $\mathcal{L}_1$ and $\mathcal{L}_1$ of the second order, $\mathcal{L}_1 = \mathcal{L}_2$. Such equations arise in the models of the heat transfer, filtration in the fissured media, quasi-stationary processes in the crystalline semiconductor.

In this paper we consider pseudo–parabolic equation generated by Sturm–Liouville operator with Caputo time-fractional derivative. We study the following equation

$$\mathcal{D}_t^\alpha[u(t,x) + \mathcal{L}u(t,x)] + \mathcal{L}u(t,x) = f(x), \tag{2}$$

for $(t,x) \in \Omega = \{(t,x)|0 < t \le T < \infty, a \le x \le b\}$, where $\mathcal{D}_t^\alpha$ is the Caputo derivative and $\mathcal{L}$ is the Sturm–Liouville operator which are defined in the next section.

This paper is devoted to the inverse problems of determining a source function in the pseudo-parabolic equation (2) by using the L-Fourier method.

In a series of articles [7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19] some recent work has been done on inverse problems and spectral problems for the diffusion and anomalous diffusion equations.

2 Preliminaries

2.1 Sturm-Liouville problem

We first describe the setting of the Sturm-Liouville operator. Let $\mathcal{L}$ be an ordinary second order differential operator in $L^2(a,b)$ generated by the differential expression

$$\mathcal{L}(u) = -u_{xx}(x), a < x < b, \tag{3}$$

and boundary conditions

$$A_1 u'(b) + B_1 u(b) = 0, A_2 u'(a) + B_2 u(a) = 0, \tag{4}$$

where $A_1^2 + A_2^2 > 0, B_1^2 + B_2^2 > 0$, and $A_j, B_j, j = 1,2$, are some real numbers.

It is known [2] that the Sturm-Liouville problem for(3) with boundary conditions (4) is self-adjoint in $L^2(a,b)$. It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2(a,b)$. So we can denote eigenvalues of the operator $\mathcal{L}$ and their eigenfunctions accordingly by λ_ξ and $e_\xi(x)$. That say us for $e_\xi(x) \in L^2(a,b)$ following identity is hold:

$$\mathcal{L}e_\xi(x) = \lambda_\xi e_\xi(x), \lambda_\xi \in \mathbb{R}. \tag{5}$$

Where $\mathcal{I}$ is a countable set and $\forall\xi \in \mathcal{I}$.

2.2 Definitions of the Caputo fractional derivative

Definition 2.1 *[6] The Riemann-Liouville fractional integral I^α of order $\alpha > 0$ for an integrable function is defined by*

$$I^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s) ds, t \in [c,d],$$

where Γ denotes the Euler gamma function.

Definition 2.2 *[6] The Riemann-Liouville fractional derivative D^α of order $\alpha \in (0,1)$ of a continuous function is defined by*

$$D^\alpha[f](t) = \frac{d}{dt} I^\alpha[f](t), t \in [c,d].$$

Definition 2.3 *[6] The Caputo fractional derivative of order $0 < \alpha < 1$ of a differentiable function is defined by*

$$\mathcal{D}_*^\alpha[f](t) = D^\alpha[f'(t)], t \in [c,d].$$

3 Formulation of the problem

Problem 3.1 *We aim to find a couple of functions $(u(t,x), f(x))$ satisfying the equation (2) , under the conditions*

$$u(0,x) = \varphi(x), x \in [a,b] \tag{6}$$

$$u(T,x) = \psi(x), x \in [a,b]. \tag{7}$$

and the homogeneous boundary conditions(4).

By using $\mathcal{L}$–Fourier analysis we obtain existence and uniqueness results for this problem.

We say a solution of Problem 3.1 is a pair of functions $(u(t,x), f(x))$ such that they satisfy equation (2) and conditions (6) , (7) and (4) where $u(t,x) \in C^1([0,T]; C^2([a,b]))$ and $f(x) \in C([a,b])$.

4 Main results

For Problem 3.1, the following theorem holds.

Theorem 4.1 *Assume that* $\varphi(x), \psi(x) \in C^2[a,b]$. *Then the solution* $u(t,x) \in C^1([0,T], C^2([a,b])), f(x) \in C([a,b])$ *of the Problem 3.1 exists, is unique, and can be written in the form*

$$u(x,t) = \varphi(x) + \sum_{\xi \in \mathcal{I}} \frac{\left(\psi_\xi^{(2)} - \varphi_\xi^{(2)}\right)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)\right) e_\xi(x)}{\lambda_\xi \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)},$$

$$f(x) = -\varphi''(x) + \sum_{\xi \in \mathcal{I}} \frac{\left(\psi_\xi^{(2)} - \varphi_\xi^{(2)}\right) e_\xi(x)}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)}$$

where $\varphi_\xi^{(2)} = (\varphi'', e_\xi)_{L^2(0,l)}, \psi_\xi^{(2)} = (\psi'', e_\xi)_{L^2(0,l)}$ and $E_{\alpha,\beta}(\lambda t)$ is Mittag-Leffler type function (see [5]):

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}.$$

Proof. First of all, we start by proving an existence result. Let us look for functions $u(t,x)$ and $f(x)$ in the forms:

$$u(t,x) = \sum_{\xi \in \mathcal{I}} u_\xi(t) e_\xi(x), \tag{8}$$

and

$$f(x) = \sum_{\xi \in \mathcal{I}} f_\xi e_\xi(x), \tag{9}$$

where $u_\xi(t)$ and f_ξ are unknown. Substituting (8) and (9) into Problem 3.1 and using(5) we obtain the following problem for the functions $u_\xi(t)$ and for the constants $f_\xi, \xi \in \mathcal{I}$:

$$\begin{cases} \mathcal{D}^\alpha u_\xi(t) + \frac{\lambda_\xi}{1+\lambda_\xi} u_\xi(t) = \frac{f_\xi}{1+\lambda_\xi}, \\ u_\xi(0) = \varphi_\xi, \\ u_\xi(T) = \psi_\xi, \end{cases} \tag{10}$$

where φ_ξ, ψ_ξ are $\mathcal{L}$-Fourier coefficients of $\varphi(x)$ and $\psi(x)$:

$$\varphi_\xi = (\varphi, e_\xi)_{L^2(a,b)},$$

$$\psi_\xi = (\psi, e_\xi)_{L^2(a,b)}.$$

General solution of this problem:

$$u_\xi(t) = \frac{f_\xi}{\lambda_\xi} + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right), \tag{11}$$

where the constants C_ξ, f_ξ are unknown. By using initial and additional conditions we can find they. We first find C_ξ:

$$u_\xi(0) = \frac{f_\xi}{\lambda_\xi} + C_\xi = \varphi_\xi,$$

$$u_\xi(T) = \frac{f_\xi}{\lambda_\xi} + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right) = \psi_\xi,$$

$$\varphi_\xi - C_\xi + C_\xi E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right) = \psi_\xi.$$

Then

$$C_\xi = \frac{\varphi_\xi - \psi_\xi}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)}.$$

f_ξ is represented as

$$f_\xi = \lambda_\xi \varphi_\xi - \lambda_\xi C_\xi.$$

Substituting $f_\xi, u_\xi(t)$ into formulas (8) and (9), we find

$$u(x,t) = \varphi(x) + \sum_{\xi \in \mathcal{I}} C_\xi \left(E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right) - 1\right) e_\xi(x).$$

Using self-adjoint property of operator $\mathcal{L}$

$$\left(-\varphi'', e_\xi\right)_{L^2(a,b)} = \left(\varphi, -e_{\xi''}\right)_{L^2(a,b)}$$

and in respect that(5) we obtain

$$\left(\varphi, e_\xi\right)_{L^2(a,b)} = \frac{\left(-\varphi'', e_\xi\right)_{L^2(a,b)}}{\lambda_\xi},$$

and for $\psi(x)$ we can write analogously. Substituting these equality into formula of C_ξ we can get that

$$C_\xi = \frac{\psi_\xi^{(2)} - \varphi_\xi^{(2)}}{\lambda_\xi \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)}.$$

Then

$$u(t,x) \quad = \varphi(x) + \sum_{\xi \in \mathcal{I}} \frac{\left(\psi_\xi^{(2)} - \varphi_\xi^{(2)}\right)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} t^\alpha\right)\right) e_\xi(x)}{\lambda_\xi \left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)}, \tag{12}$$

$$f(x) = \mathcal{L}\varphi(x) + \sum_{\xi\in\mathcal{I}} \frac{\left(\psi_\xi^{(2)} - \varphi_\xi^{(2)}\right) e_\xi(x)}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)}. \quad (13)$$

The following Mittag-Leffler function's estimate is known by [12]:

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1+|z|}, arg(z) = \pi, |z| \to \infty. \quad (14)$$

Now, we show that $u(t,x) \in C^1([0,T]; C^2([a,b]))$, $f(x) \in C([a,b])$, that is

$$\| u \|_{C^1([0,T];C^2([a,b]))} = \max_{t\in[0,T]} \| u(t,\cdot) \|_{C^2([a,b])} + \max_{t\in[0,T]} \| \mathcal{D}_t^\alpha u(t,\cdot) \|_{C^2([a,b])} < \infty,$$

and

$$\| f \|_{C([a,b])} < \infty.$$

By using(14), we get the following estimates

$$\begin{aligned} |u(t,x)| &\lesssim |\varphi(x)| + \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{\lambda_\xi\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \\ &\lesssim |\varphi(x)| + \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{\lambda_\xi}, \end{aligned} \quad (15)$$

$$\begin{aligned} |f(x)| &\lesssim |\varphi''(x)| + \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)} \\ &\lesssim |\varphi''(x)| + \sum_{\xi\in\mathcal{I}} \left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|. \end{aligned} \quad (16)$$

Where, $L \lesssim$ Q $L denotes $L \leq$ CQ for some positive constant C independent of L and Q.

By supposition of the theorem we know $\varphi^{(2)}$ and $\psi^{(2)}$ are continuous on $[a,b]$. Then by the Bessel inequality for the trigonometric series (see [1]) and by the Weierstrass M-test (see [4]), series (15) and (16) converge absolutely and uniformly in the region $\overline{\Omega}$. Now we show

$$\begin{aligned} |u_{xx}(t,x)| &\lesssim |\varphi''(x)| + \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)} \\ &\lesssim |\varphi''(x)| + \sum_{\xi\in\mathcal{I}} \left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right| < \infty, \end{aligned}$$

$$\begin{aligned} |\mathcal{D}_t^\alpha u(t,x)| &\lesssim \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{(1+\lambda_\xi)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi} T^\alpha\right)\right)} \\ &\lesssim \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{1+\lambda_\xi} < \infty, \end{aligned}$$

$$|\mathcal{D}_t^\alpha u_{xx}(t,x)| \lesssim \sum_{\xi\in\mathcal{I}} \frac{\lambda_\xi\left(\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|\right)}{(1+\lambda_\xi)\left(1 - E_{\alpha,1}\left(-\frac{\lambda_\xi}{1+\lambda_\xi}T^\alpha\right)\right)}$$

$$\lesssim \sum_{\xi\in\mathcal{I}} \left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right| + \sum_{\xi\in\mathcal{I}} \frac{\left|\varphi_\xi^{(2)}\right| + \left|\psi_\xi^{(2)}\right|}{1+\lambda_\xi} < \infty.$$

Finally, we obtain

$$\| u \|_{C^1([0,T],C^2[a,b])} \le C < \infty, C = const,$$

and

$$\| f \|_{C([a,b])} < \infty.$$

Existence of the solution is proved.

Now, we start proving uniqueness of the solution

Let us suppose that $\{u_1(t,x), f_1(x)\}$ and $\{u_2(t,x), f_2(x)\}$are solution of the Problem 3.1. Then $u(t,x) = u_1(t,x) - u_2(t,x)$ and $f(x) = f_1(x) - f_2(x)$ are solution of following problem:

$$\mathcal{D}_t^\alpha[u(t,x) - u_{xx}(t,x)] - u_{xx}(t,x) = f(x), \tag{17}$$

$$u(0,x) = 0, \tag{18}$$

$$u(T,0) = 0. \tag{19}$$

By using (12) and (13) for (17) - (19) we easily see $u(x,t) \equiv 0, f(x) \equiv 0$. Uniqness of the solution of the Problem 3.1.

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Д. Серикбаев[1], Н. Токмагамбетов[2]

[1,2]Әл-ФарабиатындағыҚазҰУ, Алматы, Қазақстан;
[2]Гент университеті, Гент, Белгия;
[1,2]Математикажәнематематикалық модельдеу институты, Алматы, Қазақстан

ШТУРМ-ЛУИВИЛЛ ОПЕРАТОРЛЫ ПСЕВДО-ПАРАБОЛАЛЫҚ ТЕҢДЕУ ҮШІН КЕРІ ЕСЕП

Аннотация.Штурм-Лиувилл операторлы псевдо-параболалық теңдеудің оң жақ бөлігін қалпына келтіру кері есебі қарастылады. Соңғы уақыт моментіндегі температура алдын ала берілген жағдайда кері есеп әр уақытта Адамар мағынасында қисынды болуы керек. Математикалық тұжырымдарға сүйенсек, псевдо-параболалық теңдеу үшін кері есеп қатарына теңдеуді шешу барысында тек кеңістік айнымалыдан тәуелді теңдеудің оң жақ функциясын табу есебі жатады. Теңдеудің классикалық шешімі бар және жалғыз екендігі дәлелденеді, дәлелдеу L-Фурье талдауы арқылы жүргізіледі. Аталған нәтижелер уақыт бойынша бөлшек туындылы псевдо-параболалық теңдеу үшін көрсетілді. Алдын ала берілген локалдық шарт бойынша псевдо-параболалық теңдеудің оң жақ функциясын анықтау есебі, әр түрлі қолданбалы ғылым және де техника саласында маңызды қолданысы бар. Сонымен қатар, мұндай проблемалар жалпы біртекті сол-инвариантты гипоэллиптикалық операторлар арқылы біртекті Ли топтары бойынша модельденеді.

Түйін сөздер: Псевдопараболалық теңдеу, Штурм-Лиувилл операторы, бөлшек Капуто туындысы, кері есеп, есептің қисындылығы.

Д. Серикбаев[1], Н. Токмагамбетов[2]

[1,2]КазНУ им. Аль-Фараби, Алматы, Казахстан; [2]Гентский университет, Гент, Белгия;
[1,2]Институт математики и математического моделирования, Алматы, Казахстан

ОБРАТНАЯ ЗАДАЧА ДЛЯ ПСЕВДО-ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ ДЛЯ ОПЕРАТОРА ШТУРМА-ЛИУВИЛЯ

Аннотация. Рассматривается класс обратных задач восстановления правой части псевдопараболичес-кого уравнения для оператора Штурма-Лиувилля. Обратная задача должна быть корректной в смысле Адамара всякий раз, когда дается условие переопределения конечной температуры. Математические утверждения включают обратные задачи для псевдопараболического уравнения, в котором, решая уравнение, мы должны найти неизвестную правую часть, зависящую только от пространственной переменной. Доказано существование и единственность классических решений задачи. Доказательство результатов существования и единственности решений проводится с помощью L-анализа Фурье. Упомя-нутые результаты представлены также для псевдопараболического уравнения с дробным временем. Обратная задача идентификации функции правой части псевдопараболического уравнения из условия локального переопределения, имеють важные приложения в различных областях прикладной науки и техники, а так же эти проблемы моделируются на однородных группах Ли с помощью однородных левоинвариантных гипоэллиптических операторов.

Ключевые слова: Псевдопараболическое уравнение, оператор Штурма-Лиувилля, дробная производная Капуто, обратная задача, корректность.

Information about authors:

Tokmagambetov Niyaz – PhD doctor, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium;

Serikbaev Daurenbek – Doctorate, Al–Farabi Kazakh National University, Almaty, Kazakhstan.

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