N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2019.2518-1726.46**
Volume 4, Number 326 (2019), 76 – 82

UDK 517.958

B. Bekbolat[1], B. Kanguzhin[2], N. Tokmagambetov[3]

[1,2,3]Al-Farabi Kazakh National University, Almaty, Kazakhstan;
[3]Ghent University, Ghent, Belgium;
[1,2,3]Institute of Mathematics and Mathematical Modeling,Almaty, Kazakhstan
bekbolat@math.kz, kanbalta@mail.ru, niyaz.tokmagambetov@gmail.com

TO THE QUESTION OF A MULTIPOINT MIXED BOUNDARY VALUE PROBLEM FOR A WAVE EQUATION

Abstract. It is well known that some problems in mechanics and physics lead to partial differential equations of the hyperbolic type. A classical example of the hyperbolic type is wave equation. When posed, the task sometimes lacks the classical boundary condition and the need arises to have a nonlocal boundary condition. Aim our work is get D'Alembert formula for mixed boundary value problem generated by a wave equation. In the classical case, given D'Alembert formula for boundary value problem generated by a wave equation. In our case, we must give D'Alembert formula for mixed boundary value problem. For this, we consider ordinary differential operator $\mathcal{L}$ withnon–local boundary conditions. We search the solution of the wave equation like a sum with eigenfunction of the operator $\mathcal{L}$. There are we use that fact, that eigenfunction of the operator $\mathcal{L}$ is Riesz basis in $L^2(0,l)$. Through this method and calculation we get D'Alembert formula.
Key words: D'Alembert formula, wave equation, mixed boundary value problem, nonlocal boundary condition.

1 Introduction

It is well known that some problems in mechanics and physics lead to partial differential equations of the hyperbolic type. When posed, the task sometimes lacks the classical boundary condition and the need arises to have a nonlocal boundary condition (see, [2, 3, 4, 5]). A simple example of such nonlocal conditions are multipoint conditions relating the value of the solution at the boundary points with the values at some interior points. For example, we refer the reader to [6, 7, 8, 9, 10].

2 Main result

Let us consider mixed boundary value problem generated by the homogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, (x,t) \in S \tag{1}$$

with inhomogeneous initial data

$$U(x,0) = f(x), U_t(x,0) = F(x), x \in [0,l], \tag{2}$$

and with non–local conditions

$$U(0,t) = 0, \sum_{j=0}^{N} \alpha_j {U_x}'(x_j,t) = 0, \tag{3}$$

where $S = \{(x,t): 0 < x < l, t > 0\}$,

$$0 = x_0 < x_1 <\ldots< x_N = l, \alpha_0 \neq 0, \alpha_N \neq 0, \sum_{j=0}^{N} \alpha_j = 1,\ l < \infty,$$

the system of point $\{x_j\}_{j=0}^{N}$ on the segment $[0, l]$ is chosen such that the relation $\frac{x_j}{x_{j+1}}$ is a rational number for all $j \geq 0$.

Also consider an ordinary differential operator $\mathcal{L}$ with the expression

$$\mathcal{L}(y) \equiv -y''(x), 0 < x < b, \quad (4)$$

with non–local boundary conditions

$$y(0) = 0, \sum_{j=0}^{N} \alpha_j y'(x_j) = 0. \quad (5)$$

By $\{\lambda_k\}_{k=1}^{\infty}$ denote eigenvalues of $\mathcal{L}$, which are zeros of the characteristic function

$$\Phi(\lambda) = \sum_{j=0}^{N} \alpha_j cos\sqrt{\lambda}x_j.$$

Theorem 1 *A solution of the boundary value problem (1)–(3) has the form*

$$U(x,t) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{F}(\tau)d\tau.$$

Proof. The system of eigen- and associated functions has the form

$$\{X_{kn}(x) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}(\frac{sin\sqrt{\lambda}x}{\sqrt{\lambda}})|_{\lambda=\lambda_n}, k = 0,1,\ldots,m_n - 1\}_{n=1}^{\infty}, \quad (6)$$

which is the Riesz basis in $L^2(0, l)$ (see [1]). Here m_n is a multiplicity of the corresponding eigenvalue λ_n for all $n \in \mathbb{N}$.

Using this fact, let us prove solvability of the problem (1)-(3). The solution of the problem (1)-(3) we will seek in the view

$$U(x,t) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}(t)X_{kn}(x), \quad (7)$$

where d_{kn} is a coefficient of the Fourier decomposition of U, depends on second argument.

By differentiating twice respect to x (7), we get

$$\frac{\partial^2 U}{\partial x^2} = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}(t)\left[-\lambda_n X_{kn}(x) - X_{k-1,n}(x)\right] =$$

$$\sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} [-\lambda_n d_{kn}(t) - d_{k+1,n}(t)]X_{kn}(x), \quad (8)$$

where $d_{m_n,n} = 0$.

By differentiating twice respect to t (7), we have

$$\frac{\partial^2 U}{\partial t^2} = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n-1} d_{kn}''(t)X_{kn}(x), \quad (9)$$

By using the condition $U(0,t) = 0$, from (8)-(9) we take

$$d_{kn}''(t) = -\lambda_n d_{kn}(t) - d_{k+1,n}(t), k < m, where d_{m_n,n} = 0. \quad (10)$$

Use initial conditions (2):

$$U(x,0)=\sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1} d_{kn}(0)X_{kn}(x)=f(x),$$

$$U_t(x,0)=\sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1} d_{kn}'(0)X_{kn}(x)=F(x).$$

We calculate Fourier coefficients using the biorthogonal system of $\{X_{kn}(x)\}$ by formulas

$$\begin{aligned} d_{kn}(0)=d_{kn}^{f}=\int_0^b f(x)\overline{h_{kn}(x)}dx,\\ d_{kn}'(0)=d_{kn}^{F}=\int_0^b F(x)\overline{h_{kn}(x)}dx. \end{aligned} \qquad (11)$$

Indeed, (10) – (11) is a Cauchy problem, and it's solution has the form

$$d_{kn}(t)=d_{kn}^{f}cos\sqrt{\lambda_n}t+d_{kn}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\xi)}{\sqrt{\lambda_n}}d_{k+1,n}d\xi. \qquad (12)$$

Taking into account $d_{m_n,n}=0$, for $k=m_n-1$ we have

$$d_{m_n-1,n}(t)=d_{m_n-1,n}^{f}cos\sqrt{\lambda_n}t+d_{m_n-1,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}.$$

And for $k=m_n-2$ we have

$$d_{m_n-2,n}(t)=d_{m_n-2,n}^{f}cos\sqrt{\lambda_n}t+$$

$$d_{m_n-2,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda}}d_{m_n-1,n}(t)d\zeta$$

$$=d_{m_n-2,n}^{f}cos\sqrt{\lambda_n}t-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda_n}}d_{m_n-1,n}^{f}cos\sqrt{\lambda_n}td\zeta$$

$$+d_{m_n-2,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}-\int_0^t\frac{sin\sqrt{\lambda}(t-\zeta)}{\sqrt{\lambda_n}}d_{m_n-1,n}^{F}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}}d\zeta.$$

We denote

$$C_{0n}(t)=cos\sqrt{\lambda_n}t, S_{0n}(t)=\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}},$$

$$C_{kn}(t)=-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\zeta)}{\sqrt{\lambda_n}}cos\sqrt{\lambda_n}\zeta d\zeta,$$

$$S_{kn}(t)=-\int_0^t\frac{sin\sqrt{\lambda_n}(t-\zeta)}{\sqrt{\lambda_n}}\frac{sin\sqrt{\lambda_n}\zeta}{\sqrt{\lambda_n}}d\zeta,$$

and

$$\begin{aligned} C_{k+1,n}(t)=\frac{1}{k!}\frac{\partial^{k+1}}{\partial\lambda^{k+1}}C_{0n}(t),\\ S_{k+1,n}(t)=\frac{1}{k!}\frac{\partial^{k+1}}{\partial\lambda^{k+1}}S_{0n}(t). \end{aligned} \qquad (13)$$

Therefore

$$d_{m_n-2,n}(t) = d^f_{m_n-2,n} cos\sqrt{\lambda_n}t + d^F_{m_n-2,n}\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} -$$

$$d^f_{m_n-1,n}C_{1,n}(t) - d^F_{m_n-1,n}S_{1,n}(t).$$

Analogically for another k the function $d_{kn}(t)$ can be written as

$$d_{kn}(t) = \sum_{j=k}^{m_n-1} [d^f_{jn}C_{j-k,n}(t) + d^F_{jn}S_{j-k,n}(t)]. \quad (14)$$

By substituting found $d_{kn}(t)$ in (7), we get

$$U(x,t) = \sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1}\sum_{j=k}^{m_n-1} [d^f_{jn}C_{j-k,n}(t) + d^F_{jn}S_{j-k,n}(t)]X_{kn}(x)$$

$$= \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} \{d^f_{jn}\sum_{k=0}^{j} C_{j-k,n}(t)X_{kn}(x) + d^F_{jn}\sum_{k=0}^{j} S_{j-k,n}(t)X_{kn}(x)\}.$$

By virtue of (6) and (13), we have a new presentation

$$U(x,t) = \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} \{d^f_{jn}\sum_{k=0}^{j}\frac{1}{(j-k)!}\frac{\partial^{j-k}}{\partial\lambda_n^{j-k}}(cos\sqrt{\lambda_n}t)\frac{1}{k!}\frac{\partial^k}{\partial\lambda_n^k}(\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}})$$

$$+d^F_{jn}\sum_{k=0}^{j}\frac{1}{(j-k)!}\frac{\partial^{j-k}}{\partial\lambda_n^{j-k}}(\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}})\frac{1}{k!}\frac{\partial^k}{\partial\lambda_n^k}(\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})\}.$$

Syne $\frac{sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} = \int_0^t cos\sqrt{\lambda_n}\tau d\tau$, then

$$U(x,t) = \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d^f_{jn}\frac{1}{j!}\sum_{k=0}^{j}\frac{j!}{k!\,(j-k)!}\frac{\partial^{j-k}}{\partial\lambda_n^{j-k}}(cos\sqrt{\lambda_n}t)\frac{\partial^k}{\partial\lambda_n^k}(\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$+\int_0^t d\tau\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d^F_{jn}\frac{1}{j!}\sum_{k=0}^{j}\frac{j!}{k!\,(j-k)!}\frac{\partial^{j-k}}{\partial\lambda_n^{j-k}}(cos\sqrt{\lambda_n}\tau)\frac{\partial^k}{\partial\lambda_n^k}(\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}).$$

And using $\sum_{k=0}^{j} C_k^j U^{(k)}(x)V^{(j-k)}(x) = (UV)^{(j)}$, we take

$$U(x,t) = \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d^f_{jn}\frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}(cos\sqrt{\lambda_n}t\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$+\int_0^t d\tau\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d^F_{jn}\frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}(cos\sqrt{\lambda_n}\tau\frac{sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}})$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x+t)}{\sqrt{\lambda_n}}\right)$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x-t)}{\sqrt{\lambda_n}}\right)$$

$$+ \frac{1}{2}\int_0^t d\tau \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x+\tau)}{\sqrt{\lambda_n}}\right)$$

$$+ \frac{1}{2}\int_0^t d\tau \sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} \frac{1}{j!}\frac{\partial^j}{\partial\lambda_n^j}\left(\frac{\sin\sqrt{\lambda_n}(x-\tau)}{\sqrt{\lambda_n}}\right).$$

So,

$$\begin{aligned} U(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x+t) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x-t) \\ &+ \frac{1}{2}\int_x^{x+t}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau)d\tau + \frac{1}{2}\int_{x-t}^{x}\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau)d\tau. \end{aligned} \tag{15}$$

When $0 \le x - t \le x + t \le b$, then sums of series are coincides with the initial data

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x+t) = f(x+t),$$

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{f} X_{jn}(x-t) = f(x-t),$$

$$\sum_{n=1}^{\infty}\sum_{j=0}^{m_n-1} d_{jn}^{F} X_{jn}(\tau) = F(\tau).$$

For $0 \le x - t \le x + t \le b$ the solution is well–known D'Alembert formula

$$U(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} F(\tau)d\tau.$$

Thus, the formula (15) can be interpreted as a generalization of the D'Alembert formula for arbitrary $0 \le x \le b, t \ge 0$.

As a result, we conclude

$$U(x,t) = \frac{\tilde{f}(x+t) + \tilde{f}(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{F}(\tau)d\tau,$$

where $\tilde{f}(x)$ and $\tilde{F}(x)$ extended from the segment $[0, l]$ to the whole real axis by the analytical continuation of the basis system

$$\left\{X_{kn}(x) = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\right\}.$$

Since $\{X_{kn}(x)\}$ is defined on $(-\infty, +\infty)$, then we can always to continue functions $f(x)$ and $F(x)$ outside of the segment $[0, l]$. From [1] follows that the system $\{X_{kn}(x)\}$ is Riesz basis in $L_2(-\infty, +\infty)$.

ACKNOWLEDGEMENTS

The authors were supported by the Ministry of Education and Science of the Republic of Kazakhstan (MESRK) Grant AP05130994. No new data was collected or generated during the course of research.

Б. Бекболат[1], Б. Кангужин[2], Н. Токмагамбетов[3],

[1,2,3]Әл-Фараби атындағы ҚазҰУ, Алматы, Қазақстан;
[3]Гент университеті, Гент, Белгия;
[1,2,3]Математика және математикалық модельдеу институты, Алматы, Қазақстан

ТОЛҚЫН ТЕҢДЕУІ ҮШІН КӨП НҮКТЕЛІ АРАЛАС ШЕКАРАЛЫҚ ЕСЕП

Аннотация.Механика мен физиканың кейбір есептері дербес туындылы дифференциалдық теңдеулердің гиперболалық түріне алып келетіні белгілі. Гиперболалық теңдеудің классикалық өкіліне толқын теңдеуі жатады. Кейде есеп шығару барысында тек шекаралық шарт жеткіліксіз болады, сондықтан қосымша локалды емес шекаралық шартта қолданылады. Біздің жұмыстың мақсаты толқын теңдеуі арқылы туындаған аралас шекаралық есептің Даламбер формуласын табу. Классикада толқын теңдеуі арқылы туындаған шекаралық есеп үшін Даламбер формуласы берілген. Біз аралас шекаралық есеп үшін Даламбер формуласын табу керекпіз. Ол үшін біз қосымша $\mathcal{L}$дифференциалдық операторын қарастырамыз. Себебі біз шешімді $\mathcal{L}$ операторының меншікті функциялары арқылы құрылған қатар арқылы іздейміз. Біз бұл жерде $\mathcal{L}$ операторының меншікті функциялары $L^2(0,l)$кеңістігінде Рисс базисы болатынын пайдаланамз. Біз осы әдіс және есептеулер арқылы Даламбер формуласын аламыз.

Түйін сөздер:Даламбер формуласы, толқын теңдеуі, аралас шекаралық есеп, локалды емес шекаралық шарт.

Б. Бекболат[1], Б. Кангужин[2], Н. Токмагамбетов[3]

[1,2,3]КазНУ им. Аль-Фараби, Алматы, Казахстан;
[3]Гентский университет, Гент, Белгия;
[1,2,3]Институт математики и математического моделирования, Алматы, Казахстан

О МНОГОТОЧЕЧНОЙ ЗАДАЧЕ СМЕШАННОЙ ГРАНИЦЫ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ

Аннотация.Хорошо известно, что некоторые проблемы механики и физики приводят к уравнениям в частных производных гиперболического типа. Классическим примером гиперболического типа является волновое уравнение. При постановке задачи иногда не хватает классического граничного условия, и возникает необходимость иметь нелокальное граничное условие. Цель нашей работы - получить формулу Даламбера для смешанной краевой задачи, порожденной волновым уравнением. В классическом случае дана формула Даламбера для краевой задачи, порожденная волновым уравнением. В нашем случае мы должны дать формулу Даламбера для смешанной краевой задачи. Для этого рассмотрим обыкновенный дифференциальный оператор $\mathcal{L}$ с нелокальными граничными условиями. Мы ищем решение волнового уравнения как сумму с собственной функцией оператора $\mathcal{L}$. Мы используем тот факт, что собственная функция оператора $\mathcal{L}$ является базисом Рисса в $L^2(0,l)$. С помощью этого метода и расчета мы получаем формулу Даламбера.

Ключевые слова:Формула Даламбера, волновое уравнение, смешанная краевая задача, нелокальные краевые условия.

Information about authors:

Kanguzhin Baltabek – Doctor of Physical and Mathematical Sciences, Professor,Al–Farabi Kazakh National University, Almaty, Kazakhstan;

Tokmagambetov Niyaz – PhD doctor, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium;

Bekbolat Bayan – Doctorate, al-Farabi Kazakh National University, Almaty, Kazakhstan.

REFERENCES

[1] Gubreev G. M., *Spectral analysis of biorthogonal expansions of functions, and exponential series*, Izv. Akad. Nauk SSSR Ser. Mat.,Vol.53, №6. 1989. P.1236–1268.

[2] Kanguzhin B. E., Anijarov A. A., *Well-Posed Problems for the Laplace Operator in a Punctured Disk*, Mathematical Notes.Vol.89, №6. 2011. P.819–829.

[3] Kanguzhin B. E., Tokmagambetov N. E., *On Regularized Trace Formulas for a Well-Posed Perturbation of the m-Laplace Operator.* Differential Equations. **51**:1583-1588 (2015).

[4] Kanguzhin B. E., Tokmagambetov N. E., *Resolvents of well-posed problems for finite-rank perturbations of the polyharmonic operator in a punctured domain.* Siberian Mathematical Journal. **57**:265-273 (2016)

[5] Kanguzhin B. E., Tokmagambetov N. E., *A regularized trace formula for a well-perturbed Laplace operator*, Doklady Mathematics.Vol.91, №1. 2015.P.1–4.

[6] Bastys A., Ivanauskas F., Sapagovas M., *An explicit solution of a parabolic equation with nonlocal boundary conditions*, Lithuanian Mathematical Journal. –Vol.45, №3. 2005. P.257–271.

[7] Zhanbing Bai, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear analysis-theory methods and applications. Vol.72, №2. 2005. P.916–924.

[8] Webb J. R. L., Gennaro I., *Positive solutions of nonlocal boundary value problems: A unified approach*, Jornal of the London Mathematical society series. Vol.74, №3. 2006. P.973–693.

[9] Webb J. R. L., Gennaro I., *Positive solutions of nonlocal boundary value problems involving integral conditions*, Node-nonlinear differential equations and applications. Vol. 15, №1-2. 2008. P.45–67.

[10] Goodrich, Christopher S., *Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions*, Computers and mathematics with applications. Vol.61, №2. 2011. P.191–202.

[11] Seitmuratov A., Zharmenova B., Dauitbayeva A., Bekmuratova A. K., Tulegenova E., Ussenova G.,*Numerical analysis of the solution of some oscillation problems by the decomposition method*,News of the national academy of sciences of the republic of Kazakhstan,Series physic-mathematical. Vol.323, №1.2019. P.28–37.ISSN 1991-346X https://doi.org/10.32014/2019.2518-1726.4