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ABOUT SINGLE OPERATOR METHOD OF SOLUTION OF A SINGULARLY PERTURBED CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION n – ORDER

Abstract. In this paper, by the method of the deviating argument, we obtain an asymptotic expansion of the solution of the Cauchy problem for an ordinary differential equation of $n-$ th order with variable coefficients, with an estimate of the residual term through the right side of the equation. Many papers devoted to this topic are of an applied nature, and their estimates of the residual term are expressed in terms of O −large or o −small, so they have a theoretical value rather than applied, as they claim. The main advantage of the proposed method is the simplicity of its algorithm, and the residual term formula, explicitly expressed through the right side of the equation, and its evaluation.

Keywords: Singular value perturbation, spectral decomposition, deviating argument, residual term estimation, self - adjoint operator, Gilbert-Schmidt theorem, completely continuous operator, Friedrich's Lemma, Cauchy problem, asymptotic expansion, small parameter.

1. Introduction. Many problems of mechanics, physics, engineering and other fields of science lead to differential and integro-differential equations with a small parameter at the highest derivative. A systematic study of such equations (at present they are called singularly perturbed) began after the appearance of the fundamental works of A. N. Tikhonov [1-3], drew the attention of many researchers to equations with a small parameter at the highest derivative. In these works, a General formulation of the Cauchy problem for systems of nonlinear ordinary differential equations with a small parameter at the highest derivative is given, and theorems on the limit transition are proved, establishing a connection between the solution of the initial singularly perturbed Cauchy problem and the solution of the unperturbed problem obtained from the initial at zero value of the parameter.

One of the important problems of the theory of singularly perturbed equations is the construction of asymptotic expansions of solutions of equations by a small parameter.

Among the asymptotic methods developed for singularly perturbed problems, it should be noted a very effective method of the boundary functions proposed By M. I. Vishik and L. A. Lyusternik [4,5] for singularly perturbed linear and partial differential equations, as well as for the singularly perturbed nonlinear ordinary differential equations, and M. I. Imanaliev [8,9] for singularly perturbed nonlinear integro-differential equations. This method is now called the " Method of boundary layer function". Further development of this method is connected with the works of V. F. Butuzova [10,11], V. A. Tupchiev [12] and V. A. Trenogin [13].

S. A. Lomov [14,15] developed a method of regularization of singular perturbations, which allows to reduce the singularly perturbed problem to the regularly perturbed ones, with the help of which it is possible to develop the foundations of the General theory of singularly perturbed equations. The method is applicable to a wide range of problems for ordinary and partial differential equations.

In this paper, we propose a new method for solving singularly perturbed problems, which originates from the spectral theory of equations with a divergent argument. The essence of the method is as follows: the solution of the problem is decomposed into a Fourier series by eigenfunctions of the corresponding boundary value problem, then the coefficients of this series are transformed by integration in parts. As a result of these transformations, we obtain a new (recurrent) representation of the solution of the original problem. Further, by the method of mathematical induction it is possible to obtain an asymptotic expansion of the solution of the problem of interest to us. The remainder of the obtained decomposition is estimated by a priori estimates. With the help of direct calculations, the generality of the obtained recurrent formula is shown, and additional conditions that appeared in the course of research are removed. This work completes a series of studies devoted to the development of a spectral method for solving ill-posed problems [17-25].

Problem statement. Consider in the space $H = L^2(0,1)$the singularly perturbed Cauchy problem

$$L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = f(x), \tag{1}$$

$$y(0) = 0, y'(0) = 0, \ldots, y^{(n-1)}(0) = 0, \tag{2}$$

where $a_i(x)$ - are real and sufficiently smooth functions on the interval $[0,1]$, $f(x) \in L^2(0,1)$, $\varepsilon > 0$ – small parameter. The question is how the solution of this problem behaves as ε → 0, depending on the behavior of the coefficients $a_i(x), i = \overline{1,n}$ and the right part$f(x)$? With the help of the spectral theory of the equation with deviating argument, to obtain the spectral decomposition of the solution of this problem in the space of the crane, and bring with it the asymptotic representation of the solution with the assessment of the remainder term through the right part of the equation.

2. Supporting proposals.

The Cauchy's problem (1) - (2) corresponds to the linear operator

$$L_\varepsilon y = \varepsilon y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = f(x),$$

with the range of definition

$$D(L_\varepsilon) = \{y(x)\epsilon C^n[0,1]; y(0) = 0, y^{'}(0) = 0, .., y^{(n-1)}(0) = 0\},$$

and with the range of values $R(L_\varepsilon) \subset C[0,1]$ contained in the linear variety of continuous functions.

We want to use theory of Hilbert – Schmidt on the spectral decomposition of a completely continuous and self-adjoint operator, therefore we give appropriate definitions and facts from the theory of linear operators.

Lemma 2.1. Let A be a densely defined operator in a Hilbert space H. Then

(a) A^* exists and is closed;

(b) A admits a closure if and only if $D(A^*)$ is tight in H, and in this case $\bar{A} = A^{**}$;

(c) if A admits a closure, then $(\bar{A})^* = A^*$;

(d) if A - admits a closure and is invertible, then A^{-1} admits a closure and $(\bar{A})^{-1} = \overline{A^{-1}}$;

(e) the continuous operator always admits a closure to $\overline{D(A)}$ by continuity.

Lemma 2.2.

(a) If a densely defined linear operator A in a Hilbert space H has a continuous inverse $A^{-1}: H \to D(A)$, then A^* - has a continuous inverse $(A^*)^{-1}: H \to D(A^*)$ and $(A^*)^{-1} = (A^{-1})^*$;

(b) If a linear operator A in a Hilbert space H is densely defined and closed and A^* has a bounded inverse, then A has a bounded inverse, and $(A^{-1})^* = (A^*)^{-1}$.

The proofs of this Lemmas 1, 2 are contained in many manuals on functional analysis [see, for example, 16].

Definition 2.1. An operator A^+ is called formally adjoint to an operator A if for all $u \in D(A)$ and $v \in D(A^+)$ the equality

$$(Au, v) = (u, A^+v),$$

It is obvious that the operator A^* adjoint to A coincides with the formally adjoint operator which has a maximal domain.

Lemma 2.3. The operator formally adjoint to the operator L_ε has the following form:

$$L_\varepsilon^+ v = (-1)^n \varepsilon v^{(n)}(x) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}[a_1(x)v(x)] +$$

$$+(-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}[a_2(x)v(x)] + \cdots - [a_{n-1}(x)v(x)]' + a_n(x)v(x),$$

$$D(L_\varepsilon^+) = \{v(x) \in C^n[0,1]; v(1) = 0, v'(1) = 0, \ldots, v^{(n-1)}(1) = 0\}.$$

Proof. If $u(x) \in D(L_\varepsilon)$ и $v(x) \in D(L_\varepsilon^+)$, so

$$\int_0^1 u^{(n)}(x)v(x)dx = \int_0^1 v(x)du^{(n-1)}(x) = v(x)u^{(n-1)}(x)\Big|_0^1 -$$

$$-\int_0^1 v'(x)u^{(n-1)}(x)dx = -\int_0^1 v'(x)u^{(n-1)}(x)dx = \cdots =$$

$$(-1)^n \int_0^1 u(x)v^{(n)}(x)dx;$$

$$\int_0^1 a_1(x)u^{(n-1)}(x)v(x)dx = \int_0^1 a_1(x)v(x)u^{(n-1)}(x)\,dx = a_1(x)v(x)u^{(n-2)}(x)\Big|_0^1$$

$$-\int_0^1 [a_1(x)v(x)]' u^{(n-2)}(x)dx = -\int_0^1 [a_1(x)v(x)]' u^{(n-2)}(x)dx = \cdots =$$

$$= (-1)^{n-1}\int_0^1 [a_1(x)v(x)]^{(n-1)} u(x)dx,$$

$$\int_0^1 a_2(x)u^{(n-2)}(x)v(x)dx = (-1)^{n-2}\int_0^1 u(x)[a_2(x)v(x)]^{(n-2)}dx, \ldots$$

$$\int_0^1 a_{n-1}(x)u'(x)v(x)dx = \int_0^1 a_{n-1}(x)v(x)du(x) = a_{n-1}(x)v(x)u(x)|_0^1 -$$

$$-\int_0^1 [a_{n-1}(x)v(x)]' u(x)dx = -\int_0^1 u(x)[a_{n-1}(x)v(x)]'dx.$$

Therefore,

$$(L_\varepsilon u, v) = \int_0^1 \{\varepsilon(-1)^n v^{(n)}(x) + (-1)^{n-1}[a_1(x)v(x)]^{(n-1)} +$$

$$(-1)^{n-2}[a_2(x)v(x)]^{(n-2)} + \cdots + \cdots - [a_{n-1}(x)v(x)]' +$$

$$+a_n(x)v(x)\}u(x)dx = (u, L^+v), =>$$

$$L^+v(x) = (-1)^n \varepsilon v^{(n)}(x) + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}[a_1(x)v(x)] +$$

$$(-1)^{n-2}\cdot\frac{d^{n-2}}{dx^{n-2}}[a_2(x)v(x)] + \cdots - [a_{n-1}(x)v(x)]' + a_n(x)v(x),$$

that's what was required to prove.

By virtue of the Friedrich's Lemma, the linear variety of infinitely differentiable and finite functions $C_0^\infty(0,1)$ is tight in space H, therefore both operators L_ε and L_ε^+ are tightly defined in this space. Then, by Lemma 1, the operator L_ε^* exists, and is closed, L_ε permits the circuit, and $\overline{L_\varepsilon} = L_\varepsilon^{**}$, $(\overline{L_\varepsilon})^* = L_\varepsilon^*$. By virtue of one of the theorems of the theory of Hilbert spaces there is a formula:

$$H = \overline{R(L_\varepsilon^*)} \otimes N(L_\varepsilon^{**}) = \overline{R(L_\varepsilon^*)} \otimes N(\overline{L}_\varepsilon),$$

therefore, for the existence of the inverse operator $(\overline{L}_\varepsilon)^{-1}$ is necessary and sufficient fulfillment of the equality $\overline{R(L_\varepsilon^*)} = H$. Then, by virtue of paragraph (d) of Lemma 1, we have the formula $(\overline{L}_\varepsilon)^{-1} = \overline{(L_\varepsilon^{-1})}$, i.e. the inverse operator to the closure of the operator L_ε can be found using the closure of the operator L_ε^{-1}, which exists due to the existence of the operator $(\overline{L}_\varepsilon)^{-1}$. If $D(\overline{L}_\varepsilon)^{-1} = H$, then by the Banach theorem about closed graph operator $(\overline{L}_\varepsilon)^{-1}$ is restricted in space H. But the problem is precisely that. Without further information, we cannot confirm this.

We show that $\overline{R(L_\varepsilon^*)} = H$, note for this that $L_\varepsilon^+ \subset L_\varepsilon^*$, therefore $R(L_\varepsilon^+) \subset R(L_\varepsilon^*)$. If $\overline{R(L_\varepsilon^+)} = H$, then the required statement follows.

Lemma 2.4. If the function is $K(x,t)$, for a fixed value of t, the first variable x is the solution of the Cauchy problem of the following homogeneous equation:

$$\left[\varepsilon\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)\right]K(x,t) = 0,$$

$$K(x,t)|_{t=x} = 0, \left.\frac{\partial K}{\partial x}\right|_{t=x} = 0, ..., \left.\frac{\partial^{(n-2)}K}{\partial x^{n-2}}\right|_{t=x} = 0, \varepsilon\left.\frac{\partial^{(n-1)}K}{\partial x^{n-1}}\right|_{t=x} = 1, \quad (3)$$

that function

$$y(x) = y(x,\varepsilon,f) = \int_0^x K(x,t)f(t)dt = \int_0^1 \theta(x-t)K(x,t)f(t)dt \quad (4)$$

for any continuous function f (t), is the solution of the Cauchy problem(1)-(2).

Proof. If f (t) is continuous on the [0,1] segment, then the function (4) is continuously differentiable and the formula holds:

$$y'(x) = K(x,x)\cdot f(x) + \int_0^x \frac{\partial K}{\partial x}f(t)dt,$$

therefore with condition (3), we have

$$y'(x) = \int_0^x \frac{\partial K}{\partial x}f(t)dt,$$

we have

$$y^{(m)}(x) = \int_0^x \frac{\partial^m K}{\partial x^m}f(t)\,dt, 1 \le m \le n-1;$$

$$y^{(n)}(x) = \left.\frac{\partial^{n-1}K}{\partial x^{n-1}}\right|_{t=x}\cdot f(x) + \int_0^x \frac{\partial^n K}{\partial x^n}f(t)dt.$$

Consequently,

$$\varepsilon y^{(n)}(x) + \sum_{k=1}^{n} a_k(x)\,y^{(n-k)}(x) = \varepsilon\left.\frac{\partial^{n-1}}{\partial x^{n-1}}\right|_{t=x}\cdot f(x) +$$

$$+\int_0^x \left[\varepsilon\frac{\partial^n K}{\partial x^n} + \sum_{k=1}^n a_k(x)\frac{\partial^{n-k}K}{\partial x^{n-k}}\right]f(t)dt = f(x).$$

Definition 2.2. The function $K(x,t)\cdot\theta(x-t)$ is called the Cauchy kernel of the integral operator (4), where $\theta(x)$ is the Heaviside function.

We will return to the study of Cauchy kernel properties a little later, and now we will deal with the solvability of the Cauchy problem. It would be tempting to deduce this statement from the formula $\overline{R(L_\varepsilon^*)} = H$, but from the form of the formally adjoint operator L_ε^+, it is obvious that this path requires a certain smoothness of the coefficients of the equation $L_\varepsilon y = f$, o we will deal with the equation itself $L_\varepsilon y = f$, especially since we have already shown the dense solvability of this equation. The uniqueness of the solution found follows from the following a priori estimates.

Lemma 2.5

If $a_1(x)$ is a continuous function on the interval [0,1], satisfying the condition

(a) $a_1(x) \ge \alpha > 0, \forall x \in [0,1]$

and on the domain of definition of the operator L_ε the inequality holds:

(b) $\left(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \ge 0, \forall y \in D(L_\varepsilon)$;

then the following a priori estimates take place:

$$\|y\| \le \|\acute{y}\| \le \|y'\| \dots \le \left\|y^{(n-2)}\right\| \le \left\|y^{(n-1)}\right\| \le \tfrac{\|L_\varepsilon y\|}{\alpha}. \qquad (6)$$

Proof. Multiplying both sides of equation (1), scalar by the function $y^{(n-1)}(x)$, we have

$$\left(L_\varepsilon y, y^{(n-1)}\right) = \left(\varepsilon y^n, y^{(n-1)}\right) + \left(a_1 y^{(n-1)}, y^{(n-1)}\right) +$$
$$\left(\sum_{k=2}^{n} a_k(x)\, y^{(n-k)}, y^{(n-1)}\right) = \left(f, y^{(n-1)}\right)$$

then

$$\left(\varepsilon y^{(n)}, y^{(n-1)}\right) = \varepsilon \int_0^1 y^{(n)} y^{(n-1)}(x)dx = \varepsilon \int_0^1 y^{(n-1)}(x) d\, y^{(n-1)}(x) =$$
$$= \frac{\varepsilon\left[y^{(n-1)}(x)\right]^2}{2}\Bigg|_0^1 = \frac{\varepsilon\left[y^{(n-1)}(1)\right]^2}{2} \ge 0,$$

so

$$\alpha\left\|y^{(n-1)}\right\|^2 \le \int_0^1 a_1(x)\left[y^{(n-1)}(x)\right]^2 dx \le \left(L_\varepsilon y, y^{(n-1)}\right) \le \left(f, y^{(n-1)}\right) \le$$
$$\le \|f\| \cdot \left\|y^{(n-1)}\right\|;\ \alpha\left\|y^{(n-1)}\right\| \le \|f\|;\ \left\|y^{(n-1)}(x)\right\| \le \tfrac{\|f\|}{\alpha} = \tfrac{\|L_\varepsilon y\|}{\alpha};$$

As, $y(0) = 0$, then $y(x) = \int_0^x \acute{y}(t)dt$,

$$|y(x)| \le \left(\int_0^x 1^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 \acute{y}^2(t)dt\right)^{\frac{1}{2}};\ |y(x)|^2 \le \int_0^x dt \int_0^x \acute{y}^2(t)dt \le x\int_0^x \acute{y}^2(t)dt,$$
$$|y(x)|^2 \le x\int_0^x \acute{y}^2(t)dt \le \int_0^1 \acute{y}^2(t)dt, \Rightarrow \|y\|^2 \le \|\acute{y}\|^2, \Rightarrow \|y\| \le \|\acute{y}\|.$$

In a similar way, we have

$$\|\acute{y}\| \le \|y'\| \dots \le \left\|y^{(n-2)}\right\| \le \left\|y^{(n-1)}\right\| \le \tfrac{\|L_\varepsilon y\|}{\alpha}. \qquad (5)$$

Thus, there is an inverse operator L_ε^{-1}, therefore, by virtue of item (d) of Lemma 1, L_ε^{-1} admits a closure and $\overline{(L_\varepsilon^{-1})} = (\bar{L}_\varepsilon)^{-1}$. Lemma 4 that $R(L_\varepsilon)$ coincides with the linear diversity continuous on [0,1] functions that is dense in the space H, therefore $\overline{R(L_\varepsilon)} = H$, but $L_\varepsilon \subset \bar{L}_\varepsilon, \Rightarrow R(L_\varepsilon) \subset R(\bar{L}_\varepsilon)$, so a fortiori $\overline{R(\bar{L}_\varepsilon)} = H$. By virtue of a priori estimates (5) $\overline{R(\bar{L}_\varepsilon)} = R(\bar{L}_\varepsilon)$. Indeed, if $y \in D(\bar{L}_\varepsilon)$, then there is a sequence $\{y_n\} \in D(L_\varepsilon)$, such that $y_n \to y$, $L_\varepsilon y_n = f_n \to f$, then it follows from (5) that the sequence $\left\{y_n^{(k)}\right\}$ $(k = 1,2,\dots n-1)$, $n = 1,2,\dots$ in the space H, which means that $y(x) \in W_2^{n-1}[0,1]$ и $y_n(x) \to y(x)$ in the space $W_2^{n-1}[0,1]$. Passing to the limit in inequalities (5), we get:

$$\|y\| \le \|\acute{y}\| \le \dots \le \left\|y^{(n-1)}\right\| \le \tfrac{\|\bar{L}_\varepsilon y\|}{\alpha}. \qquad (6)$$

If $z(x) \in \overline{R(\bar{L}_\varepsilon)}$, then there exists a sequence $\{z_n(x)\} \subset R(\bar{L}_\varepsilon)$, such that $z_n(x) \to z(x)$ in H. Then the sequence $z_n(x) = \bar{L}_\varepsilon y_n$ fundamental in H, and in virtue of a priori estimates, the sequence $\{y_k\}$is fundamental in the space of $W_2^{n-1}[0,1]$, which means that $y_k \to y$, $y_k' \to y'$, $\dots, y_k^{(n-1)} \to y^{(n-1)}$, $\bar{L}_\varepsilon y_n \to z(x)$, i.e. there exists a function $y(x) \in D(\bar{L}_\varepsilon)$ such that $\bar{L}_\varepsilon y_n = z(x)$, that is, $z(x) \in R(\bar{L}_\varepsilon)$, as required.

Thus, $(\bar{L}_\varepsilon)^{-1}$exists and is defined on the whole space H, since

$$D(\bar{L}_\varepsilon)^{-1} = R(\bar{L}_\varepsilon) = \overline{R(\bar{L}_\varepsilon)} = H,$$

then, by the Banach theorem, the operator $(\overline{L}_\varepsilon)^{-1}$ is bounded; this is also obvious from the obtained inequalities (6). By virtue of clause c) of Lemma 1, $(\overline{L}_\varepsilon)^* = L_\varepsilon^*$, therefore by virtue of clause a) of Lemma 2, the operator L_ε^* has a continuous inverse

$(L_\varepsilon^*)^{-1}: H \to D(L_\varepsilon^*)$ and $(L_\varepsilon^*)^{-1} = [(L_\varepsilon^{-1})^{-1}]^* = \left(\left(\overline{L_\varepsilon^{-1}}\right)\right)^* = (L_\varepsilon^{-1})^*$.

From inequalities (6) it follows that

$$\sqrt{\|y\|^2 + \|\acute{y}\|^2} \le \left(\frac{\|\overline{L}_\varepsilon y\|^2}{\alpha^2} + \frac{\|\overline{L}_\varepsilon y\|^2}{\alpha^2}\right)^{\frac{1}{2}} \le \frac{\sqrt{2}}{\alpha}\|\overline{L}_\varepsilon y\|,$$

those, the operator $(\overline{L}_\varepsilon)^{-1}$ translates a bounded set into a compact one, therefore it is completely continuous, by Schauder's theory, the operator $(L^{-1})^*$ is also completely continuous.

Definition 2.3. The closures of the operator L_ε are called the Cauchy operator and denote by C_ε, i.e. $C_\varepsilon = \overline{L_\varepsilon}$.

Lemma 2.6. Under the conditions of Lemma 5, the Cauchy operator C_ε is bounded invertible, and the inverse operator ${C_\varepsilon}^{-1}$ is completely continuous in the space H, moreover, the equality $(L_\varepsilon^*)^{-1} = (L_\varepsilon^{-1})^*$.

Lemma 2.7. If

$$\begin{cases} L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{m=0}^{n-1} a_{n-m}(x)\, y^{(m)}(x), \\ D(L_\varepsilon) = \{y(x) \in C^n[0,1], y(0) = 0, y'(0) = 0, \ldots, y^{(n-1)}(0) = 0\}, \end{cases}$$

$$\begin{cases} L_\varepsilon^+ z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x) z(x)]^{(k)}, \\ D(L_\varepsilon^+) = \{z(x) \in C^n[0,1]; z(1) = 0, z'(1) = 0, \ldots, z^{(n-1)}(1) = 0\}, \end{cases}$$

and the operator S is defined by the equality $Su(x) = u(1-x)$, then the equality $SL_\varepsilon = L_\varepsilon^+ S$ holds if and only if

$$a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2, \ldots, n-1. \qquad (7)$$

Proof. According to the Leibniz's formula

$$[a_{n-k} v]^{(k)} = \sum_{m=0}^{k} c_k^m v^{(m)}\, a_{n-k}^{(k-m)},$$

therefore

$$\sum_{k=0}^{n-1} (-1)^k [a_{n-k} v(x)]^{(k)} = \sum_{k=0}^{n-1} (-1)^k \sum_{m=0}^{k} c_k^m a_{n-k}^{(k-m)}(x)\, v^{(m)}(x) =$$

$$= \sum_{m=0}^{n-1} \left[\sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x)\right] \cdot v^{(m)}(x).$$

Therefore,

$$L_\varepsilon^+ z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{m=0}^{n-1} \left[\sum_{k=m}^{n-1} (-1)^k c_k^m a_{n-k}^{(k-m)}(x)\right] \cdot z^{(m)}(x).$$

Then

$$SL_\varepsilon y = \varepsilon y^{(n)}(1-x) + \sum_{m=0}^{n-1} a_{n-m}(1-x)\, y^{(m)}(1-x),$$

$$L_\varepsilon^+ Sy = (-1)^n \varepsilon (Sy)^{(n)} + \sum_{m=0}^{n-1}\left[\sum_{k=m}^{n-1}(-1)^k c_k^m a_{n-k}^{(k-m)}(x)\right]\cdot (Sy)^{(m)} =$$

$$= \varepsilon y^{(n)}(1-x) + \sum_{m=0}^{n-1}\left[\sum_{k=m}^{n-1}(-1)^k c_k^m a_{n-k}^{(k-m)}(x)\right](-1)^m y^{(m)}(1-x).$$

Equating the corresponding coefficients of these expressions, we obtain the formula (7).

Lemma 2.8. If

(a) $Su(x) = u(1-x)$;

(b) $L_\varepsilon y(x) = \varepsilon y^{(n)} + \sum_{m=0}^{n-1} a_{n-m}(x)\, y^{(m)}(x); y \in D(L_\varepsilon)$;

(c) $a_{n-m}(1-x) = \sum_{k=m}^{n-1}(-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x)\,, m = 0,1,2,\dots,n-1,$

then the operator SL_ε is symmetric in the space H.

Proof. If $y(x) \in D(L_\varepsilon)$, i.e. $y(x)C^n[0,1]$, and $y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0$, then $Sy(x) = y(1-x) \in D(L_\varepsilon^+)$. In fact, it is obvious that $Sy(x) \in C^n[0,1]$, and $[Sy(x)]^{(m)} = (-1)^m y^{(m)}(1-x)$, therefore, $[Sy(x)]^{(m)}\big|_{x=1} = 0,\ m = 0,1,2,\dots n-1$. By virtue of the previous lemma, the equality $SL_\varepsilon = L_\varepsilon^+ S$, holds; therefore, for all $u(x)$ and $v(x) \in D(L_\varepsilon)$ we have $(SL_\varepsilon u, v) = (L_\varepsilon u, Sv) = (u, L_\varepsilon^+ Sv) = (u, SL_\varepsilon v)$.

Lemma 2.9. If

(a) $Su(x) = u(1-x)$;

(b) $L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{m=0}^{n-1} a_{n-m}(x)\, y^{(m)}(x)$;

$y(x) \in D(L_\varepsilon) = \{y(x)C^n[0,1];\ y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0\}$,

then equality takes place

$$\overline{SL_\varepsilon} = S\bar{L}_\varepsilon,$$

where the bar ($\bar{\ }$), as usual, means the operator's closure operation.

Proof.

Suppose that the operator SL_ε is not closable, then there exists a sequence $u_n \in D(L_\varepsilon)$ such that $u_n \to 0$, $SL_\varepsilon u_n \to f \in H$ and $f \neq 0$. Then $u_n \in D(L_\varepsilon)$, $L_\varepsilon u_n = SSL_\varepsilon u_n \to Sf \neq 0$, therefore L_ε is also not closable. Similarly, the non-closure of the operator L_ε implies the non-closure of the operator SL_ε. Therefore, the operator SL_ε is closable if and only if we close the operator L_ε. If $u \in D(\bar{L}_\varepsilon)$, then there is a sequence $u_n \in D(SL_\varepsilon)$ such that $u_n \to u$, $L_\varepsilon u_n \to \bar{L}_\varepsilon u$, therefore, $u_n \in D(SL_\varepsilon)$ and $SL_\varepsilon u_n \to S\bar{L}_\varepsilon u$.. Therefore, $u \in D(\overline{SL_\varepsilon})$ and $S\bar{L}_\varepsilon u = \overline{SL_\varepsilon} u$. Thus, if $u \in D(\bar{L}_\varepsilon)$, then $u \in D(\overline{SL_\varepsilon})$ and the equality $\overline{SL_\varepsilon} u = S\bar{L}_\varepsilon u$ holds.

Conversely, let $u \in D(\overline{SL_\varepsilon})$. Then there is a sequence $u_n \in D(SL_\varepsilon)$, such that $u_n \to u$ and $SL_\varepsilon u_n \to \overline{SL_\varepsilon} u$, since $D(SL_\varepsilon) = D(L_\varepsilon)$, then $u_n \in D(L_\varepsilon)$ and $L_\varepsilon u_n = S\,\underbrace{SL_\varepsilon u_n} \to S(\overline{SL_\varepsilon} u)$, and this means that $u \in D(\bar{L}_\varepsilon)$ and $\bar{L}_\varepsilon u = S\overline{SL_\varepsilon} u$, i.e. $S\bar{L}_\varepsilon = \overline{SL_\varepsilon}$.

Consequence 2.1. The operator SL_ε is self-adjoint in the essential in space, i.e. $(\overline{SL_\varepsilon})^* = \overline{SL_\varepsilon}$.

Proof. The operator SL_ε is symmetric, therefore, the operator $\overline{SL_\varepsilon}$ is also symmetric. Since $\overline{SL_\varepsilon} = S\bar{L}_\varepsilon$ and $R(\bar{L}_\varepsilon) = H$, then $R(S\bar{L}_\varepsilon) = H, \Rightarrow R(\overline{SL_\varepsilon}) = H$. From the symmetry of the operator SL_ε it follows that $SL_\varepsilon \subset (SL_\varepsilon)^*$, passing to the closure, and taking into account the closedness of the adjoint operator, we get the inclusion $\overline{SL_\varepsilon} \subset (SL_\varepsilon)^*$, and since $R(\overline{SL_\varepsilon}) = H$, then $R(SL_\varepsilon)^* = H$ and $R(\overline{SL_\varepsilon}) = R(SL_\varepsilon)^*$. Therefore, taking into account the invertibility of these operators, we have

$$D(\overline{SL_\varepsilon}) = D(SL_\varepsilon)^* \text{ и поэтому } \overline{SL_\varepsilon} = (SL_\varepsilon)^*.$$

So $(\overline{SL_\varepsilon})^* = (SL_\varepsilon)^{**} = \overline{SL_\varepsilon}$, that's what was required to prove.

Lemma 2.10. If the k - th coefficient of equation (1) is $n-k$ $(k = 0,1,2,\dots,n)$ times continuously differentiable on the interval [0,1], and satisfies the following conditions:

(a) $a_1(x) \geq \alpha > 0$;

(b) $\left(\sum_{k=2}^{n} a_k(x) y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$;

(c) $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2,\dots,n-1$,

where c_k^m are binomial coefficients, then the SC_ε operator is self-adjoint in the space H, and has a completely continuous inverse, where C_ε is the Cauchy operator.

3. Main Results.

Theorem 3.1. If the k -th coefficient of equation (1) $n-k$ $(k = 0,1,2,\dots,n)$ is continuously differentiable on the interval [0,1] and satisfies the following conditions:

(a) $a_1(x) \geq \alpha > 0$;

(b) $\left(\sum_{k=2}^{n} a_k(x) y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$;

(c) $a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2,\dots,n-1$,

where c_k^m are binomial coefficients, then the Cauchy's problem (1) - (2) is strongly solvable and this strong solution has the following representation:

$$y(x,\varepsilon,f) = \sum_{n=1}^{\infty} \frac{(Sf,\varphi_n)}{\lambda_n} \varphi_n(x), \tag{8}$$

where λ_n $(n = 1,2,\dots)$ - are eigenvalues, and $\varphi_n(x)$ $(n = 1,2,\dots)$ are eigenfunctions of the operator SL_ε, the S operator is defined by the formula:

$$Su(x) = u(1-x).$$

Proof. By Lemma 10, the operator $(SC_\varepsilon)^{-1}$ is completely continuous and self-adjoint, therefore, according to the Hilbert – Schmidt theorem, the decomposition takes place

$$(SC_\varepsilon)^{-1} f = \sum_{n=1}^{\infty} \frac{(f,\varphi_n)}{\lambda_n} \varphi_n(x) + \varphi_0(x),$$

where $\varphi_0(x) \in ker(SC_\varepsilon)^{-1}$ and $\{\varphi_n(x)\}$ $(n = 1,2,\dots)$ - are orthonormal eigenvectors of the operator $(SC_\varepsilon)^{-1}$, and λ_n^{-1} $(n = 1,2,\dots)$the corresponding eigenvalues of the same operator, then $(SC_\varepsilon)^{-1}\varphi_0 = 0, \Rightarrow \varphi_0 = SC_\varepsilon (SC_\varepsilon)^{-1} \varphi_0 = 0$, therefore

$$(SC_\varepsilon)^{-1} f(x) = \sum_{n=1}^{\infty} \frac{(f,\varphi_n)}{\lambda_n} \varphi_n(x).$$

If for some function $f(x) \in H$ there is the equality $(f,\varphi_n) = 0$ $(n = 1,2,\dots)$, then $(SC_\varepsilon)^{-1} f = 0, \Rightarrow f = 0$, the system of eigenfunctions $\{\varphi_n\}$ is complete in H. Since, by virtue of the self-adjointness of the operator $(SC_\varepsilon)^{-1}$, this system is orthogonal, after normalization it forms an orthonormal basis of the space H.

We now return to the Cauchy's problem (1) - (2).

$$L_\varepsilon y(x) = \varepsilon y^{(n)}(x) + \sum_{k=1}^{n} a_k(x)\, y^{(n-k)}(x) = f(x), x \in (0,1],$$

$$y(0) = 0, y'(0) = 0, \dots, y^{(n-1)}(0) = 0,$$

or in operator form:

$$L_\varepsilon y = f.$$

Acting by the operator S on both sides of the equation, we get

$$SL_\varepsilon y = Sf.$$

Therefore, for all $y(x) \in D(L_\varepsilon)$, the equality

$$SC_\varepsilon y = S\bar{L}_\varepsilon y = SL_\varepsilon y = Sf,$$

So

$$y(x) = y(x, \varepsilon, f) = (SC_\varepsilon)^{-1}Sf = \sum_{n=1}^{\infty}((SC_\varepsilon)^{-1}Sf, \varphi_n)\varphi_n(x) =$$

$$= \sum_{n=1}^{\infty}(Sf, (SC_\varepsilon)^{-1}\varphi_n)\varphi_n(x) = \sum_{n=1}^{\infty}\frac{(Sf, \varphi_n)}{\lambda_n}\varphi_n(x).$$

The equation for the functions is:

$$(SC_\varepsilon)^{-1}\varphi_n = \frac{\varphi_n(x)}{\lambda_n}, \lambda_n \neq 0, n = 1,2, ...$$

or

$$C_\varepsilon^{-1}S\varphi_n = \frac{\varphi_n(x)}{\lambda_n}, (\bar{L}_\varepsilon)^{-1}S\varphi_n = \frac{\varphi_n(x)}{\lambda_n}.$$

So $\psi_n(x) = S\varphi_n(x)$, we have

$$(\bar{L}_\varepsilon)^{-1}\psi_n(x) = \frac{S\psi_n(x)}{\lambda_n}$$

or

$$S\psi_n(x) = \lambda_n(\bar{L}_\varepsilon)^{-1}\psi_n(x) = \lambda_n\overline{(L_\varepsilon^{-1})}\psi_n(x) = \lambda_n\int_0^x K(x,t)\,\psi_n(t)dt.$$

If $\psi_n(t) \in L^2(0,1)$, then from this equation we see that ⟦$S\psi_n \in W_2^n[0,1]$, then $\varphi_n(x) \in W_2^n[0,1]$. If $\varphi_n(x) \in W_2^n[0,1]$, then $S\psi_n(x) \in W_2^{2n}[0,1]$, then $\varphi_n(x) \in W_2^{2n}[0,1]$, continuing this process, we obtain that $\varphi_n(x) \in C^\infty$, i.e. infinitely differentiable. Therefore, any function belongs to the domain $D(L_\varepsilon)$, therefore

$$\lambda_n\varphi_n = SC_\varepsilon\varphi_n = S\bar{L}_\varepsilon\varphi_n = SL_\varepsilon\varphi_n, =>$$

$$L_\varepsilon\varphi_n(x) = \lambda_n S\varphi_n(x), n = 1,2, ...$$

Therefore,

$$\begin{cases} \varepsilon\varphi_m^{(n)}(x) + a_1(x)\varphi_m^{(n-1)}(x) + \dots + a_n(x)\varphi_m(x) = \lambda_m\varphi_m(1-x) = \lambda_m S\varphi_m(x), \\ \varphi_m(0) = 0, \varphi_m'(0) = 0, \dots, \varphi_m^{(n-1)}(0) = 0, \end{cases}$$

For further clarity, we will study the properties of the operator B, where

$$\begin{cases} Bu(x) = a_1(x)u^{(n-1)}(x) + a_2(x)u^{(n-2)}(x) + \dots + a_n(x)u(x), \\ D(B) = \{u(x) \in C^{n-1}(0,1) \cap C^{n-2}[0,1]; u(0) = 0, u'(0) = 0, \dots, u^{(n-2)}(0) = 0, \end{cases}$$

denote these equations as (9)-(10).

Theorem 3.2.

(a) If k -th coefficient of operator (9)-(10) $n-k$ times is continuously differentiable on the interval [0,1], then one of the formally adjoint operators of operator B has the form:

$$\begin{cases} B^{+}v = \sum_{k=1}^{n} (-1)^{n-k}[a_k(x)v(x)]^{(n-k)}, \\ v(1) = 0, v'(1) = 0, \dots, v^{(n-2)}(1) = 0; \end{cases}$$

(b) if there is equality

$$a_{n-m}(1-x) = \sum_{k=m}^{n-1} (-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2, \dots, n-1,$$

where c_k^m- are binomial coefficients then

$$SBu = B^{+}Su, \forall u(x) \in D(B).$$

Proof. If $u \in D(B)$ and $v(x) \in D(B^{+})$, so

(a) $(Bu, v) = \left(\sum_{k=1}^{n} a_k(x)u^{(n-k)}, v\right) = \sum_{k=1}^{n}\left(a_k(x)u^{(n-k)}, v\right);$

$$\left(a_k u^{(n-k)}, v\right) = \left(a_k v, u^{(n-k)}\right) = \int_0^1 a_k v u^{(n-k)} dx = \int_0^1 a_k v du^{(n-k-1)} =$$

$$= a_k v u^{(n-k-1)}\Big|_0^1 - \int_0^1 (a_k v)' u^{(n-k-1)} dx = a_k v u^{(n-k-1)}\Big|_0^1 -$$

$$- \int_0^1 (a_k v)' du^{(n-k-2)} dx = a_k v u^{(n-k-1)}\Big|_0^1 - (a_k v)' u^{(n-k-2)}\Big|_0^1 +$$

$$+ \int_0^1 (a_k v)'' u^{(n-k-2)} dx = \dots = \sum_{m=0}^{n-k-1} (a_k v)^{m} (-1)^{m} u^{(n-k-1-m)}\Big|_0^1 +$$

$$+(-1)^{n-k} \int_0^1 (a_k v)^{(n-k)} u(x) dx,$$

Therefore,

$$(Bu, v) = \sum_{k=1}^{n} \sum_{m=0}^{n-k-1} (a_k v)^{m} (-1)^{m} u^{(n-k-1-m)}\Big|_0^1 + \sum_{k=1}^{n} (-1)^{n-k}\left(u, (a_k v)^{(n-k)}\right)$$

$$= [u, v] + \left(u, \sum_{k=1}^{n} (-1)^{n-k} (a_k v)^{(n-k)}\right),$$

where

$$[u, v] = \sum_{k=1}^{n} \sum_{m=0}^{n-k-1} (a_k v)^{m} (-1)^{m} u^{(n-k-1-m)}(x)\Big|_0^1.$$

If

$$u(0) = 0, u'(0) = 0, \dots, u^{(n-2)}(0) = 0;$$
$$v(1) = 0, v'(1) = 0, \dots, v^{(n-2)}(1) = 0,$$

therefore $[u, v] = 0$, so $(Bu, v) = (u, B^{+}v)$, where

$$B^{+}v = \sum_{k=1}^{n} (-1)^{n-k} (a_k v)^{(n-k)}.$$

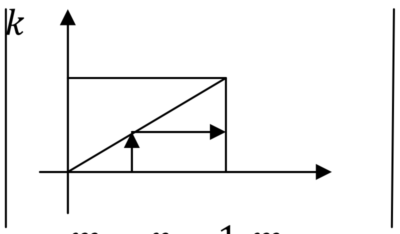
(b) If $u(x) \in D(B)$, we have it $Su(x) \in D(B^{+})$ and expression $B^{+}Su$ it has sense

$$B^{+}Su = \sum_{k=1}^{n}(-1)^{n-k}\,(a_k Su)^{(n-k)} = \left|\begin{matrix} n-k=m, \\ k=n-m \end{matrix}\right| =$$

$$\sum_{m=n-1}^{0}(-1)^{m}(a_{n-m}Su)^{(m)} = \sum_{m=0}^{n-1}(-1)^{m}(a_{n-m}Su)^{(m)} =$$

$$= \sum_{m=0}^{n-1}(-1)^{m}\sum_{k=0}^{m} a_{n-m}^{(m-k)} c_m^k (Su)^{(k)} = \sum_{m=0}^{n-1}(-1)^{m+k}\sum_{k=0}^{m} a_{n-m}^{(m-k)} c_m^k Su^{(k)} =$$

$$= \left(\sum_{k=0}^{n-1}\sum_{m=k}^{n-1}(-1)^{m+k} a_{n-m}^{(m-k)} c_m^k\right) Su^{(k)};$$



$m = n-1, m$

$$SBu = S\sum_{k=0}^{n-1} a_{n-k}u^{(k)} = \sum_{k=0}^{n-1} a_{n-k}(1-x)\,Su^{(k)} =$$

$$= \sum_{k=0}^{n-1}\left(\sum_{m=1}^{n-1}(-1)^{m+k} c_m^k a_{n-m}^{(m-k)}\right) Su^{(k)} = B^{+}Su.$$

By Lemma 10, the operator $\overline{SB}$ is in the space H. Therefore, $(\overline{SB})^{*} = \overline{SB} = S\bar{B}$, $(S\bar{B})^{*} = S\bar{B}$, $(\bar{B})^{*}S^{*} = S\bar{B}$, $(\bar{B})^{*}S = S\bar{B}$, $B^{*}S = S\bar{B}$, $B^{*} = S\bar{B}S$, $(B^{*})^{-1} = S^{-1}(\bar{B})^{-1}S^{-1} = S(\bar{B})^{-1}S$, $S(B^{*})^{-1} = (\bar{B})^{-1}S$, $S(B^{-1})^{*} = \overline{(B^{-1})}S$, $(B^{-1})^{*}S = S\overline{(B^{-1})}$,that's what was required to prove.

Theorem 3.3. If the k - th coefficient of equation (1) is $n-k$ $(k = 0,1,2,\dots,n)$ it is continuously different once on the interval [0,1] and satisfies the following conditions:

(a) $a_1(x) \geq \alpha > 0$;

(b) $\left(\sum_{k=2}^{n} a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$;

(c) $a_{n-m}(1-x) = \sum_{k=m}^{n-1}(-1)^{m+k} c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2,\dots,n-1,$

where c_k^m -are binomial coefficients, then the formula holds:

$$\psi(x) = \sum_{m=1}^{\infty}\frac{\varepsilon\varphi_m(1)}{\lambda_m}\varphi_m(x), \qquad (11)$$

where

$$\varepsilon\varphi_m^{(n)}(x) + a_1(x)\varphi_m^{(n-1)}(x) + \dots + a_n(x)\varphi_m(x) = \lambda_m S\varphi_m(x);$$

$$\varphi_m(0) = 0, \varphi_m^{'}(0) = 0, \dots, \varphi_m^{(n-1)}(0) = 0,$$

and $\psi(x)$ - is a solution of a homogeneous equation

$$\varepsilon\psi^{(n)}(x)+a_1(x)\psi^{(n-1)}(x)+\cdots+a_n(x)\psi(x)=0 \tag{12}$$

satisfying the initial conditions:

$$\psi(0)=0,\psi'(0)=0,\dots,\psi^{(n-2)}(0)=0,\psi^{(n-1)}(0)=1.$$

Proof. Noting that $\psi(x)\in D(B)$ we rewrite equations (12) in the form $\varepsilon\psi^{(n)}(x)+B\psi(x)=0$. Noting also that $\varphi_m(x)\in D(B)$ and $\varphi_m'(x)\in D(B)$we rewrite the equations of the functions as

$$\varepsilon\varphi_m^{(n)}(x)+B\varphi_m(x)=\lambda_m S\varphi_m(x),\ S\varphi_m(x)=\varphi_m(1-x).$$

Using these two formulas, we calculate the Fourier coefficients of this function $SB\psi$.

$$(SB\psi,\varphi_m)=\left(SB\psi,\lambda_m B^{-1}S\varphi_m-\varepsilon B^{-1}\varphi_m^{(n)}\right)=\lambda_m(SB\psi,B^{-1}S\varphi_m)-$$

$$-\varepsilon\left(SB\psi,B^{-1}\varphi_m^{(n)}\right)=\lambda_m((B^{-1})^*SB\psi,S\varphi_m)-\varepsilon\left((B^{-1})^*SB\psi,\varphi_m^{(n)}\right)=$$

$$=\lambda_m\left(S\overline{B^{-1}}B\psi,S\varphi_m\right)-\varepsilon\left(S\overline{B^{-1}}B\psi,\varphi_m^{(n)}\right)=\left|B\psi\in D(B^{-1})\subset D\overline{(B^{-1})}\right|=$$

$$=\lambda_m(SB^{-1}B\psi,S\varphi_m)-\varepsilon\left(SB^{-1}B\psi,\varphi_m^{(n)}\right)=\lambda_m(\psi,\varphi_m)-\varepsilon\left(S\psi,\varphi_m^{(n)}\right);$$

Using integration by parts, we transform the scalar product $\left(S\psi,\varphi_m^{(n)}\right)$.

$$\left(S\psi,\varphi_m^{(n)}\right)=\int_0^1 S\psi d\,\varphi_m^{(n-1)}(x)=S\psi\cdot\varphi_m^{(n-1)}(x)\Big|_0^1-\int_0^1(S\psi)'\varphi_m^{(n-1)}(x)dx=$$

$$=\psi(1-x)\varphi_m^{(n-1)}(x)\Big|_0^1-\int_0^1(S\psi)'\varphi_m^{(n-1)}(x)dx=-\int_0^1(S\psi)'\varphi_m^{(n-1)}(x)dx;$$

$$\left(S\psi,\varphi_m^{(n)}\right)=\sum_{k=0}^{n-1}(-1)^k(S\psi)^{(k)}\varphi_m^{(n-1-k)}(x)|_0^1+(-1)^n\int_0^1(S\psi)^{(n)}\varphi_m(x)dx=$$

$$=\sum_{k=0}^{n-1}(-1)^k\,(-1)^k\psi^{(k)}(1-x)\,\varphi_m^{(n-1-k)}(x)\Big|_0^1+(-1)^{n+n}\cdot$$

$$\cdot\int_0^1\psi^{(n)}(1-x)\varphi_m(x)dx=\sum_{k=0}^{n-1}\psi^{(k)}(1-x)\,\varphi_m^{(n-1-k)}(x)\Big|_0^1+$$

$$+\int_0^1\psi^{(n)}(1-x)\varphi_m(x)dx=\psi^{(n-1)}(0)\varphi_m(1)+\left(S\psi^{(n)},\varphi_m\right)=$$

$$=\varphi_m(1)+\left(S\psi^{(n)},\varphi_m\right);$$

Operated by the operator S on both sides of the equation

$$\varepsilon\psi^{(n)}(x)+B\psi(x)=0,$$

we have

$$\varepsilon S\psi^{(n)}+SB\psi(x)=0,\ S\psi^{(n)}=-\tfrac{SB\psi}{\varepsilon}.$$

Therefore,

$$\left(S\psi^{(n)},\varphi_m\right)=\left(=-\tfrac{SB\psi}{\varepsilon},\varphi_m\right),$$

so

$$\left(S\psi,\varphi_m^{(n)}\right)=\varphi_m(1)-\left(\tfrac{SB\psi}{\varepsilon},\varphi_m\right).$$

So

$$\underbrace{(SB\psi,\varphi_m)}=\lambda_m(\psi,\varphi_m)-\varepsilon\varphi_m(1)+\underbrace{(SB\psi,\varphi_m)},=>\lambda_m(\psi,\varphi_m)=\varepsilon\varphi_m(1),$$

$$(\psi,\varphi_m)=\frac{\varepsilon\varphi_m(1)}{\lambda_m},\psi(x)=\sum_{m=1}^{\infty}(\psi,\varphi_m)\cdot\varphi_m(x)=\sum_{m=1}^{\infty}\frac{\varepsilon\varphi_m(1)}{\lambda_m}\varphi_m(x),$$

that's what was required to prove.

Apparently, the formula (11) has independent value, for example, for the solution of inverse problems.

Now, using the formula (8) we derive a recurrence relation for the solution of the Cauchy problem (1)-(2). For convenience, we'll rewrite it first

$$y(x,\varepsilon,f)=\textstyle\sum_{m=1}^{\infty}\frac{(Sf,\varphi_m)}{\lambda_m}\varphi_m(x). \tag{8}$$

Using integration by parts, we transform the coefficients of this series.

$$(Sf,\varphi_m)=\left(Sf,\lambda_m B^{-1}S\varphi_m-\varepsilon B^{-1}\varphi_m^{(n)}\right)=\lambda_m(Sf,B^{-1}S\varphi_m)-$$

$$-\varepsilon\left(Sf,B^{-1}\varphi_m^{(n)}\right)=\lambda_m((B^{-1})^*Sf,S\varphi_m)-\varepsilon\left((B^{-1})^*Sf,\varphi_m^{(n)}\right)=$$

$$\left|(B^{-1})^*S=S\overline{(B^{-1})}\right|=\lambda_m\left(S\overline{B^{-1}}f,S\varphi_m\right)-\varepsilon\left(S\overline{B^{-1}}f,\varphi_m^{(n)}\right)=$$

$$=\lambda_m(B^{-1}f,\varphi_m)-\varepsilon\left(S(\bar{B})^{-1}f,\varphi_m^{(n)}\right);$$

$$\left(S\overline{B^{-1}}f,\varphi_m^{(n)}\right)=\sum_{k=0}^{n-1}(-1)^k\left(S\overline{B^{-1}}f\right)^{(k)}\varphi_m^{(n-1-k)}\Big|_0^1+$$

$$+(-1)^n\int_0^1\left(S\overline{B^{-1}}f\right)^{(n)}\varphi_m(x)dx=\sum_{k=0}^{n-1}S\left(\overline{B^{-1}}f\right)^{(k)}\varphi_m^{(n-1-k)}(x)\Big|_0^1+$$

$$+\left(S\left(\overline{B^{-1}}f\right)^{(n)},\varphi_m\right)=\left(\overline{B^{-1}}f\right)^{(n-1)}(0)\varphi_m(1)+\left(S\left(\overline{B^{-1}}f\right)^{(n)},\varphi_m\right),=>$$

$$(Sf,\varphi_m)=\lambda_m\left(\overline{B^{-1}}f,\varphi_m\right)-\varepsilon\left(\overline{B^{-1}}f\right)^{(n-1)}(0)\varphi_m(1)-\varepsilon\left(S\left(\overline{B^{-1}}f\right)^{(n)},\varphi_m\right);$$

Therefore,

$$y(x,\varepsilon,f)=\overline{B^{-1}}f-\left(\overline{B^{-1}}f\right)^{(n-1)}(0)\sum_{m=1}^{\infty}\frac{\varepsilon\varphi_m(1)}{\lambda_m}-\varepsilon y\left(x,\varepsilon,\frac{d^n}{dx^n}\overline{B^{-1}}f\right)=$$

$$=\overline{B^{-1}}f(x)-\left(\overline{B^{-1}}f\right)^{(n-1)}(0)\psi(x)-\varepsilon y\left(x,\varepsilon,\tfrac{d^n}{dx^n}\overline{B^{-1}}f\right). \tag{13}$$

Note that the function $\overline{B^{-1}}f$ as a strong solution to the Cauchy problem belongs to the space $W_2^{n-1}[0,1]$; therefore, for the validity of the formula obtained, it suffices to require that $f(x)\in W_2^1[0,1]$. Denoting,

$$D^0 = I, D = \frac{d^n}{dx^n}\overline{B^{-1}},$$

rewrite the formula as:

$$y(x,\varepsilon,f) = B^{-1}D^0f(x) - (B^{-1}D^0f)^{(n-1)}(0)\psi(0) - \varepsilon y(x,\varepsilon,Df).$$

Further,

$$y(x,\varepsilon,Df) = B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x) - \varepsilon y(x,\varepsilon,D^2f), =>$$

$$y(x,\varepsilon,f) = B^{-1}D^0f(x) - (B^{-1}D^0f)^{(n-1)}(0)\psi(x) -$$

$$-\varepsilon\left[B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x) - y(x,\varepsilon,D^2f)\right] = B^{-1}D^0f(x) -$$

$$-(B^{-1}D^0f)^{(n-1)}(0)\psi(x) - \varepsilon\left[B^{-1}Df(x) - (B^{-1}Df)^{(n-1)}(0)\psi(x)\right] +$$

$$+\varepsilon^2 y(x,\varepsilon,D^2f).$$

Continuing this process by the method of mathematical induction, we obtain

$$y(x,\varepsilon,f) = \sum_{k=0}^{n-1}(-1)^k\left[B^{-1}D^kf(x) - \left(B^{-1}D^kf\right)^{(n-1)}(0)\psi(x)\right]\varepsilon^k +$$
$$+(-1)^n\varepsilon^n y(x,\varepsilon,D^nf),$$

where $\|y(x,\varepsilon,D^nf)\| \le \frac{\|D^nf\|}{\alpha}$. From this formula it is seen that, if $f(x) \in W_2^n[0,1]$, then $D^nf \in L^2(0,1)$ and $y(x,\varepsilon,D^nf) \in W_2^n[0,1]$; the function $\psi(x)$, at least n - times continuously differentiable. $D^kf(x) \in W_2^{n-k}[0,1]$, $B^{-1}D^kf(x) \in W_2^{2n-k-1}$. For $k = n-1$, $B^{-1}D^{n-1}f(x) \in W_2^n[0,1]$. We formulate the result.

Theorem 3.4. If the k - th coefficient of equation (1) $n-k$ times is continuously differentiable on the interval [0,1] and satisfies the following conditions:

(a) $a_1(x) \ge \alpha > 0, \forall x \in [0,1]$;

(b) $\left(\sum_{k=2}^n a_k(x)y^{(n-k)}(x), y^{(n-1)}(x)\right) \ge 0, \forall y \in D(L_\varepsilon)$;

(c) $a_{n-m}(1-x) = \sum_{k=m}^{n-1}(-1)^{m+k}c_k^m a_{n-k}^{(k-m)}(x), m = 0,1,2,\dots,n-1,$ (14)

where c_k^m - are binomial coefficients and the right part belongs to the space $W_2^n[0,1]$, then the solution of the Cauchy problem (1) - (2) also belongs to the space $W_2^n[0,1]$ and admits an asymptotic expansion of the form:

$$y(x,\varepsilon,f) = \sum_{k=0}^{n-1}(-1)^k\left[B^{-1}D^kf(x) - \left(B^{-1}D^kf\right)^{(n-1)}(0)\psi(x)\right]\varepsilon^k +$$
$$+(-1)^n\varepsilon^n y(x,\varepsilon,D^nf),$$

where

$$\|y(x,\varepsilon,D^nf)\| \le \frac{\|D^nf\|}{\alpha},\ D^0 = I,\ D = \frac{d^n}{dx^n}\overline{B^{-1}}.$$

$$\begin{cases} Bz(x) = a_1(x)z^{(n-1)}(x) + a_2(x)z^{(n-2)}(x) + \dots + a_n(x)z(x), \\ z(0) = 0, z'(0) = 0, \dots, z^{(n-2)}(0) = 0; \end{cases}$$

$$\begin{cases} \varepsilon\psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \cdots + a_n(x)\psi(x) = 0, \\ \psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1. \end{cases}$$

4. Discussions.

Remark 4.1. Whether it is possible to prove the recurrent formula (13) directly, by direct computations, if possible, we would get rid of the conditions (14) that are needed for the self-conjugacy of the operator SC_ε.

By virtue of the known formula, we have

$$\varepsilon y\left(x, \varepsilon, \left(\overline{B^{-1}} f\right)^{(n)}\right) = \varepsilon \int_0^x K(x,t)\left(\overline{B^{-1}} f(t)\right)^{(n)} dt =$$

$$= \varepsilon \left[\sum_{m=0}^{n-1} (-1)^m \frac{\partial^m K}{\partial t^m} \left(\overline{B^{-1}} f\right)^{n-1-m} \Bigg|_0^x + (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt \right];$$

If $f \in L^2(0,1)$, then $\overline{B^{-1}} f \in W_2^{n-1}[0,1]$, i.e. This function has absolutely continuous derivatives up to the $n-2$ - order and a derivative of the $n-1$ - order that is summable with square. Thus, all derivatives $\left(\overline{B^{-1}} f\right)^{(k)}$ up to $n-2$ - nd order inclusive are continuous and formulas $\left(\overline{B^{-1}} f\right)^{(k)}(0)$ make sense and, moreover, they all vanish.

A little earlier we proved the formula $(L_\varepsilon^*)^{-1} = (L_\varepsilon^{-1})^*$, now we use this formula.

$$L_\varepsilon^{-1} f(x) = \int_0^x K(x,t) f(t) dt = \int_0^1 \theta(x-t)\, K(x,t) f(t) dt,$$

$$(L_\varepsilon^{-1})^* g(x) = \int_0^1 K^*(x,t) g(t) dt = \int_0^1 \theta(x-t) K(t,x) g(t) dt = \int_x^1 K(t,x) g(t) dt;$$

We act by the operator L_ε^* on both sides of this equality, in the end we get

$$g(x) = L_\varepsilon^* \cdot \int_x^1 K(t,x) g(t) dt.$$

Now let's calculate the right side, for convenience

$$z(x) = \int_x^1 K(t,x) g(t) dt$$

we have

$$z'(x) = -K(t,x)g(t)|_{t=x} + \int_x^1 \frac{\partial K}{\partial x}(t,x) g(t) dt = \int_x^1 \frac{\partial K}{\partial x}(t,x) g(t) dt,$$

$$z''(x) = -\frac{\partial K}{\partial x}(t,x) g(t)\Big|_{t=x} + \int_x^1 \frac{\partial^2 K}{\partial x^2}(t,x) g(t) dt = \int_x^1 \frac{\partial^2 K}{\partial x^2}(t,x) g(t) dt, \dots,$$

$$z^{(n-1)}(x) = \int_x^1 \frac{\partial^{n-1} K}{\partial x^{n-1}}(t,x) g(t) dt,$$

$$z^{(n)}(x) = -\frac{\partial^{n-1} K}{\partial x^{n-1}}(t,x) g(t)\Big|_{t=x} + \int_x^1 \frac{\partial^n K}{\partial x^n}(t,x) g(t) dt;$$

Therefore

$$L_\varepsilon^* z(x) = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x) z(x)]^{(k)};$$

$$[a_{n-k}(x)z(x)]^{(k)} = \sum_{j=0}^{k} a_{n-k}^{(k-j)}(x)\, z^{(j)} c_k^j = \sum_{j=0}^{k} a_{n-k}^{(k-j)}(x) c_k^j \int_x^1 \frac{\partial^j K}{\partial x^j}(t,x) g(t) dt =$$

$$= \int_x^1 \sum_{j=0}^{k} a_{n-k}^{(k-j)}(x) c_k^j \frac{\partial^j K}{\partial x^j} g(t) dt = \int_x^1 [a_{n-k}(x) K(t,x)]^{(k)} g(t) dt, 1 \le k \le n-1;$$

Therefore,

$$L_\varepsilon^* z = (-1)^n \varepsilon z^{(n)} + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x) z(x)]^{(k)} =$$

$$(-1)^n \varepsilon \int_x^1 \frac{\partial^n K}{\partial x^n}(t,x) g(t) dt - (-1)^n \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t,x) g(t)\Big|_{t=x} + \cdots =$$

$$= \int_x^1 \left[(-1)^n \varepsilon \frac{\partial^n K}{\partial x^n}(t,x) + \sum_{k=0}^{n-1} (-1)^k [a_{n-k}(x) K(t,x)]^{(k)} \right] g(t) dt -$$

$$-(-1)^n \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t,x) g(t)\Big|_{t=x} = g(x).$$

Due to the arbitrariness of $g(x)$, we conclude that

$$(-1)^{n-1} \varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} K(t,x) = 1$$

and

$$L_\varepsilon^+ K(t,x) = 0,$$

i.e. the first variable, the Cauchy kernel is the solution of the homogeneous equation $L_\varepsilon K(x,t) = 0$ and the second variable is a solution of the homogeneous equation $L^+ K(t,x) = 0$.

Now using the formula

$$\int_0^x v(x) u^{(n)}(t) dt = \sum_{m=0}^{n-1} (-1)^m v^{(m)} u^{(n-1-m)}(t)|_0^x + (-1)^n \int_0^x v^{(n)}(t) u(t) dt,$$

convert expressions: $\varepsilon y\left(x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f\right)$.

$$\varepsilon y\left(x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f\right) = \varepsilon \int_0^x K(x,t) \frac{d^n}{dx^n} \overline{B^{-1}} f(t) dt =$$

$$= \varepsilon \left[\sum_{m=0}^{n-1} (-1)^m \frac{\partial^m K}{\partial t^m} \left(\overline{B^{-1}} f\right)^{(n-1-m)}(t)\Big|_0^x + (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt \right] =$$

$$= \varepsilon \left[(-1)^{n-1} \frac{\partial^{n-1} K}{\partial t^{n-1}} \left(\overline{B^{-1}} f\right)(t)\Big|_{t=x} - K(x,t) \left(\overline{B^{-1}} f\right)^{(n-1)}(0) \right] +$$

$$+\varepsilon (-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t) dt;$$

From the equation

$$(-1)^n \varepsilon \frac{\partial^n K}{\partial t^n} + B^+ K(x,t) = 0$$

we have

$$(-1)^n \varepsilon \frac{\partial^n K}{\partial t^n} = -B^+ K(x,t).$$

so

$$\varepsilon(-1)^n \int_0^x \frac{\partial^n K}{\partial t^n} \overline{B^{-1}} f(t)dt = -\int_0^x B^+ K(x,t)\, \overline{B^{-1}} f(t)dt =$$

$$= -\left[\int_0^x K(x,t) B\overline{B^{-1}} f(t)dt + \left[K, \overline{B^{-1}} f\right]\right] = -\underbrace{\left[K, \overline{B^{-1}} f\right]}_{0} - \int_0^x K(x,t) f(t)dt =$$

$$= -\int_0^x K(x,t) f(t)dt\,;$$

$$\int_0^x B^+ K\, \overline{B^{-1}} f(t)dt = \int_0^x \overline{B^{-1}} f(t) B^+ K dt = (B^{-1}f, B^+K);$$

$$(Bu,v) = [u,v] + (u, B^+ v);$$

Gentle $u = \overline{B^{-1}} f$,$v = K$, we get

$$\left(B\overline{B^{-1}} f, K\right) = \left[\overline{B^{-1}} f, K\right] + (B^{-1}f, B^+K),$$

$$[u,v] = \textstyle\sum_{k=1}^{n} \sum_{m=0}^{n-k-1} (a_k v)^{(m)} (-1)^m u^{(n-k-1-m)}(x),$$

$$\left[\overline{B^{-1}} f, K\right] = \sum_{k=1}^{n} \sum_{m=0}^{n-k-1} (a_k K)^{(m)} (-1)^m \left(\overline{B^{-1}} f\right)^{(n-k-1-m)} \Bigg|_0^x = 0.$$

So

$$K(x,t)|_{t=x} = 0, \frac{\partial K}{\partial t}\bigg|_{t=x} = 0, \dots, \frac{\partial^{(n-2)} K}{\partial t^{(n-2)}}\bigg|_{t=x} = 0,$$

$$\left(\overline{B^{-1}} f\right)(0) = 0, \left(\overline{B^{-1}} f\right)'(0) = 0, \dots, \left(\overline{B^{-1}} f\right)^{(n-2)}(0) = 0.$$

Therefore,

$$\varepsilon y\left(x, \varepsilon, \frac{d^n}{dx^n} \overline{B^{-1}} f\right) = \varepsilon \left[(-1)^{n-1} \frac{\partial^{n-1} K}{\partial t^{n-1}}\bigg|_{t=x} \overline{B^{-1}} f - K(x,0)\left(\overline{B^{-1}} f\right)^{(n-1)}(0)\right] -$$

$$-\int_0^x K(x,t) f(t)dt = \overline{B^{-1}} f(x) - \left(\overline{B^{-1}} f\right)^{(n-1)}(0)\psi(x) - y(x,\varepsilon,f),$$

where $\psi(x)$ is a solution to the Cauchy's problem

$$\varepsilon \psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \dots + a_n(x)\psi(x) = 0,$$

$$\psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1.$$

Theorem 3.4 we can now reformulate.

Theorem 4.1. If the k - th coefficient of equation (1) is continuously differentiable $n-k$ $(k = 0,1,2,\dots,n)$ times on the interval [0,1] and satisfies the following conditions:

(a) $a_1(x) \geq \alpha > 0, \forall x \in [0,1]$;

(b) $\left(\sum_{k=2}^{n} a_k(x) y^{(n-k)}(x), y^{(n-1)}(x)\right) \geq 0, \forall y \in D(L_\varepsilon)$

and the right part $f(x) \in W_2^n[0,1]$, the solution of the Cauchy problem(1)-(2) also belongs to the space $W_2^n[0,1]$ and admits an asymptotic expansion of the form:

$$y(x,\varepsilon,f) = \sum_{k=0}^{n-1}(-1)^k \left[B^{-1}D^k f(x) - \left(B^{-1}D^k f\right)^{(n-1)}(0)\psi(x)\right]\varepsilon^k + \\ +(-1)^n \varepsilon^n y(x,\varepsilon,D^n f),$$

where

$$\|y(x,\varepsilon,D^n f)\| \leq \frac{\|D^n f\|}{\alpha},\ D^0 = I,\ D = \frac{d^n}{dx^n}\overline{B^{-1}},$$

$$\begin{cases} Bz(x) = a_1(x)z^{(n-1)}(x) + a_2(x)z^{(n-2)}(x) + \dots + a_n(x)z(x), \\ z(0) = 0, z'(0) = 0, \dots, z^{(n-2)}(0) = 0; \end{cases}$$

$$\begin{cases} \varepsilon\psi^{(n)}(x) + a_1(x)\psi^{(n-1)}(x) + \dots + a_n(x)\psi(x) = 0, \\ \psi(0) = 0, \psi'(0) = 0, \dots, \psi^{(n-2)}(0) = 0, \psi^{(n-1)}(0) = 1. \end{cases}$$

5. Summary. The spectral theory of equations with a deviating argument can be successfully applied in the study of singularly perturbed problems. The residual term formula can be used to control current errors in the numerical solution of such problems.

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ЖОҒАРҒЫ РЕТТІ КӘДІМГІ ДИФФЕРЕНЦАЛДЫҚ ТЕҢДЕУДІҢ СИНГУЛЯР ӘСЕРЛЕНГЕН КОШИ ЕСЕБІН ШЕШУДІҢ ОПЕРАТОРЛЫҚ ӘДІСІ ТУРАЛЫ

Аннотация. Бұл еңбекте аргументін ауытқыту әдісімен кәдімгі $\boldsymbol{n}$ −ретті дифференциалдық теңдеудің сингуляр әсерленген Коши есебінің шешімі асимтотикалық қатарға таратылды. Ал қатардың қалдығы теңдеудің оң жағындағы бос мүшесі арқылы бағаланды. Бұл саланың көптеген еңбектері кәделі саналады және олар техникалық мәселелермен тығыз байланысты, мүмкін, сондықтан болар, алынған бағамдар O - үлкен немесе o - кіші шамалары арқылы өрнектелген, сондықтан олар тек теориялық мазмұнға ие, сондықтан нақты кәдеге жарамайды, соған қарамастан мұндай жұмыстар жетіп артылады. Ұсынылып отырған еңбектің негізгі артықшылығы, алгоритімінің қарапайымдылығы мен қалдықтың формуласы болса керек, ол теңдеудің бос мүшесі арқылы өрнектелген және нақты бағаланған.

Түйін сөздер: Сингуляр әсерленген, спектрәлді таралым, ауытқыған аргумент, қалдық мүшенің бағамы, жалқы оператор, Гилберт пен Шмидтің теоремасы, әсіре үзіксіз оператор, Фридрихстың леммасы, Кошидің есебі, асимптотикалық таралым, мардымсыз параметр.

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ОБ ОДНОМ ОПЕРАТОРНОМ МЕТОДЕ РЕШЕНИЯ СИНГУЛЯРНО ВОЗМУЩЕННОЙ ЗАДАЧИ КОШИ ДЛЯ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $\boldsymbol{n}$-ГО ПОРЯДКА

Аннотация. В настоящей работе, методом отклоняющегося аргумента, получено асимптотическое разложение решения задачи Коши для обыкновенного дифференциального уравнения $n-$ го порядка с переменными коэффициентами, с оценкой остаточного члена через правую часть уравнения. Многие работы посвященные к этой теме носят прикладной характер, и полученные им оценки остаточного члена выражены в терминах O −большое, или o −малое, поэтому имеют теоретическое значение, нежели прикладное, как они утверждают.Основным достойнством предлагаемого нами метода яляется простота его алгортитма, и формула остаточного члена, явно выраженная через правую часть уравнения, и его оценка.

Ключевые слова: Сингулярное возмущение, спектральное разложение, отклоняющиеся аргумент, оценка остаточного члена, самосопряженный оператор, теорема Гилберта - Шмидта, вполне непрерывный оператор, лемма Фридрихса, задача Коши, асимптотическое разложение, малый параметр.

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