ISSN 2518-1726 (Online),
ISSN 1991-346X (Print)

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ
Әль-фараби атындағы Қазақ ұлттық университетінің

Х А Б А Р Л А Р Ы

ИЗВЕСТИЯ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК
РЕСПУБЛИКИ КАЗАХСТАН
Казахский национальный университет
имени Аль-фараби

N E W S

OF THE NATIONAL ACADEMY OFSCIENCES
OF THE REPUBLIC OF KAZAKHSTAN
Al-farabi kazakh
national university

SERIES
PHYSICO-MATHEMATICAL

5 (327)

SEPTEMBER-OCTOBER 2019

PUBLISHED SINCE JANUARY 1963

PUBLISHED 6 TIMES A YEAR

ALMATY, NAS RK

«ҚР ҰҒА Хабарлары. Физика-математикалық сериясы».

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Меншіктенуші: «Қазақстан Республикасының Ұлттық ғылым академиясы» РҚБ (Алматы қ.)
Қазақстан республикасының Мәдениет пен ақпарат министрлігінің Ақпарат және мұрағат комитетінде 01.06.2006 ж. берілген **№5543-Ж** мерзімдік басылым тіркеуіне қойылу туралы куәлік

Мерзімділігі: жылына 6 рет.
Тиражы: 300 дана.

Редакцияның мекенжайы: 050010, Алматы қ., Шевченко көш., 28, 219 бөл., 220, тел.: 272-13-19, 272-13-18, http://physics-mathematics.kz/index.php/en/archive

---

Типографияның мекенжайы: «Аруна» ЖК, Алматы қ., Муратбаева көш., 75.

«Известия НАН РК. Серия физико-математическая».

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Собственник: РОО «Национальная академия наук Республики Казахстан» (г. Алматы)
Свидетельство о постановке на учет периодического печатного издания в Комитете информации и архивов Министерства культуры и информации Республики Казахстан **№5543-Ж**, выданное 01.06.2006 г.

Периодичность: 6 раз в год.
Тираж: 300 экземпляров.

Адрес редакции: 050010, г. Алматы, ул. Шевченко, 28, ком. 219, 220, тел.: 272-13-19, 272-13-18,
http://physics-mathematics.kz/index.php/en/archive

Адрес типографии: ИП «Аруна», г. Алматы, ул. Муратбаева, 75.

News of the National Academy of Sciences of the Republic of Kazakhstan. Physical-mathematical series.

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Owner: RPA "National Academy of Sciences of the Republic of Kazakhstan" (Almaty)
The certificate of registration of a periodic printed publication in the Committee of information and archives of the Ministry of culture and information of the Republic of Kazakhstan N 5543-Ж, issued 01.06.2006

Periodicity: 6 times a year
Circulation: 300 copies

Editorial address: 28, Shevchenko str., of. 219, 220, Almaty, 050010, tel. 272-13-19, 272-13-18,
http://physics-mathematics.kz/index.php/en/archive

Address of printing house: ST "Aruna", 75, Muratbayev str, Almaty

N E W S
OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2019.2518-1726.59**
Volume 5, Number 327 (2019), 59 – 69

UDK 517.9

A.Sh. Shaldanbayev[1] , A.A. Shaldanbayeva[2] , B.A. Shaldanbay[3]

[1]Silkway International University, Shymkent, Kazakhstan;
[2]Regional social-innovative University, Shymkent, Kazakhstan;
[3]South Kazakhstan State University M.O.Auezov, Shymkent, Kazakhstan shaldanbaev51@mail.ru, altima_a@mail.ru, baglan.shaldanbayev@bk.ru

INVERSE PROBLEM OF A STURM-LIOUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

Abstract. In this paper we prove a uniqueness theorem, in a single spectrum, for the Sturm-Liouville operator with non-separated boundary value conditions and real continuous and symmetric potential. The research method is different from all known methods, and based on internal symmetry of the operator generated by invariant subspaces.
Keywords: Sturm - Liouville operator, spectrum, Sturm - Liouville inverse problem, Borg theorem, Ambartsumian theorem, Levinson theorem, non-separated boundary value conditions, symmetric potential, invariant subspaces.

1. Introduction. By inverse problems of spectral analysis, we understand the tasks of restoring a linear operator according to its one or other spectral characteristics.

The first significant result in this direction was obtained in 1929 by V.A. Ambartsumyan [1]. He proved the following theorem.

We denote eigenvalues of the following Sturm - Liouville operator by $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$:

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0, y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If $\lambda_n = n^2$ $(n = 0,1,2, \dots)$, then $q(x) \equiv 0$.

The first mathematician who drew attention to importance of this Ambartsumian result was the Swedish mathematician Borg. He also performed the first systematic research of one of the important inverse problems, namely, the inverse problem for the classical Sturm – Liouville operator of the form (1.1) by the spectra [2]. Borg showed that, in the general case, one spectrum of the Sturm-Liouville operator does not define it, so Ambartsumian's result is an exception to the general rule. In the same paper [2], Borg shows that two spectra of the Sturm – Liouville operator (with different boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Borg Theorem.

Let the equation

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$
$$-z'' + p(x)z = \lambda z, \tag{1.3}$$

have the same spectrum under the following boundary conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases} \tag{1.4}$$

and under the following boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0, \pi]$, if

$$\delta \cdot \delta' = 0, |\delta| + |\delta'| > 0.$$

Shortly after the Borg's work important studies on the theory of inverse problems were performed by Levinson [3], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm – Liouville operator

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.5}$$

restored by one spectrum. Ambartsumian and Levinson theorems were developed in [4] - [6].

Inverse spectral analysis problems for Sturm – Liouville operators with non-separated boundary conditions

A number of works B.M. Levitan [7] - [8] are devoted to reconstructing the Sturm-Liouville operator from one and two spectra.

Note that inverse problems of spectral analysis for Sturm – Liouville operators with non-separated boundary conditions were studied in [9] - [10] by other methods, with the results obtained being expressed through conformal mappings, and difficult to verify.

From later works in this direction, we note [11] - [13].

This paper is devoted to the generalization of Ambartsumian [1] and Levinson [3] theorems to non-decaying boundary conditions, in particular, our results contain results of these authors. Method of this paper appeared i development of spectral methods for solving ill-posed problems of mathematical physics [14] - [25].

2. Research Method.

Idea of this paper is very simple. Carefully studying the content of [1, 3], we realized that both of these operators have an invariant subspace. If for the linear operator L the following formulas hold:

$$LP = PL^*, QL = L^*Q,$$

where P, Q are orthogonal projectors, satisfying the condition $P + Q = I$, then the operators L and L^* have invariant subspaces, sometimes restriction of these operators to these invariant subspaces, with certain conditions, form a Borg pair.

3. Research Results.

In Hilbert space $H = L^2(0, \pi)$ we consider the Sturm – Liouville operator

$$Ly = -y'' + q(x)y, x \in (0, \pi); \tag{3.1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{3.2}$$

where $q(x)$ is a continuous complex function, a_{ij} $(i = 1,2;\ j = 1,2,3,4)$ are arbitrary complex coefficients, and by Δ_{ij} $(i = 1,2;\ j = 1,2,3,4)$ we denote minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

We suppose that $\Delta_{24} \neq 0$, then the Sturm – Liouville operator (3.1) – (3.2) takes the following form

$$Ly = -y'' + q(x)y, x \in (0, \pi); \tag{3.1}$$

$$\begin{cases} \Delta_{14}y(0) + \Delta_{24}y'(0) + \Delta_{34}y(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0 \end{cases} \tag{3.3}$$

and its conjugate operator L^{+} takes the form

$$L^{+}z = -z'' + q(x)z, x \in (0,\pi); \tag{3.1$^+$}$$

$$\begin{cases} \overline{\Delta_{14}}z(0) + \overline{\Delta_{24}}z'(0) + \overline{\Delta_{12}}z(\pi) = 0, \\ \overline{\Delta_{34}}z(0) + \overline{\Delta_{32}}z(\pi) - \overline{\Delta_{24}}z'(\pi) = 0. \end{cases} \tag{3.3$^+$}$$

Let P and Q be orthogonal projectors, defined by the formulas

$$Pu(x) = \frac{u(x)+u(\pi-x)}{2}, Qv(x) = \frac{v(x)-v(\pi-x)}{2} \tag{3.4}$$

The main result of this paper is the following theorem.

Theorem 3.1. If $\Delta_{24} \neq 0$, and

1) $LP = PL^{+}$;
2) $QL = L^{+}Q$;
3) $\Delta_{12} = -\Delta_{12}$;

then the Sturm – Liouville operator (3.1) – (3.3) is restored by one spectrum.

4. Discussion.

In this section we prove the theorem and discuss the obtained results. The following Lemma 4.1 and Lemma 4.2 can have independent values.

Lemma 4.1. If for a linear and discrete operator L we have:

1) $LP = PL^{*}$;
2) $QL = L^{*}Q$;
3) $P + Q = I$;

where P, Q are orthogonal projectors, and I is a unique operator, then all its eigenvalues are real.

Proof.

Let $LP = PL^{*}$, $QL = L^{*}Q$; then

$$(LP)^{*} = P^{*}L^{*} = PL^{*} = LP;$$
$$(QL)^{*} = L^{*}Q^{*} = L^{*}Q = QL;$$

i.e. operators LP and QL are self-adjoint, therefore their eigenvalues are real.

If $Ly = \lambda y, y \neq 0$, then $QLy = \lambda Qy$, $L^{+}Qy = \lambda Qy$, $L^{+}Q(Qy) = \lambda Qy$, $QL(Qy) = \lambda Qy$ if $Qy \neq 0$, then λ is a real quantity; if $Qy = 0$, then $y = Py \neq 0$, and $LPy = \lambda Py, LP(Py) = \lambda Py$. Consequently, λ is again real quantity.

The following lemmas shows that the spectrum $\sigma(L)$ of the operator L is divided into two parts, therefore the operator L, apparently, is also divided into two parts. Later we will see that this is exactly what happens, and more precisely, these parts form a Borg pair under a certain condition.

Lemma 4.2. If L is a linear discrete operator satisfying the conditions:

1) $LP = PL^{*}$;
2) $QL = L^{*}Q$;
3) $P + Q = I$;

where P, Q are orthogonal projectors, and I is identity operator, then we get

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2).$$

where $L_1 = LP, L_2 = QL, \sigma(L)$ is a spectrum of the operator L.

Proof.

If $Ly = \lambda y, y \neq 0$, then $QLy = \lambda Qy, L^{+}Qy = \lambda Qy$, $L^{+}Q(Qy) = \lambda Qy$, $L_2Qy = \lambda Qy$. If $Qy \neq 0$, hence $\lambda \in \sigma(L_2)$. If $Qy = 0$, then we get $y = Py \neq 0$ and $LPy = \lambda Py$, $LP(Py) = \lambda Py, L_1Py = \lambda Py$. Consequently, $\lambda \in \sigma(L_1)$.

Thus, $\sigma(L) \subset \sigma(L_1) \cup \sigma(L_2)$.

If $\lambda \neq 0$, and $\lambda \in \sigma(L_1) \cup \sigma(L_2)$, then

a) If $\lambda \in \sigma(L_1)$, then $\exists u \neq 0$, such that $u \in H_1, L_1u = \lambda u, LPu = \lambda u, \rightarrow Lu = \lambda u$. Consequently, $\lambda \in \sigma(L)$.

б) If $\lambda \in \sigma(L_2)$, then $\exists v \in H_2$, $v \neq 0$ such that $L_2v = \lambda v$, $QLv = \lambda v$, $L^{+}Qv = \lambda v, L^{+}v = \lambda v$. Consequently, $\lambda \in \sigma(L^{+}) = \sigma(L)$.

в) If $0 \in \sigma(L_1) \cup \sigma(L_2)$, then If $0 \in \sigma(L_1)$, therefore $L_1 u = 0, u \in H_1$, $LPu = 0, => Lu = 0, => 0 \in \sigma(L)$. If $0 \in \sigma(L_2)$, then $L_2 v = 0$, $v \in H_2$, $QLv = 0, => L^+ Qv = 0, L^+ v = 0, => 0 \in \sigma(L^+) = \sigma(L)$.

The following Lemma 4.3 nad Lemma 4.4 clarify the boundary conditions of Sturm-Liouville operators with invariant subspaces.

Lemma 4.3. If

а) $\Delta_{24} \neq 0$;

б) $LP = PL^+$;

then we have

1) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)} = \frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}} = \frac{\Delta_{14}+\Delta_{34}}{\Delta_{24}}$;

2) $q(\pi - x) = q(x)$;

3) $\overline{q(x)} = q(x)$;

and the operators L and L^+ take the following forms:

$Ly = -y'' + q(x)y, x \in (0, \pi)$;

$$\begin{cases} \frac{\Delta_{12}-\Delta_{14}}{\Delta_{24}}[y(0) - y(\pi)] - y'(0) - y'(\pi) = 0, \\ \Delta_{12} y(0) + \Delta_{32} y(\pi) - \Delta_{24} y'(\pi) = 0. \end{cases}$$

$L^+ z = -z'' + q(x)z, x \in (0, \pi)$;

$$\begin{cases} \frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}[z(0) + z(\pi)] + z'(0) + z'(\pi) = 0, \\ \overline{\Delta_{14}} z(0) + \overline{\Delta_{24}} z'(0) + \overline{\Delta_{12}} z(\pi) = 0. \end{cases}$$

Proof.

Let $a_{24} \neq 0$, then the Sturm-Liouville operator has the form

$$Ly = -y'' + q(x)y, x \in (0, \pi); \tag{4.1}$$

$$\begin{cases} \Delta_{14} y(0) + \Delta_{24} y'(0) + \Delta_{34} y(\pi) = 0, \\ \Delta_{12} y(0) + \Delta_{32} y(\pi) - \Delta_{24} y'(\pi) = 0, \end{cases} \tag{4.2}$$

and its conjugate operator L^+ takes the form

$L^+ z = -z'' + \overline{q(x)} z, x \in (0, \pi)$; (4.1)$^+$

$$\begin{cases} \overline{\Delta_{14}} z(0) + \overline{\Delta_{24}} z'(0) + \overline{\Delta_{12}} z(\pi) = 0, \\ \overline{\Delta_{34}} z(0) + \overline{\Delta_{32}} z(\pi) - \overline{\Delta_{24}} z'(\pi) = 0. \end{cases} \tag{4.2$^+$}$$

By P and Q we denote the following orthogonal projectors

$$Pu(x) = \frac{u(x)+u(\pi-x)}{2}, Qv(x) = \frac{v(x)-v(\pi-x)}{2}. \tag{4.3}$$

If $LP = PL^+$, then $y = Pz \in D(L)$, where $z \in D(L^+)$, therefore

$$y(x) = \frac{z(x) + z(\pi - x)}{2}, y'(x) = \frac{z'(x) - z'(\pi - x)}{2};$$

$$\begin{cases} \Delta_{14} \frac{z(0) + z(\pi)}{2} + \Delta_{24} \frac{z'(0) - z'(\pi)}{2} + \Delta_{34} \frac{z(0) + z(\pi)}{2} = 0, \\ \Delta_{12} \frac{z(0) + z(\pi)}{2} + \Delta_{32} \frac{z(0) + z(\pi)}{2} - \Delta_{24} \frac{z'(\pi) - z'(0)}{2} = 0; \end{cases}$$

$$\begin{cases} (\Delta_{14} + \Delta_{34}) \frac{z(0) + z(\pi)}{2} + \Delta_{24} \frac{z'(0) - z'(\pi)}{2} = 0, \\ (\Delta_{12} + \Delta_{32}) \frac{z(0) + z(\pi)}{2} + \Delta_{24} \frac{z'(0) - z'(\pi)}{2} = 0; \end{cases}$$

From $LP = PL^{+}$ we get that $\Delta_{12} + \Delta_{32} = \Delta_{14} + \Delta_{34}$, hence these boundary conditions melt into one boundary condition:

$$(\Delta_{12} + \Delta_{32})\frac{z(0)+z(\pi)}{2} + \Delta_{24}\frac{z'(0)+z'(\pi)}{2} = 0. \quad (4.4)$$

Summing up these boundary conditions (4.2)$^{+}$, we have

$$(\overline{\Delta_{14}} + \overline{\Delta_{34}})z(0) + (\overline{\Delta_{12}} + \overline{\Delta_{32}})z(\pi) + \overline{\Delta_{24}}[z'(0) - z'(\pi)] = 0,$$

$$(\overline{\Delta_{12}} + \overline{\Delta_{32}})\frac{z(0)+z(\pi)}{2} + \overline{\Delta_{24}}[z'(0) - z'(\pi)] = 0. \quad (4.5)$$

Combining (4.4) with (4.5), we get the following system of equations:

$$\begin{cases} (\Delta_{12} + \Delta_{32})\dfrac{z(0) + z(\pi)}{2} + \Delta_{24}\dfrac{z'(0) - z'(\pi)}{2} = 0. \\ (\overline{\Delta_{12}} + \overline{\Delta_{32}})\dfrac{z(0) + z(\pi)}{2} + \overline{\Delta_{24}}\dfrac{z'(0) - z'(\pi)}{2} = 0. \end{cases}$$

This system of equations has a non-trivial solution, thus

$$\begin{vmatrix} \Delta_{12} + \Delta_{32} & \Delta_{24} \\ \overline{\Delta_{12}} + \overline{\Delta_{32}} & \overline{\Delta_{24}} \end{vmatrix} = 0,$$

or

$$\overline{\left(\frac{\Delta_{12} + \Delta_{32}}{\Delta_{24}}\right)} = \frac{\Delta_{12} + \Delta_{32}}{\Delta_{24}} = \frac{\Delta_{14} + \Delta_{34}}{\Delta_{24}}.$$

Therefore, operator L^{+} has the form:

$$L^{+}z = -z'' + q(x)z, x \in (0, \pi);$$

$$\begin{cases} \dfrac{\Delta_{12} + \Delta_{32}}{\Delta_{24}}[z(0) + z(\pi)] + z'(0) + z'(\pi) = 0, \\ \overline{\Delta_{14}}z(0) + \overline{\Delta_{24}}z'(0) + \overline{\Delta_{12}}z(\pi) = 0; \end{cases}$$

where

$$\overline{\left(\frac{\Delta_{12} + \Delta_{32}}{\Delta_{24}}\right)} = \frac{\Delta_{12} + \Delta_{32}}{\Delta_{24}} = \frac{\Delta_{14} + \Delta_{34}}{\Delta_{24}}.$$

Now we find the operator L, subtracting the second row of the formula (4.2) from the first row, we obtain

$$(\Delta_{12} - \Delta_{14})y(0) + (\Delta_{32} - \Delta_{34})y(\pi) - \Delta_{24}[y'(0) + y'(\pi)] = 0,$$

$$(\Delta_{12} - \Delta_{14})y(0) + (\Delta_{14} - \Delta_{12})y(\pi) - \Delta_{24}[y'(0) + y'(\pi)] = 0,$$

$$(\Delta_{12} - \Delta_{14})[y(0) - y(\pi)] - \Delta_{24}[y'(0) + y'(\pi)] = 0.$$

Consequently, operator L has the form

$$\begin{cases} \dfrac{\Delta_{12} - \Delta_{14}}{\Delta_{24}}[y(0) - y(\pi)] - y'(0) - y'(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0. \end{cases}$$

Further, from the formula $LP = PL^{+}$ we get

$$LPz = PL^{+}z, \forall z \in D(L^{+}),$$

$$LPz = L^{\circ}\frac{z(x) + z(\pi - x)}{2} = -\frac{z''(x) + z''(\pi - x)}{2} + q(x)\frac{z(x) + z(\pi - x)}{2};$$

$$PL^{+}z = P^{\circ}\left[-z'' + \overline{q(x)}z\right] = -\frac{z''(x) + z''(\pi - x)}{2} +$$

$$+\frac{\bar{q}(x)z(x) + \bar{q}(\pi - x)z(\pi - x)}{2};$$

$$q(x)z(x) + q(x)z(\pi - x) = \bar{q}(x)z(x) + \bar{q}(\pi - x)z(\pi - x),$$

$$[q(x) - \bar{q}(x)]z(x) + [q(x) - \bar{q}(\pi - x)]z(\pi - x) = 0, \text{(4.6)}$$

$$[q(\pi - x) - \bar{q}(\pi - x)]z(\pi - x) + [q(\pi - x) - \bar{q}(x)]z(x) = 0;$$

$$\begin{vmatrix} q(x) - \bar{q}(x) & q(x) - \bar{q}(\pi - x) \\ q(\pi - x) - \bar{q}(x) & q(\pi - x) - \bar{q}(\pi - x) \end{vmatrix} = 0;$$

$$[q(x) - \bar{q}(x)] \cdot [q(\pi - x) - \bar{q}(\pi - x)] = [q(x) - \bar{q}(\pi - x)] \cdot [q(\pi - x) - \bar{q}(x)]$$

$$q(x)q(\pi - x) - q(x)\bar{q}(\pi - x) - \bar{q}(x)q(\pi - x) + \bar{q}(x)\bar{q}(\pi - x) =$$

$$= q(x)q(\pi - x) - q(x)\bar{q}(x) - \bar{q}(\pi - x)q(\pi - x) + \bar{q}(\pi - x)\bar{q}(x);$$

$$q(x)\bar{q}(\pi - x) + \bar{q}(x)q(\pi - x) = q(x)\bar{q}(x) + \bar{q}(\pi - x)q(\pi - x),$$

$$q(x)[\bar{q}(\pi - x) - \bar{q}(x)] + q(\pi - x)[\bar{q}(x) - \bar{q}(\pi - x)] = 0,$$

$$[\bar{q}(x) - \bar{q}(\pi - x)] \cdot [q(\pi - x) - q(x)] = 0,$$

$$|q(x) - q(\pi - x)|^{2} = 0, => \; q(x) = q(\pi - x). \quad (4.7)$$

From (4.6) and (4.7) it follows that

$$[q(x) - \bar{q}(x)]z(x) + [q(x) - \bar{q}(x)]z(\pi - x) = 0,$$

$$[q(x) - \bar{q}(x)][z(x) + z(\pi - x)] = 0, => q(x) - \bar{q}(x) = 0.$$

Lemma 4.4. If

a) $\Delta_{24} \neq 0$;

b) $QL = L^{+}Q$,

then

1) $\overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)} = \frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}} = \frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}$;

2) $q(\pi - x) = q(x)$;

3) $\bar{q}(x) = q(x)$,

and the operators L and L^{+} take the forms:

$$Ly = -y'' + q(x)y, x \in (0, \pi);$$

$$\begin{cases} \frac{\Delta_{12}-\Delta_{14}}{\Delta_{24}}[y(0) - y(\pi)] - y'(0) - y'(\pi) = 0, \\ \Delta_{12}y(0) + \Delta_{32}y(\pi) - \Delta_{24}y'(\pi) = 0. \end{cases}$$

$$L^{+}z = -z'' + q(x)z, x \in (0, \pi);$$

$$\begin{cases} \overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)}[z(0) + z(\pi)] + z'(0) - z'(\pi) = 0, \\ \overline{\Delta_{14}}z(0) + \overline{\Delta_{24}}z'(0) + \overline{\Delta_{12}}z(\pi) = 0. \end{cases}$$

Proof.

If $QL = L^{+}Q$ and $y \in D(L)$, тthen $z = Qy \in D(L^{+})$, thus we get

$$z(x) = \frac{y(x) - y(\pi - x)}{2}, z'(x) = \frac{y'(x) + y'(\pi - x)}{2};$$

$$\begin{cases}\overline{\Delta_{14}}\dfrac{y(0)-y(\pi)}{2}+\overline{\Delta_{24}}\dfrac{y'(0)+y'(\pi)}{2}+\overline{\Delta_{12}}\dfrac{y(\pi)-y(0)}{2}=0,\\ \overline{\Delta_{34}}\dfrac{y(0)-y(\pi)}{2}+\overline{\Delta_{32}}\dfrac{y(\pi)-y(0)}{2}-\overline{\Delta_{24}}\dfrac{y'(\pi)+y'(0)}{2}=0;\end{cases}$$

$$\begin{cases}(\Delta_{14}-\Delta_{12})\dfrac{y(0)-y(\pi)}{2}+\overline{\Delta_{24}}\dfrac{y'(0)+y'(\pi)}{2}=0,\\ (\overline{\Delta_{34}}-\overline{\Delta_{32}})\dfrac{y(0)-y(\pi)}{2}-\Delta_{24}\dfrac{y'(0)+y'(\pi)}{2}=0.\end{cases}$$

Due to the formula $\Delta_{14}-\Delta_{12}=\Delta_{32}-\Delta_{34}$ these boundary conditions melt into one boundary condition

$$(\overline{\Delta_{14}}-\overline{\Delta_{12}})\frac{y(0)-y(\pi)}{2}+\overline{\Delta_{24}}\frac{y'(0)+y'(\pi)}{2}=0. \quad (4.8)$$

Subtracting the second row of the formula (4.2) from its first row, we have

$$(\Delta_{14}-\Delta_{12})y(0)+(\Delta_{34}-\Delta_{32})y(\pi)+\Delta_{24}[y'(0)+y'(\pi)]=0,$$

$$(\Delta_{14}-\Delta_{12})[y(0)-y(\pi)]+\Delta_{24}[y'(0)+y'(\pi)]=0,$$

$$(\Delta_{14}-\Delta_{12})\frac{[y(0)-y(\pi)]}{2}+\Delta_{24}\frac{[y'(0)+y'(\pi)]}{2}=0. \quad (4.9)$$

Combining (4.8) with (4.9), we get the system of equations

$$\begin{cases}(\overline{\Delta_{14}}-\overline{\Delta_{12}})\dfrac{y(0)-y(\pi)}{2}+\overline{\Delta_{24}}\dfrac{y'(0)+y'(\pi)}{2}=0,\\ (\Delta_{14}-\Delta_{12})\dfrac{y(0)-y(\pi)}{2}+\Delta_{24}\dfrac{y'(0)+y'(\pi)}{2}=0.\end{cases}$$

This system has a non-trivial solution, therefore

$$\begin{vmatrix}\overline{\Delta_{14}}-\overline{\Delta_{12}} & \overline{\Delta_{24}}\\ \Delta_{14}-\Delta_{12} & \Delta_{24}\end{vmatrix}=0, => \overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)}=\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}=\frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}.$$

Thus, operator L has the form

$$Ly=-y''+q(x)y, x\in(0,\pi);$$

$$\begin{cases}\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}[y(0)-y(\pi)]+y'(0)+y'(\pi)=0,\\ \Delta_{12}y(0)+\Delta_{32}y(\pi)-\Delta_{24}y'(\pi)=0;\end{cases}$$

where

$$\overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)}=\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}=\frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}.$$

Combining boundary conditions $(4.2)^{+}$, we receive:

$$(\overline{\Delta_{14}}+\overline{\Delta_{34}})z(0)+(\overline{\Delta_{12}}+\overline{\Delta_{32}})z(\pi)+\overline{\Delta_{24}}[z'(0)-z'(\pi)]=0,$$

$$(\overline{\Delta_{12}}+\overline{\Delta_{32}})[z(0)+z(\pi)]+\overline{\Delta_{24}}[z'(0)-z'(\pi)]=0,$$

$$\frac{\overline{\Delta_{12}}+\overline{\Delta_{32}}}{\overline{\Delta_{24}}}[z(0)+z(\pi)]+z'(0)-z'(\pi)=0.$$

Consequently,

$$L^{+}z=-z''+q(x)z, x\in(0,\pi);$$

$$\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)}[z(0)+z(\pi)]+z'(0)-z'(\pi)=0,$$

$$\overline{\Delta_{14}}z(0)+\overline{\Delta_{24}}z'(0)+\overline{\Delta_{12}}z(\pi)=0.$$

Further, from the formula $QL = L^{+}Q$, we have

$$QLy = Q^{\circ}[-y''+q(x)y] = -\frac{y''(x)-y''(\pi-x)}{2}+$$

$$+\frac{q(x)y(x)-q(\pi-x)y(\pi-x)}{2};$$

$$L^{+}Qy = L^{+}\left[\frac{y(x)-y(\pi-x)}{2}\right]=$$

$$=-\frac{y''(x)-y''(\pi-x)}{2}+\bar{q}(x)\frac{y(x)-y(\pi-x)}{2};$$

$$q(x)y(x)-q(\pi-x)y(\pi-x)=\bar{q}(x)y(x)-\bar{q}(x)y(\pi-x),$$

$[q(x)-\bar{q}(x)]y(x)+[\bar{q}(x)-q(\pi-x)]y(\pi-x)=0,$ (4.10)

$$[q(\pi-x)-\bar{q}(\pi-x)]y(\pi-x)+[\bar{q}(\pi-x)-q(x)]y(x)=0;$$

$$\begin{vmatrix} q(x)-\bar{q}(x) & \bar{q}(x)-q(\pi-x) \\ \bar{q}(\pi-x)-q(x) & q(\pi-x)-\bar{q}(\pi-x) \end{vmatrix}=0,$$

$$[q(x)-\bar{q}(x)]\cdot[q(\pi-x)-\bar{q}(\pi-x)]-$$

$$-[\bar{q}(x)-q(\pi-x)][\bar{q}(\pi-x)-q(x)]=0,$$

$$q(x)q(\pi-x)-q(x)\bar{q}(\pi-x)-\bar{q}(x)q(\pi-x)+\bar{q}(x)\bar{q}(\pi-x)=$$

$$=\bar{q}(x)\bar{q}(\pi-x)-\bar{q}(x)q(x)-q(\pi-x)\bar{q}(\pi-x)+q(\pi-x)q(x),$$

$$q(x)\bar{q}(\pi-x)+\bar{q}(x)q(\pi-x)=\bar{q}(x)q(x)+q(\pi-x)\bar{q}(\pi-x),$$

$$q(x)[\bar{q}(\pi-x)-\bar{q}(x)]+q(\pi-x)[\bar{q}(x)-\bar{q}(\pi-x)]=0,$$

$$[\bar{q}(x)-\bar{q}(\pi-x)][q(\pi-x)-q(x)]=$$

$$=-|q(x)-q(\pi-x)|^2=0, \Rightarrow\ q(x)=q(\pi-x). \quad (4.7)$$

From (4.7) and (4.10), we have

$$[q(x)-\bar{q}(x)]y(x)+[\bar{q}(x)-q(x)]y(\pi-x)=0,$$

$$[q(x)-\bar{q}(x)][y(x)-y(\pi-x)]=0, \Rightarrow\ q(x)-\bar{q}(x)=0.$$

Comparing the results of Lemma 4.3 and Lemma 4.4, we obtain the following theorem

Theorem 4.1. If

а) $\Delta_{24}\neq 0$;

б) $LP = PL^{+}$;

в) $QL = L^{+}Q$,

then restriction of the operator L to the subspace $H_1 = PH$ has the form

$$L_1u = -u''+q(x)u, x\in\left(0,\frac{\pi}{2}\right),$$

$$\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}u(0)+u'(0)=0, u'\left(\frac{\pi}{2}\right)=0; \quad (4.11)$$

and restriction of the operator L^{+} to the subspace $H_2 = QH$ has the form

$$L_2 v = -v'' + q(x)v, x \in \left(0, \frac{\pi}{2}\right),$$

$$\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}} v(0) + v'(0) = 0, v\left(\frac{\pi}{2}\right) = 0; \quad (4.12)$$

where

1) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)} = \frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}} = \frac{\Delta_{14}+\Delta_{34}}{\Delta_{24}}$;
2) $\overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)} = \frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}} = \frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}$;
3) $q(\pi - x) = q(x)$;
4) $\bar{q}(x) = q(x)$.

Equating the coefficients of the boundary conditions (4.11) and (4.12), we have

$$\Delta_{12} + \Delta_{32} = \Delta_{14} - \Delta_{12}, => \Delta_{12} = \Delta_{14} - \Delta_{12} - \Delta_{32} =$$
$$= -(\Delta_{12} + \Delta_{32} - \Delta_{14}) = -\Delta_{34}.$$

Then operators L_1 and L_2 have the following forms

$$L_1 u = -u'' + q(x)u, x \in \left(0, \frac{\pi}{2}\right),$$
$$(\Delta_{14} - \Delta_{12})u(0) + \Delta_{24}u'(0) = 0, u'\left(\frac{\pi}{2}\right) = 0;$$
$$L_2 v = -v'' + q(x)v, x \in \left(0, \frac{\pi}{2}\right),$$
$$(\Delta_{14} - \Delta_{12})v(0) + \Delta_{24}v'(0) = 0, v\left(\frac{\pi}{2}\right) = 0.$$

If spectrum of the operator L is known, then by Lemma 4.2 spectra of the operators L_1 and L_2, considered on the segment $\left[0, \frac{\pi}{2}\right]$, are known. Then, by the Borg theorem, the operator L is defined uniquely on the interval $\left[0, \frac{\pi}{2}\right]$, and due to the evenness and periodicity of the functions $q(x)$, on the whole interval $[0, \pi]$.

ӘОЖ 517.9

А.Ш. Шалданбаев[1], А.А. Шалданбаева[2] , Б.А. Шалданбай[3]

[1]Халықаралық Silkway университеті, Шымкент қ., Казақстан;
[2]Аймақтық әлеуметтік-инновациялық университеті, Шымкент қ., Казақстан;
[3]М.О.Ауезов атындағы Оңтүстік Қазақстан мемлекеттік университеті, Шымкент қ., Казақстан

ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

Аннотация. Бұл еңбекте потенциалы симметриялы,нақты әрі үзіксіз, ал шекаралық шарттары ажырамайтын Штурм-Лиувилл операторын бір спектр арқылы анықтауға болатыны көрсетілді. Зерттеу әдісі бұрынғы әдістердің ешбіріне ұқсамайды, және ол оператордың ішкі симметриясына негізделген, ал ол өз кезегінде инвариантты кеңістіктердің салдары.

Түйін сөздер: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, ажырамайтын шекаралық шарттар, симметриялы потенциал,инвариантты кеңістіктер.

УДК 517.9

А.Ш.Шалданбаев[1], А.А.Шалданбаева[2], Б.А.Шалданбай[3]

[1]Международный университет Silkway, г. Шымкент, Казахстан;
[2]Региональный социально-инновационный университет, г. Шымкент, Казахстан;
[3]Южно-Казахстанский Государственный университет им.М.Ауезова, г. Шымкент, Казахстан

ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕРАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

Аннотация. В данной работе доказана теорема единственности по одному спектру для оператора Штурма-Лиувилля с неразделенными краевыми условиями и вещественным непрерывным и симметричным потенциалом. Метод исследования отличается от всех известных методов и основан на внутреннюю симметрию оператора, порожденного инвариантными подпространствами.

Ключевые слова: оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства.

Information about authors:
Shaldanbayev A.Sh. – doctor of physico-mathematical Sciences, associate Professor, head of the center for mathematical modeling, «Silkway» International University, Shymkent; http://orcid.org/0000-0002-7577-8402;
Shaldanbayeva A.A. - "Regional Social-Innovative University", Shymkent; https://orcid.org/0000-0003-2667-3097;
Shaldanbay B.A. - M.Auezov South Kazakhstan State University, Shymkent; https://orcid.org/0000-0003-2323-0119.

REFERENCES

[1] Ambartsumyan V.A. Uber eine Frage der Eigenwert-theorie, Zsch.f.Physik, 53(1929), 690-695.
[2] Borg G. Eine Umbehrung der Sturm – Liovillschen Eigenwertaufgabe, Acta Math., 78, №2(1946)
[3] Levinson N. The inverse Sturm – Lioville problem, Math.Tidsskr. B., 1949, 25-30.
[4] Hald O.H. The Inverse Sturm-Liouville Problem with symmetric Potentials, Acta mathematica 1978,141:1,263 .
[5] Isaakson E. L., Trubowitz - Pure Appl. Math. 1983. V. 36. N 6. P. 763-783.
[6] Isaakson E. L., McKean H. P., Trubowitz - Pure Appl. Math. 1984. V. 37. N 1. P. 1-12.
[7] Levitan B.M. On definition of the Sturm-Liouville operator by two spectra. News of the AS USSR, ser. Math., 28(1964), 63-78.
[8] Levitan B.M. On definition of the Sturm-Liouville operator by one and two spectra. News of the AS USSR, ser. Math., 42. №1. 1964.
[9] Plaksina O.A., Inverse problems of spectral analysis for Sturm – Liouville operators with non-separated boundary conditions, Math. Sb., 1986, Vol. 131 (173), Number 1(9), 3–26.
[10] Plaksina O.A., Inverse problems of spectral analysis for Sturm – Liouville operators with non-separated boundary conditions. II, Math. Sat, 1988, vol. 136 (178), Number 1(5), 140–159.
[11] Esin Inan Eskitascioglu, Mehmet Acil. An inverse sturm-liouville problem with a generalized symmetric potential., Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 41, pp. 1-7. ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu.
[12] Martin Bohnera, Hikmet Koyunbakanb.Inverse Problems for Sturm–Liouville Difference Equations. Filomat 30:5 (2016), 1297–1304 DOI 10.2298/FIL1605297B.
[13] Münevver Tuz.On inverse sturm- liouville problems with symmetric potentials., ITM Web of Conferences 13, 01035 (2017) DOI: 10.1051/itmconf/20171301035.
[14] Akylbayev M.I., Beysebayeva A., Shaldanbayev A. Sh., On the Periodic Solution of the Goursat Problem for a Wave Equation of a Special Form with Variable Coefficients. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 1, Number 317 (2018), 34 – 50.
[15] Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh., Beisebaeva A.Zh., The Spectral Decomposition of Cauchy Problem's Solution for Laplace Equation. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 5, Number 321 (2018), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10
[16] T.Sh. Kal'menov, A.Sh. Shaldanbaev, On a criterion of solvability of the inverse problem of heat conduction. Journal of Inverse and Ill-posed Problems. 18:5, (2010). 471_492.
[17] Shaldanbayev A., Shomanbayeva M., Kopzhassarova A., Solution of a singularly perturbed Cauchy problem for linear systems of ordinary differential equations by the method of spectral decomposition, International Conference on Analysis and Applied Mathematics (ICAAM 2016) Серия книг: AIP Conference Proceedings Том:1759 Номер статьи: 020090 DOI: 10.1063/1.4959704, 2016.
[18] Orazov I., Shaldanbayev A., Shomanbayeva M., Solution of a singularly perturbed Cauchy problem using a method of a deviating argument, Advancements in Mathematical Sciences (AMS 2015) Серия книг: AIP Conference Proceedings Том: 1676

Номер статьи: 020072 DOI:10.1063/1.4930498, 2015, WOS:000371818700072, International Conference on Advancements in Mathematical Sciences (AMS),NOV 05-07, 2015, Antalya, Turkey.

[19] Shaldanbayev A., Orazov I., Shomanbayeva M., On the restoration of an operator of Sturm-Liouville by one spectrum, International Conference on Analysis and Applied Mathematics (ICAAM 2014). Серия книг: AIP Conference Proceedings Том: 1611 Стр. 53-57, DOI: 10.1063/1.4893803, 2014, WOS:000343720600010, 2-nd International Conference on Analysis and Applied Mathematics (ICAAM), SEP 11-13, 2014,Shymkent, Kazakhstan.

[20] Orazov I., Shaldanbayev A., Shomanbayeva M., About the Nature of the Spectrum of the Periodic Problem for the Heat Equation with a Deviating Argument, Abstract and Applied Analysis Номер статьи: 128363 DOI: 10.1155/2013/128363. Опубликовано: 2013,WOS:000325557100001.

[21] Imanbaeva, A.B., Kopzhasarova, A.A. and Shaldanbaev, A.Sh. Asymptotic expansion of solution of a singularly perturbed Cauchy problem for a system of ordinary differential equations with constant coefficients. “News of NAS RK. Physics and Mathematics Series”, 2017, No. 5, 112-127.

[22] Kopzhasarova A.A., Shaldanbaev A.Sh., Imanbaeva A.B. Solution of singularly perturbed Cauchy problem by the similarity method. “News of NAS RK. Physics and Mathematics Series”, 2017, no 5,127-134.

[23] Amir Sh. Shaldanbayev, Manat T. Shomanbayeva , Solution of singularly perturbed Cauchy problem for ordinary differential equation of second order with constant coefficients by Fourier method ,Citation: AIP Conference Proceedings 1880, 040017 (2017); doi: 10.1063/1.5000633 View online: http://dx.doi.org/10.1063/1.5000633 View Table of Contents: http://aip.scitation.org/toc/apc/1880/1 Published by the American Institute of Physics.

[24] Shaldanbayev A.Sh., Shaldanbayeva A.A., Shaldanbay B.A., On projectional orthogonal basis of a linear non-self - adjoint operator. N e w s of the national academy of sciences of the republic of Kazakhstan Physic - mathematical series 7, Issn 1991-346X https://doi.org/10.32014/2019.2518-1726.15, Volume 2, Number 324 (2019), 79 – 89.

[25] Akylbayev M.I., Shaldanbayev A.Sh., Orazov I., Beysebayeva A., About single operator method of solution of a singularly perturbed Cauchy problem for an ordinary differential equation n – order. N e w s of the national academy of sciences of the republic of Kazakhstan Physico-mathematical series, Issn 1991-346X https://doi.org/10.32014/2019.2518-1726.8, Volume 2, Number 324 (2019), 17 – 36.

Publication Ethics and Publication Malpractice
in the journals of the National Academy of Sciences of the Republic of Kazakhstan

For information on Ethics in publishing and Ethical guidelines for journal publication see http://www.elsevier.com/publishingethics and http://www.elsevier.com/journal-authors/ethics.

Submission of an article to the National Academy of Sciences of the Republic of Kazakhstan implies that the described work has not been published previously (except in the form of an abstract or as part of a published lecture or academic thesis or as an electronic preprint, see http://www.elsevier.com/postingpolicy), that it is not under consideration for publication elsewhere, that its publication is approved by all authors and tacitly or explicitly by the responsible authorities where the work was carried out, and that, if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder. In particular, translations into English of papers already published in another language are not accepted.

No other forms of scientific misconduct are allowed, such as plagiarism, falsification, fraudulent data, incorrect interpretation of other works, incorrect citations, etc. The National Academy of Sciences of the Republic of Kazakhstan follows the Code of Conduct of the Committee on Publication Ethics (COPE), and follows the COPE Flowcharts for Resolving Cases of Suspected Misconduct (http://publicationethics.org/files/u2/New_Code.pdf). To verify originality, your article may be checked by the Cross Check originality detection service http://www.elsevier.com/editors/plagdetect.

The authors are obliged to participate in peer review process and be ready to provide corrections, clarifications, retractions and apologies when needed. All authors of a paper should have significantly contributed to the research.

The reviewers should provide objective judgments and should point out relevant published works which are not yet cited. Reviewed articles should be treated confidentially. The reviewers will be chosen in such a way that there is no conflict of interests with respect to the research, the authors and/or the research funders.

The editors have complete responsibility and authority to reject or accept a paper, and they will only accept a paper when reasonably certain. They will preserve anonymity of reviewers and promote publication of corrections, clarifications, retractions and apologies when needed. The acceptance of a paper automatically implies the copyright transfer to the National Academy of Sciences of the Republic of Kazakhstan.

The Editorial Board of the National Academy of Sciences of the Republic of Kazakhstan will monitor and safeguard publishing ethics.

Правила оформления статьи для публикации в журнале смотреть на сайтах:

www:nauka-nanrk.kz

http://physics-mathematics.kz/index.php/en/archive

ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

Редакторы *М. С. Ахметова, Т.А. Апендиев, Д.С. Аленов*
Верстка на компьютере *А.М. Кульгинбаевой*

Подписано в печать 10.10.2019.
Формат 60х881/8. Бумага офсетная. Печать – ризограф.
9,6 п.л. Тираж 300. Заказ 5.

Национальная академия наук РК
050010, Алматы, ул. Шевченко, 28, т. 272-13-18, 272-13-19