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ON THE SPECTRAL PROPERTIES OF A WAVE OPERATOR PERTURBED BY A LOWER-ORDER TERM

Abstract. The incorrectness of the minimal wave operator is well known, since zero is an infinite-to-one eigenvalue for it. As our study showed, the situation changes if the operator is perturbed by a low-order term containing the spectral parameter as a coefficient, and eventually the problem takes the form of a beam of operators. The resulting beam of operators is easily factorized by first-order functional-differential operators which spectral properties are easily studied by the classical method of separation of variables. Direct application of the method of separation of variables to the original beam of operators encounters the insurmountable difficulties.

Keywords: deviating argument, beam of operators, strong solvability, spectrum, functional - differential operator.

Introduction.

Investigations of the Dirichlet problem for the string oscillation equation in a bounded domain go back to J. Hadamard [1], who for the first time noted nonuniqueness of solution of the Dirichlet problem for a wave equation in a rectangle. Burgin and Duffin [2] considered the Dirichlet problem for the equation $Uxx = Utt$ in the rectangle $\{0< x < X ; 0<t<T\}$. It was described that the nonuniqueness of a solution in a specified space arises if and only if the ratio X/T is rational. By using Laplace transformation, they showed that if the number X/T is irrational, then there is the uniqueness of the solution of the problem in the class of continuously differentiable functions with the second derivatives integrable according to Lebesgue.

Later these results were refined and generalized by various authors (see, for example, [3], [4], [5], [6]). S.L. Sobolev [7] constructed an example of a well-posed boundary-value problem in a rectangle for a hyperbolic system of equations. Yu.M. Berezanskii [8] constructed a class of regions with angles, a change in the domain inside which leads to a continuous change in the solution of the Dirichlet problem.

For domains with a smooth boundary in smooth spaces, only the question of the uniqueness of the solution of the Dirichlet problem was studied (see, for example, the paper of Aleksandryan [9]). In paper [3] V.I. Arnol'd, applying his results on the mapping of the circle into itself, refines the results of [2], indicating that the proof of theorems on the existence of classical solutions of the Dirichlet problem can be carried over to the case of an ellipse. A number of studies by T.Sh. Kal'menov and M.A. Sadybekov are also devoted to boundary value problems of hyperbolic equations [10] - [12].

In [13], using the new general method, the properties of solutions of the Cauchy problem, as well as of the first, second and third boundary-value problems in the disk for a second-order hyperbolic equation with constant coefficients are investigated. The application of this method to higher-order equations can be found in [14]. A new and relatively simple method for constructing a system of polynomial solutions of the Dirichlet problem for second-order hyperbolic equations with constant coefficients in the disk is proposed in [15], and it is also proposed to construct a complete set of eigenfunctions for the Dirichlet problem for the string oscillation equation. The eigenfunctions constructed in this paper coincide with the eigenfunctions constructed earlier in the work of R.A. Aleksandaryan [9]. In this paper, the spectral properties of a beam of operators with a wave operator in the principal part were studied by the methods of [16]–[21].

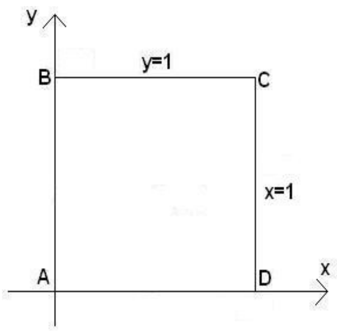

Figure 1

1. Setting of a Problem.

Let $\Omega \subset R^2$ be a quadrangle bounded by following segments: $AB: 0 \le y \le 1, x = 0$; $BC: 0 \le x \le 1$, $y = 1$; $CD: 0 \le y \le 1$, $x = 1$; $DA: 0 \le x \le 1$, $y = 0$. /See fig.1/

We denote by $C^{1,1}(\Omega)$ the set of functions $u(x,t)$ that are twice continuously differentiable with respect to the variables x and t. The boundary of the area Ω is a set of segments $\Gamma = AB \cup BC \cup CD \cup DA$.

We investigate the spectral properties of the operator beam

$$u_{xx} - u_{yy} = -2\lambda u_x + \lambda^2 u, \tag{1}$$

$$u\big|_{y=0} = 0, \tag{2}$$

$$u\big|_{x=0} = \alpha u\big|_{x=1}, \quad |\alpha| = 1. \tag{3}$$

2. Research Methods.

We shall need the following lemmas.

Lemma 2.1. [22]. Let A be a densely defined operator in a Hilbert space H. Then

(a) A^* exists and is closed;

(b) A admits a closure if and only if $D(A^*)$ is dense in H, and in this case $\overline{A} = A^{**}$.

Lemma 2.2. The set of functions that are finite in the domain Ω is dense in the space $L^2(\Omega)$ [23].

Lemma 2.3. If the symmetric operator A has a complete system of eigenvectors, then the closure of this operator $\overline{A}$ is self-adjoint in H, in other words, the operator A is an essentially self-adjoint in H [22].

Lemma 2.4. Operator

$$Lu = iu_x(x,y) + u_y(x,1-y) \tag{4}$$

$$D(L) = \{u \in C^{1,1}(\Omega) \cap C(\overline{\Omega}); u\big|_{y=0} = 0, u\big|_{x=0} = \alpha u\big|_{x=1}, |\alpha| = 1\}. \tag{5}$$

is a symmetric operator in space $L^2(\Omega)$.

{**The proof** is omitted, because it is done in the standard way}.

Proof. Let $u, v \in D(L)$, then

$$(Lu, v) = \int_0^1\int_0^1 [iu_x(x,y) + u_y(x,1-y)]\cdot \overline{v}(x,y)dxdy = \int_0^1\int_0^1 iu_x(x,y)\overline{v}(x,y)dxdy +$$

$$\int_0^1\int_0^1 u_y(x,1-y)\overline{v}(x,y)dxdy = J_1 + J_2 \ .$$

Using Fubini's theorem and integrating by parts, we transform the integrals J_1, J_2.

$$J_1 = \int_0^1\int_0^1 iu_x(x,y)\overline{v}(x,y)dxdy = \int_0^1\left[\int_0^1 iu_x(x,y)\overline{v}(x,y)dx\right]dy = \int_0^1\left[\int_0^1 i\overline{v}(x,y)d_x u\right]dy =$$

$$\int_0^1\left[i\overline{v}(x,y)u(x,y)\Big|_0^1 - i\int_0^1 u(x,y)\overline{v}_x(x,y)dx\right]dy = \int_0^1\left[\int_0^1 u(x,y)\overline{iv_x(x,y)}dx\right]dy =$$

$$\int_0^1\int_0^1 u(x,y)\overline{iv_x(x,y)}dxdy;$$

$$J_2 = \int_0^1\int_0^1 u_y(x,1-y)\overline{v}(x,y)dxdy = \int_0^1\left[\int_0^1 u_y(x,1-y)\overline{v}(x,y)dy\right]dx = \int_0^1\left[-\int_0^1 \overline{v}(x,y)d_y u(x,1-y)\right]dx =$$

$$\int_0^1\left[-\overline{v}(x,y)u(x,1-y)\Big|_0^1 + \int_0^1 u(x,1-y)\overline{v}_y(x,y)dy\right]dx = \int_0^1\left[\int_0^1 u(x,1-y)\overline{v_y}(x,y)dy\right]dx =$$

$$\int_0^1\int_0^1 u(x,y)\overline{v_y}(x,1-y)dxdy;$$

Consequently, $$(Lu, v) = \int_0^1\int_0^1 u(x,y)\overline{[iv_x(x,y) + v_y(x,1-y)]}dxdy = (u, Lv)$$

Lemma 2.5. The spectral problem

$$Lw = -w''(y) = \nu^2 w(y),$$
$$w(0) = w'(1) = 0$$

has an infinite set of eigenvalues

$$\nu_n = n\pi - \frac{\pi}{2}, \quad n = 1,2,...$$

and corresponding eigenfunctions

$$w_n(y) = \sqrt{2}\sin\left(n\pi - \frac{\pi}{2}\right)y, \ n = 1,2,...,$$

which form an orthonormal basis of the space $L^2(0,1)$ [24].

Lemma 2.6. The spectral problem

$$iv_x = \mu v(x), v(0) = \alpha v(1), |\alpha| = 1 \tag{6}$$

has an infinite set of real eigenvalues

$$\mu_m = arg\alpha + 2m\pi i, \ m = 0, \pm 1, \pm 2, ... \tag{7}$$

and their corresponding eigenfunctions

$$v_m(x) = exp[-i(arg\alpha + 2m\pi)x],\ m = 0, \pm 1, \pm 2, \dots \tag{8}$$

which form an orthonormal basis of space $L^2(0,1)$ [16].

Lemma 2.7. If the orthogonal systems $\{\Phi_n(x)\}$ and $\{\psi_n(x)\}$, $n = 1,2, \dots$ are complete in space $L^2(0,1)$, then their product $\{\Phi_n(x) \cdot \psi_n(x)\}$, $m, n = 1,2, \dots$ is complete in the space $L^2(0,1)$, where $\Omega = [0,1] \times [0,1]$ [25].

3. Results of the research.

Theorem 1. The boundary value problem

$$Lu = iu_x(x,y) + u_y(x,1-y) = f(x,y), \tag{9}$$

$$u\big|_{y=0} = 0,\ u\big|_{x=0} = \alpha u\big|_{x=1}, \quad |\alpha| = 1, \tag{10}$$

has an infinite set of real eigenvalues

$$\lambda_{mn} = \arg\alpha + 2m\pi + (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right),\ ,\ ,\ n = 1,2,\dots\ m = 0,\pm 1,\pm 2,\dots \tag{11}$$

and the corresponding eigenfunctions

$$u_{mn}(x,y) = \sqrt{2}\exp[-i(\arg\alpha + 2m\pi)x] \cdot \sin\left(n\pi - \frac{\pi}{2}\right)y \tag{12}$$

which form an orthonormal basis of space $L^2(0,1)$.

Proof. Let $Su(x,y) = u(x,1-y)$, then

$$iu_x(x,y) + u_y(x,1-y) = \left(i\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}\right)u(x,y)$$

we use this formula in the calculations.

Let $u_{mn}(x,y) = \sqrt{2}\exp[-i(\arg\alpha + 2m\pi)x] \cdot \sin\left(n\pi - \frac{\pi}{2}\right)y$, $n = 1,2,\dots$, $m = 0,\pm 1,\pm 2,\dots$.

Then the following formulas hold:

$$i\frac{\partial}{\partial x}u_{mn}(x,y) = (\arg\alpha + 2m\pi)u_{mn}(x,y);$$

$$S\frac{\partial}{\partial y}u_{mn}(x,y) = \sqrt{2}\left(n\pi - \frac{\pi}{2}\right)\exp[-i(\arg\alpha + 2m\pi)] \cdot \cos\left(n\pi - \frac{\pi}{2}\right)(1-y) =$$

$$\sqrt{2}\left(n\pi - \frac{\pi}{2}\right)\exp[-i(\arg\alpha + 2m\pi)x] \cdot (-1)^{n+1}\sin\left(n\pi - \frac{\pi}{2}\right)y = (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right)u_{mn}(x,y),$$

$$\left(i\frac{\partial}{\partial x} + S\frac{\partial}{\partial y}\right)u_{mn}(x,y) = \left[\arg\alpha + 2m\pi + (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right)\right]u_{mn}(x,y).$$

Consequently, $$i\frac{\partial}{\partial x}u_{mn}(x,y) + u_{mny}(x,1-y) = \lambda_{mn}u_{mn}(x,y),$$

where

$$\lambda_{mn} = \arg\alpha + 2m\pi + (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right),\ n = 1,2,\dots,\ m = 0,\pm 1,\pm 2,\dots.$$

Theorem 2. The operator

$$Lu = iu_x(x,y) + u_y(x,1-y), \tag{13}$$

$$D(L) = \{u \in C^{1,1}(\Omega) \cap C(\overline{\Omega}); u|_{y=0} = 0, u|_{x=0} = \alpha u|_{x=1}, |\alpha| = 1\} \tag{14}$$

is essentially self-adjoint in the space $L^2(\Omega)$.

Proof. This theorem is a simple consequence of Theorem 1, Lemma 2.3 and Lemma 2.4.

Up to the present time, we have not deliberately spoken about the spectrum of the operator L, because this concept is inherent only in a closed operator, and our operator has until now been not closed. In virtue of the Theorem 2 the equality $\overline{L} = L^*$ holds.

Further, by an operator L we mean the closure of the operator (13) – (14) and investigate its spectrum. The eigenvalues of this operator have the form:

$$\lambda_{mn} = \arg\alpha + 2m\pi + (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right),\ n = 1,2,...,\ m = 0,\pm1,\pm2,... .$$

a) Let $n = 2k$, $k = 1,2,...$,then

$$\lambda_{m,2k} = \arg\alpha + 2m\pi - \left(2k\pi - \frac{\pi}{2}\right) = \pi\left[\frac{\arg\alpha}{\pi} + 2(m-k) + \frac{1}{2}\right] = \pi\left[\frac{\arg\alpha}{\pi} + \frac{1}{2} + 2(m-k)\right].$$

The value $2(m-k)$ runs through all even numbers, the argument α lies within $0 \le \arg\alpha \prec 2\pi$, therefore

$$0 \le \frac{\arg\alpha}{\pi} \prec 2,\quad \frac{1}{2} \le \frac{\arg\alpha}{\pi} + \frac{1}{2} \prec \frac{5}{2}.$$

Between the numbers $\frac{1}{2}$ and $\frac{5}{2}$ there is only one even number 2, which is reached at $\frac{\arg\alpha}{\pi} = \frac{3}{2}$.

b) Let $n = 2k-1$, $k = 1,2,...$, then

$$\lambda_{m,2k-1} = \arg\alpha + 2m\pi(2k-1)\pi - \frac{\pi}{2} = \pi\left[\frac{\arg\alpha}{\pi} + 2m + 2k - 1 - \frac{1}{2}\right] = \pi\left[\frac{\arg\alpha}{\pi} - \frac{3}{2} + 2(m+k)\right];$$

The quantity $2(m+k)$ runs through all even integers. The following inequality holds:

$$-\frac{3}{2} \le \frac{\arg\alpha}{\pi} - \frac{3}{2} \prec \frac{1}{2}.$$

Between the numbers $-\frac{3}{2}$ and $\frac{1}{2}$ there is only one even number 0, which is reached when $\frac{\arg\alpha}{\pi} = \frac{3}{2}$.

Theorem 3. The spectrum of the operator (13)-(14) consists of two series of infinite eigenvalues:

a) $\lambda_m^{(1)} = \pi\left[\frac{\arg\alpha}{\pi} + \frac{1}{2} + 2m\right], m = 0,\pm1,\pm2,...;$

b) $\lambda_m^{(2)} = \pi\left[\frac{\arg\alpha}{\pi} - \frac{3}{2} + 2m\right],\ m = 0,\pm1,\pm2,...$

i.e. each of these values is taken an infinite number of times, the corresponding eigenfunctions form an orthonormal basis of the space $L^2(\Omega)$.

The inverse operator L^{-1} exists if and only if

$$\frac{\arg\alpha}{\pi} \neq \frac{3}{2} \qquad (15)$$

Theorem 4. The boundary-value problem

$$Lu = iu_x(x,y) + u_y(x,1-y) = f(x,y), (x,y) \in \Omega \ ;$$
$$u\big|_{y=0} = 0,\ u\big|_{x=0} = \alpha u\big|_{x=1}, \quad |\alpha| = 1,$$

is strongly solvable in space $L^2(\Omega)$ if and only if

$$\frac{\arg\alpha}{\pi} \neq \frac{3}{2}.$$

When condition (15) is satisfied, the inverse operator $(\overline{L})^{-1}$ exists, is bounded, but not compact, since there is a continuous spectrum of the operator $\overline{L}$.

Theorem 5. The spectral problem (1)-(3) has an infinite set of eigenvalues

$$\lambda_{mn} = \arg\alpha + 2m\pi + (-1)^{n+1}\left(n\pi - \frac{\pi}{2}\right),\ n = 1,2,...,\ m = 0,\pm1,\pm2,...,$$

and the corresponding eigenfunctions

$$u_{mn}(x,y) = \sqrt{2}\exp[-i(\arg\alpha + 2m\pi)x]\cdot\sin\left(n\pi - \frac{\pi}{2}\right)y,$$

which form an orthonormal basis of space $L_2(\Omega)$.

The spectrum of the beam of operators (1)-(3) consists of two series of infinite eigenvalues:

a) $\lambda_m^{(1)} = \pi\left[\frac{\arg\alpha}{\pi} + \frac{1}{2} + 2m\right], m = 0,\pm1,\pm2,...;$

b) $\lambda_m^{(2)} = \pi\left[\frac{\arg\alpha}{\pi} - \frac{3}{2} + 2m\right],\ m = 0,\pm1,\pm2,...$

each of these values is taken an infinite number of times, the corresponding eigenfunctions form an orthonormal basis of the space $L^2(\Omega)$.

The proof of the theorem follows easily from the above Theorems 1 - 4. For this it is sufficient to note that the equation coincides with the equation (1), where

$$\mathrm{Lu} = \mathrm{iu_x}(\mathrm{x},\mathrm{y}) + \mathrm{u_y}(\mathrm{x},1-\mathrm{y}) = \mathrm{i}\left(\frac{\partial}{\partial \mathrm{x}} + \mathrm{S}\frac{\partial}{\partial \mathrm{x}}\right)\mathrm{u}(\mathrm{x},\mathrm{y}),$$

and $\mathrm{Su}(\mathrm{x},\mathrm{y}) = \mathrm{u}(\mathrm{x},1-\mathrm{y})$.

4. Conclusions.

Note that the operator (13)-(14) is a two-dimensional generalization of the operator discussed in [16] - [17], which has found application to singularly perturbed Cauchy problem [18], to the operator of the heat conductivity in [19], and to the ill-posed problems of mathematical physics in [20] - [21], and Volterra problems in [26] - [27].

The spectral properties of the wave operator change dramatically if it is perturbed by the low order term containing the spectral parameter, in particular, it turns out to be reversible for certain values of the coefficient of the boundary condition. Zero is an infinite eigenvalue of the Dirichlet problem of the wave equation. Adding a low-order term with a spectral parameter and expanding the domain of definition change the situation, the operator becomes reversible.

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КІШІ МҮШЕЛІ ТОЛҚЫН ОПЕРАТОРЫНЫҢ СПЕКТРӘЛДІК ҚАСИЕТТЕРІ

Аннотация. Толқындық кішік операторды жайсыз екені көпке мәлім, себебі нөл нүктесі шексіз еселі меншікті мән.Біздің зерттеулеріміздің көрсетуінше,жағдайды өзгертуге болады, бұл, үшін операторды спектрәлді параметрге көбейтілген кіші мүшемен тітіркендіру жеткілікті,сонда есебіміз операторлар шоғырының кейпіне енеді.Бұл операторлар шоғыры қарапайым операторлардың көбейтіндісіне жіктеледі,ал ол операторлар оңай зерттеледі.

Тірек сөздер: ауытқыған аргумент, әлді шешілу, спектр, операторлар шоғыры, функционал-дифферен-циал оператор.

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О СПЕКТРАЛЬНЫХ СВОЙСТВАХ ВОЛНОВОГО ОПЕРАТОРА, ВОЗМУЩЁННОГО МЛАДШИМ ЧЛЕНОМ

Аннотация. Некорректность минимального волнового оператора общеизвестна, так как нуль для него является бесконечнократным собственным значением. Как показали наши исследования, положение изменится, если возмутить его младшим членом, содержащим в качестве коэффициента спектральный параметр, в итоге задача принимает вид операторного пучка. Полученный пучок операторов легко факторизуется с помощью функционально– дифференциальных операторов первого порядка, спектральные свойства которых легко изучаются классическим методом разделения переменных. Непосредственное применение метода разделения переменных к исходному пучку операторов наталкивается на непреодолимые трудности.

Ключевые слова: отклоняющиеся аргумент, сильная разрешимость, спектр, пучок операторов, функционально-дифференциальный оператор

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